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Lorentz invariance from classical particle paths in quantum field theory of electric and magnetic charge

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We establish the Lorentz invariance of the quantum field theory of electric and magnetic charge. This is *a priori* implausible because the theory is the second-quantized version of a classical field theory which is inconsistent if the minimally coupled charged fields are smooth functions. For our proof we express the generating functional for the gauge-invariant Green's functions of quantum electrodynamics—with or without magnetic charge—as a path integral over the trajectories of classical charged point particles. The electric-electric and electric-magnetic interactions contribute factors $\exp(JDJ)$ and $\exp(JD'K)$, where J and K are the electric and magnetic currents of classical point particles and D is the usual photon propagator. The propagator D' involves the Dirac string but $\exp(JD'K)$ depends on it only through a topological integer linking string and classical particle trajectories. The charge quantization condition $(e_i g_j - g_i e_j)/4\pi$ = integer then suffices to make the gauge-invariant Green's functions string independent. By implication, our formulation shows that if the Green's functions of quantum electrodynamics are expressed, as usual, as functional integrals over classical charged fields, the smooth field configurations have measure zero and all the support of the Feynman measure lies on the trajectories of classical point particles.

I. INTRODUCTION

The generalized Maxwell equations which describe the interaction of the electromagnetic field $F_{\mu\nu}$ with the currents J_μ and K_μ of, respectively, electric and magnetic charges are[1]

$$\partial\cdot F=J\,, \tag{1.1}$$

$$\partial\cdot F^d=K\,, \tag{1.2}$$

where $F^d_{\mu\nu}\equiv\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$ is the dual field. In relativistic particle classical mechanics, these equations, together with the generalized Lorentz force law, are Lorentz invariant for any values of the electric and magnetic charges. In quantum mechanics, however, the fields do not provide a complete description and a vector potential A_μ is also required.[2,3] Then, a consistent (rotationally invariant) theory is possible when $K\neq 0$ only if $F_{\mu\nu}$ is physically equivalent to $\partial_\mu A_\nu-\partial_\nu A_\mu\equiv(\partial\wedge A)_{\mu\nu}$, and that requires that the charge quantization condition[4]

$$\epsilon_{ij}\equiv(e_i g_j-e_j g_i)=4\pi n_{ij},\quad n_{ij}=0,\pm1,\pm2,\ldots\,, \tag{1.3}$$

among the electric charges e_i and magnetic charges g_i be satisfied, as originally shown by Dirac.[5] The reason that the constraint (1.3) arises upon quantization of the unconstrained classical theory is that an action formulation of the classical theory is required for quantization, and an action must include the vector potential (or something equivalent). The classical action can be defined only modulo each factor ϵ_{ij}, and that leads to the weak quantization condition[6]

$$\epsilon_{ij}=2\pi n_{ij}v\,, \tag{1.4}$$

with v a fixed but undetermined constant. Upon quantization, (1.4) becomes strengthened to (1.3), in which $v=\hbar=1$. But, unlike the classical particle theory, the classical field theory is *never* consistent, even without an action principle. The classical charged field φ_a carrying charges e_a and g_a is minimally coupled by means of the affine connection D^a_μ (presumably $D^a_\mu=\partial_\mu+ie_aA_\mu+ig_aB_\mu$) with curvature $[D^a_\mu,D^a_\nu]=i(e_aF_{\mu\nu}+g_aF^d_{\mu\nu})$. The Jacobi identity $\sum[D_\kappa,[D_\lambda,D_\mu]]=0$ is violated for we find instead, from Eqs. (1.1) and (1.2),

$$\sum[D_\kappa,[D_\lambda,D_\mu]]=i(e_aK_\nu-g_aJ_\nu)\,.$$

In the classical theory of charged fields, the currents are smooth functions, and (1.4) is of no avail in making the right-hand side of this relation effectively vanish. It is the pointlike nature of the charged particles which leads to the consistency of the first-quantized theory. Now, quantum field theory (the second-quantized theory) is obtained by quantization of the inconsistent classical field theory, not the consistent particle theory, and so it appears highly suspect. Of course, the classical field theory acquires various particle aspects after quantization, and the possibility exists that these aspects are sufficiently strong so as to reinstate Lorentz invariance. In this paper we will argue that this is exactly what happens.[7]

It is extremely convenient to use functional methods to study problems of the above type. For example, the role of the classical particle theory as the prequantized quantum mechanics is dramatically illustrated by Feynman's[8] representation of the Schrödinger wave function as an integral over all classical trajectories weighted by the exponential of the classical particle action. Gauge-invariant observables are seen in this way to be rotationally invariant because of (1.3) and the fact that the action changes only by ϵ_{ij} under rotations.[9] Analogously, the Green's functions of the quantum field theory may be expressed as functional integrals over classical fields weighted by the exponential of the classical field action. It thus appears that even the gauge-invariant Green's functions will not be Lorentz invariant because of the noninvariance of the classical field theory. Consistency of the nonrelativistic quantum mechanics is crucially dependent on the point nature of the charges whereas most of the contributions to the functional field integrals would appear to correspond to spread-out distributions of charge. What we will show is that, in spite of their appearance, these functional integrals have their essential support on the trajectories of classical point particles. Relativistic invariance then follows.

There have been a number of investigations of quantum field theories of electric and magnetic charge.[9-15] What has emerged from this work is that (1.3) is certainly necessary for the consistency of the theory. The sufficiency of (1.3) to guarantee Lorentz invariance has, however, never been demonstrated, even formally. Schwinger's[12] original argument for Lorentz invariance depended on delicate limiting procedures which implied in addition that the integers n_{ij} in (1.3) are divisible by four. Schwinger[15] has more recently abandoned this approach and rescinded his claim about the n_{ij} and, consequently, by implication about Lorentz invariance. A local formulation of the quantum field theory is given in Ref. 14. This formalism displays what, beyond (1.3), is *sufficient* for relativistic invariance. This turns out to be a particular condition on the physical states which is not subject to analysis at the present time. The perturbative expansion of the theory also sheds no light on this issue since it is manifestly noninvariant, only because, we hope, (1.3) cannot hold in finite orders of perturbation theory.

A convincing demonstration that the existence of magnetic monopoles is consistent with the combined principles of quantum mechanics and relativity has thus been lacking. The purpose of this paper is to indicate how to fill this gap. Using formal but standard functional methods, we explicitly establish the Lorentz invariance of the gauge-invariant Green's functions. Our results thus strongly suggest the consistency of the quantum field theory, and we can conclude that the reason that monopoles have not yet been found lies elsewhere than in their possible inconsistency with relativity and quantum mechanics.

Our analysis for the case when the charge-bearing fields are Lorentz scalars uses Feynman's path-integral representation[16] for the exponential of the external field propagator. An analogous representation for Lorentz spinor charged fields has not been heretofore derived and so we have had to deduce such a representation. We present, in fact, two such representations. The first is appealing because the current of the spinor particles has the same form as for scalar particles. Unfortunately the relevant quantities appear to have only a rather formal existence, as discussed in Sec. IV, and for this reason we pass to a second representation. Fortunately this one is on the same footing as the one for scalar fields, except that we must average over not only all possible classical trajectories, but also over all possible classical first moments defined on these trajectories.

Our study of Lorentz invariance for the quantum field theory rests heavily on the invariance of the classical action formalism and of the first-quantized theories and so we devote Sec. II to a review of these simpler theories. This section also contains an extension of the classical theory to include anomalous first moments in the charged currents and an illustration of the noninvariance of the classical field theory. Section III is a short review of the second-quantized theories of monopoles. Our path-integtal representations are deduced in Sec. IV. This section includes a general discussion of how to represent the time-ordered exponential of a matrix integral as an unordered functional integral. Our proof of Lorentz invariance of the quantum field theories is given in Sec. V. We explicitly treat only the generating functional of the conserved current Green's functions. Using the path-integral representations of Sec. IV, we establish Lorentz invariance for both spin-0 and spin-$\frac{1}{2}$ charged fields. The final Sec. VI contains some concluding remarks.

II. CLASSICAL AND FIRST-QUANTIZED THEORIES

Classical relativistic particle electromagnetodynamics is characterized by the generalized Maxwell equations (1.1) and (1.2), in which the currents

$$J_\mu(x) = \sum_{i=1}^{l} e_i \int_{\Gamma_i} dz_\mu \delta^4(x-z), \tag{2.1}$$

$$K_\mu(x)=\sum_{i=1}^{l} g_i \int_{\Gamma_i} dz_\mu \delta^4(x-z) \tag{2.2}$$

arise from charged point particles, and by the generalized Lorentz force law

$$m_i \ddot{z}_i = [e_i F(z_i) + g_i F^d(z_i)] \cdot \dot{z}_i. \tag{2.3}$$

Here e_i, g_i, m_i, and $\Gamma_i = \{z_i(s);\ -\infty \le s \le +\infty\}$ are, respectively, the electric charge, magnetic charge, mass, and trajectory of the ith particle, $i=1,\ldots,l$; s is proper time; $\dot{z}=dz/ds$, etc. This theory is obviously Lorentz invariant for any charges. The solution describing a single monopole of strength g at rest at $\vec{r}=0$ has the usual Coulomb form

$$F_{cij} = \frac{g}{4\pi}\epsilon_{ijk}\frac{r_k}{r^3},\quad i,j=1,2,3,\quad F_{cio}=0. \tag{2.4}$$

The corresponding magnetic field is

$$\vec{H}_c = \frac{g}{4\pi}\frac{\vec{r}}{r^3}. \tag{2.5}$$

The above equations of motion follow[6] from a number[10,17,14] of different-looking but actually equivalent[9] action principles. For example, the action given in Ref. 17 is essentially

$$I_n^{(c)} = I^{(c)} + I_n \tag{2.6}$$

with

$$I^{(c)} = -\sum_i m_i \int ds \tag{2.7}$$

and

$$I_n = \int d^4x\left[\tfrac{1}{4}F^2 - \tfrac{1}{2}F\cdot(\partial\wedge A) - J\cdot A - K\cdot B_n\right], \tag{2.8}$$

where

$$B_n(x) = \int dw \cdot F^d(x-w) = (n\cdot\partial)^{-1} n \cdot F^d(x). \tag{2.9}$$

Here $w_\mu(\tau)$ defines a path from $w_\mu(o)=0$ to $w_\mu(\infty)=\infty$ which, for simplicity, is chosen to be a straight line $w_\mu(\tau)=\tau n_\mu$ ($n^2=-1$). In the second equality $(n\cdot\partial)^{-1}$ is the integral operator with kernel[14]

$$(n\cdot\partial)^{-1}(x) = \tfrac{1}{2}\int_0^\infty d\eta\,[\delta^4(x-n\eta) - \delta^4(x+n\eta)]. \tag{2.10}$$

The antisymmetric form of this kernel has been chosen to invoke dual invariance, and this choice will be maintained throughout this paper.[4]

The first Maxwell equation (1.1) follows from (2.6) upon variation of A and the second equation (1.2) is an immediate consequence of the relation

$$F = \partial\wedge A - G^d, \tag{2.11}$$

where

$$G = (n\cdot\partial)^{-1} n \wedge K, \tag{2.12}$$

which follows from (2.6) upon variation of F. But to obtain the correct Lorentz force law (2.3) from variation of Γ_i, (2.6) must be defined modulo each ϵ_{ij} (Refs. 6, 18) (otherwise $I^{(c)}$, which changes by ϵ_{ij} when the trajectory of the ith particle sweeps through the string attached to the jth particle, will not be a continuous function of the trajectories Γ_i) and in the line integrals in (2.8), a contour prescription must be used when a trajectory Γ_i intersects a string[6] (otherwise the equation $m_i\ddot{z}_i = [e_i(\partial\wedge A) + g_i(\partial\wedge B_n)]\cdot\dot{z}_i$, which differs from (2.3) when a particle hits a string, will result from variation of Γ_i). When the actions of Refs. 10 and 14 are similarly amended, they can be shown to be equivalent to (2.6).[9]

Note that in (2.11) the physical field F is seen to differ from $\partial\wedge A$, which carries an unphysical string singularity, by the singular function G^d, which removes this string singularity. For example, the static Coulomb field (2.4) is given by (2.11) with the Dirac symmetric vector potential

$$A_{\hat{n}}(\vec{r}) = \frac{g}{8\pi r}\left(\frac{\hat{n}\times\vec{r}}{r-\hat{n}\cdot\vec{r}} - \frac{\hat{n}\times\vec{r}}{r+\hat{n}\cdot\vec{r}}\right). \tag{2.13}$$

The Coulomb magnetic field (2.5) is given by

$$\vec{H}_c = \vec{\nabla}\times\vec{A}_{\hat{n}} - \vec{h}_{\hat{n}}, \tag{2.14}$$

with

$$\vec{h}_{\hat{n}}(\vec{r}) = -\tfrac{1}{2}g\hat{n}\int_0^\infty d\lambda[\delta^3(\vec{r}-\hat{n}\lambda) - \delta^3(\vec{r}+\hat{n}\lambda)]. \tag{2.15}$$

Although the Lorentz invariance of (2.6), defined as in Ref. 6, is obvious from the Lorentz invariance of the consequent equations of motion, it is instructive to demonstrate this invariance directly. Consider the combined string rotation and (singular) gauge transformation[9]

$$\begin{aligned} &A \to A + \partial\lambda,\ B_n \to B_{n'} = (n'\cdot\partial)^{-1} n'\cdot F^d,\\ &F\to F,\quad \Gamma_i \to \Gamma_i. \end{aligned} \tag{2.16}$$

The function $\lambda(x)$ is determined by the condition

$$\partial\wedge\partial\lambda = \{[(n'\cdot\partial)^{-1}n' - (n\cdot\partial)^{-1}n]\wedge K\}^d, \tag{2.17}$$

which requires that it have a discontinuity $\pm g_j/2$ through the surface $\Sigma_j = \Gamma_j \times \{\tau' n' \times \tau n:\ 0\le\tau,\tau'\le\infty\}$ for each j. The contribution $\int J\cdot A$ to (2.8) then changes under (2.16) by $\sum_{ij}\epsilon_{ij}N_{ij}$, where N_{ij} is the number of times (positive, negative, or zero) that Γ_i intersects Σ_j (or, equivalently, Γ_j intersects Σ_i). $I_n^{(c)}$ is thus invariant to (2.16) if it is defined, as in Ref. 6, modulo each ϵ_{ij}. $I_n^{(c)}$ must therefore be considered as a map onto a circle of some radius r, and consistency then demands that the quantization conditions (1.4) are satisfied.

We consider next a generalization of the above classical theories in which magnetic and electric moments contribute to the currents:

$$J_\mu(x)=\sum_i \int_0^\infty d\tau\,[e_i\dot{z}_\mu(\tau)+\sigma_{i\mu\nu}(\tau)\partial^\nu]\delta^4(x-z)\,, \tag{2.18}$$

$$K_\mu(x)=\sum_i \int_0^\infty d\tau[g_i\dot{z}_\mu(\tau)+\lambda_{i\mu\nu}(\tau)\partial^\nu]\delta^4(x-z)\,. \tag{2.19}$$

We wish to show that these generalized theories are also Lorentz invariant, even after quantization, with no further consistency conditions beyond (1.3), i.e., for arbitrary moment functions $\sigma_{\mu\nu}$ and $\lambda_{\mu\nu}$. (In particular, the theory is consistent if all ϵ_{ij} vanish for any σ and λ.) The basic reason is that the moments couple to the fields rather than to the potentials. This can be seen simply by noting that a trajectory Γ_i that loops around a string of flux $\frac{1}{2}g_j$ now gives

$$\begin{aligned}\int dx\, J\cdot A &= e_i \oint dz\cdot A+\tfrac{1}{2}\int d\tau\sigma_{i\mu\nu}(\tau)(\partial\wedge A)^{\mu\nu}\\ &\quad+\text{dual contribution}\\ &\to \tfrac{1}{2}e_i g_j+\tfrac{1}{2}\int d\tau\sigma_{i\mu\nu}(\tau)(n\cdot\partial)^{-1}(n\wedge K)^{d\mu\nu}\\ &\quad+\text{dual contribution}\,,\end{aligned} \tag{2.20}$$

where (2.11) and (2.12) have been used. The contribution of the second term clearly vanishes if particle trajectories do not hit strings or, more generally, if a contour prescription is used as in Ref. 6 if a trajectory does hit a string. To be more precise, consider the effect of the string-changing gauge transformation (2.16) on the action (2.6) in which now

$$\begin{aligned}\int d^4x(J\cdot A+K\cdot B_n) &=\sum_i \int d\tau\{\dot{z}_i\cdot(e_iA+g_iB_n)\\ &\quad+\tfrac{1}{2}[\sigma_i\cdot(\partial\wedge A)+\lambda_i\cdot(\partial\wedge B_n)]\}\,.\end{aligned} \tag{2.21}$$

The induced change in the first-moment terms is, e.g.,

$$\begin{aligned}\tfrac{1}{2}\int d\tau\sigma_{\mu\nu}(\tau)(\partial\wedge\partial\lambda)^{\mu\nu}\\ =\tfrac{1}{2}\int d\tau\sigma^d_{\mu\nu}(\tau)[n'(n'\cdot\partial)^{-1}-n(n\cdot\partial)^{-1}]\wedge K\,,\end{aligned} \tag{2.22}$$

which vanishes if trajectories do not intersect strings or if a suitable contour prescription is used.

The theory obtained by first quantization of the nonrelativistic limit of the above classical theory is described by the Schrödinger equation

$$(2m)^{-1}(i\vec{\nabla}-e\vec{A}_{\vec{n}})^2\Phi_{\vec{n}}=E\Phi_{\vec{n}}\,. \tag{2.23}$$

Here $\Phi_{\vec{n}}(\vec{r})$ is the time-independent quantum-mechanical wave function and $\vec{A}_{\vec{n}}$ is related to a given solution of the classical equations of motion by (2.11). A specific example is (2.13). Given (1.3), a string rotation in (2.23) is equivalent to a gauge transformation, and so the theory is rotationally invariant.[5,19,20] The string independence of the theory can also be seen in a way which emphasizes the functional methods which we will use in the following sections. According to Feynman,[8]

$$\Phi_n(x)=N\int d\Gamma_x \exp[iI_n^{(c)}(\Gamma_x)/\hbar]\,, \tag{2.24}$$

where the integration is over all classical trajectories Γ_x terminating at x. A gauge-invariant observable such as

$$\rho_n(x)\equiv\Phi_n^\dagger(x)\Phi_n(x)=N^2\int d\gamma_x \exp[iI_n^{(c)}(\gamma_x)/\hbar] \tag{2.25}$$

is represented by an integral over all closed trajectories γ_x through x. Since a string change changes $I_n^{(c)}(\gamma_x)$ by $\sum\epsilon_{ij}N_{ij}$, (2.25) will be string independent given (1.3).[9]

A classical field theory differs from the above classical particle theories in that the currents are given not by the pointlike forms (2.1) and (2.2) but by continuous distributions. For example, the currents appropriate to l charged scalar fields $\phi_i(x)$ are

$$J=\sum_{i=1}^{l}[ie_i\phi_i^\dagger\overleftrightarrow{\partial}\phi_i-2e_i(e_iA+g_iB)\phi_i^\dagger\phi_i] \tag{2.26}$$

and

$$K=\sum_{i=1}^{l}[ig_i\phi_i^\dagger\overleftrightarrow{\partial}\phi_i-2g_i(e_iA+g_iB)\phi_i^\dagger\phi_i]\,. \tag{2.27}$$

The theory is then described by the Maxwell equations (1.1) and (1.2) together with the equations

$$[(i\partial - e_i A - g_i B_n)^2 - m_i^2]\phi_i = 0 \tag{2.28}$$

[with (2.9)–(2.12)] for the charged fields.

As shown in the Introduction, because of violation of the Jacobi identity, this field theory can never be consistent for $K \neq 0$. This inconsistency cannot be avoided for continuous distribution of charge by a charge quantization condition and singular gauge transformations. For example, the equation (2.17) for the gauge function which compensates a string change has no solution for a smooth K such as (2.27) because the left-hand side of the equation must vanish if the right-hand side is nonsingular. We will show in the remainder of this paper that the corresponding quantum field theory is nevertheless Lorentz invariant when (1.3) is satisfied.

III. SECOND-QUANTIZED THEORIES

We will consider quantum field theories in which the electric- and magnetic-charge-bearing fields are either spin 0 or spin $\frac{1}{2}$. The equations of motion and action in the spin-0 case are formally the same as the classical forms, except that a gauge-fixing term and quartic self-interaction terms

$$I' \equiv -\sum_{ij} \lambda_{ij}(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) \tag{3.1}$$

must be added to the action. For the spin-$\frac{1}{2}$ case, the electric and magnetic currents are simply

$$J_\mu = \sum_i e_i \overline{\psi}_i \gamma_\mu \psi_i , \tag{3.2}$$

$$K_\mu = \sum_i g_i \overline{\psi}_i \gamma_\mu \psi_i , \tag{3.3}$$

and the charged fields obey the generalized Dirac equation[21]

$$(i\partial\!\!\!/ - m_i - e_i A\!\!\!/ - g_i B\!\!\!/)\psi_i = 0 . \tag{3.4}$$

A suitable action is

$$I_n^{(1)} = I_n + I^{(1)} , \tag{3.5}$$

where I_n is still given by (2.8) (plus a gauge-fixing term) and

$$I^{(1)} = \sum_i \int d^4x \overline{\psi}_i (i\partial\!\!\!/ - m_i)\psi_i . \tag{3.6}$$

The Green's functions in these quantum field theories are given by functional integrals over the corresponding classical fields. For example,

$$\langle 0|T\theta_1 \cdots \theta_l|0\rangle = N \int dA\, dF\, (\Pi d\psi_i d\overline{\psi}_i)\theta_1 \cdots \theta_l \times \exp[iI_n^{(1)}(A, F, \psi_1 \cdots \overline{\psi}_N)] , \tag{3.7}$$

where the θ_k are any local operators in the spin-$\frac{1}{2}$ theory and N is a normalization constant. Such integrals are well defined (apart from renormalization) provided either a gauge-fixing term is included in $I_n^{(1)}$ or a Faddeev-Popov[22] gauge-fixing factor is included in the measure.

In the above formalisms, the independent variables are the potential A, the electromagnetic field F, and the charged fields. The potential $B = B_n$ is explicitly given by Eq. (2.9) as a nonlocal function of F, and all of the n dependence of the actions resides in this function in I_n [Eq. (2.8)]. It is more convenient for us at this point to use the alternative formalism of Ref. 14. Although equivalent[9] to the previous formalisms, it has the virtue of manifest locality. The actions are

$$S_n^{(0)} = S_n + S^{(0)} , \quad S_n^{(1)} = S_n + S^{(1)} , \tag{3.8}$$

with

$$S_n = -\tfrac{1}{2} \int d^4x \{[n \cdot (\partial \wedge A)] \cdot [n \cdot (\partial \wedge B)^d] + [n \cdot (\partial \wedge A)]^2 - [\partial(n \cdot A)]^2 + (A \to B, B \to -A)\} , \tag{3.9}$$

and

$$S^{(0)} = \sum_i \int d^4x [(-i\partial - e_i A - g_i B)\phi_i^\dagger \times (i\partial - e_i A - g_i B)\phi_i - m_i^2 \phi_i^\dagger \phi_i - \lambda_i (\phi_i^\dagger \phi_i)^2] \tag{3.10}$$

for spin 0, and

$$S^{(1)} = \sum_i \int d^4x \overline{\psi}_i (i\partial\!\!\!/ - m_i - e_i A\!\!\!/ - g_i B\!\!\!/)\psi_i \tag{3.11}$$

for spin $\frac{1}{2}$. Now the action is a local function of the independent variables A, B, and the charged fields, and the n dependence resides in the explicit n's in S_n, which includes a gauge-fixing term. Variation of A and B in (3.8) gives the correct field equations (1.1) and (1.2), with

$$F = n \wedge [n \cdot (\partial \wedge A)] - \{n \wedge [n \cdot (\partial \wedge B)]\}^d , \tag{3.12}$$

and with currents (2.26) and (2.27) for spin 0 or (3.2) and (3.3) for spin $\frac{1}{2}$, and also gives the gauge-fixing equations

$$\partial^2 n \cdot A = \partial^2 n \cdot B = 0 . \tag{3.13}$$

Variation of the charged fields in (3.8) gives the correct equations (2.28) (with $-2\sum_j \lambda_{ij}\varphi_i\varphi_j^\dagger\phi_j$) or (3.4). Finally, because of the explicit n dependence, the canonical generalized angular momentum tensor $M_{\lambda\mu\nu}$ which arises from (3.8) is not conserved. Rather[14]

$$\partial \cdot M = -n \wedge \{n \cdot [(n\cdot\partial)^{-1}J \wedge (n\cdot\partial)^{-1}K]^d\}. \quad (3.14)$$

The above theory can be quantized as usual and developed in a perturbative expansion in which the gauge field propagators are given by the noncovariant expressions

$$D_{AA}^{\mu\nu}(q) = D_{BB}^{\mu\nu}(q) = [-g^{\mu\nu} + (q^\mu n^\nu + q^\nu n^\mu)(n\cdot q)^{-1}] \times (q^2 + i\epsilon)^{-1}, \quad (3.15)$$

$$D_{AB}^{\mu\nu}(q) = -D_{BA}^{\mu\nu}(q) = (n \wedge q)^{d\mu\nu}(n\cdot q)^{-1}(q^2+i\epsilon)^{-1}. \quad (3.16)$$

The $i\epsilon$ prescription for $(n\cdot q)^{-1}$ is irrelevant in (3.15) because of current conservation, but must be the principal-value prescription in (3.16) in order to maintain dual invariance.[4,9] The resulting noncovariant perturbation expansion can be shown to be unitary, consistent with the Faddeev-Popov formalism, dual invariant, and renormalizable.[9] For the exact theory, on states for which (3.14) vanishes, the canonical Poincaré generators satisfy the Lie algebra of the Poincaré group and, if (1.3) is satisfied, can be integrated to give a group representation.[14] The key question is whether there are enough states on which $\partial\cdot M = 0$. We will not investigate this question directly but will instead explicitly establish the Poincaré invariance of the gauge-invariant Green's functions.

The Green's functions for local operators $\theta_1 \cdots \theta_l$ in, for example, the spinor theory are given by the functional integral

$$\langle 0|T\theta_1\cdots\theta_l|0\rangle = N\int dA\, dB\,(d\psi_1\cdots d\bar\psi_N)\theta_1\cdots\theta_l \times \exp[iS_n^{(1)}(A,B,\psi_1\cdots\bar\psi_N)]. \quad (3.17)$$

For gauge-invariant θ's, this expression is equivalent to (3.7).[9] For such θ's, we will show in Sec. V that (3.17) is n independent.

IV. PATH-INTEGRAL REPRESENTATIONS

In this section, as an important preliminary to our proof of Lorentz invariance, we will derive path-integral representations of matrix elements of certain unitary operators. We consider the standard representation of the canonical commutation relations

$$[X_\mu, P_\nu] = -ig_{\mu\nu} \quad (4.1)$$

among the operators X_μ and P_μ. We will use both the position representation $(|x\rangle = |x_0\rangle|x_1\rangle|x_2\rangle|x_3\rangle)$

$$\langle x|X = x\langle x|, \quad \langle x|P = i\partial\langle x|, \quad (4.2)$$

and the momentum representation

$$\langle p|P = p\langle p|, \quad \langle p|X = -i\partial_p\langle p|, \quad (4.3)$$

with

$$\langle x|p\rangle = \frac{1}{(2\pi)^2}e^{-ip\cdot x}. \quad (4.4)$$

The normalizations are

$$\langle x|x'\rangle = \delta^4(x-x'), \quad \langle p|p'\rangle = \delta^4(p-p), \quad (4.5)$$

and the completeness relations read

$$\int d^4x|x\rangle\langle x| = 1, \quad \int d^4p|p\rangle\langle p| = 1. \quad (4.6)$$

The matrix element of interest in the spin-0 theory is

$$U(\tau;x,x') = \langle x|\exp(\tfrac{1}{2}i\tau\{[P-a(X)]^2 - m^2\})|x'\rangle, \quad (4.7)$$

where $a_\mu(x)$ is an arbitrary function. The representation given long ago by Feynman is[16]

$$U(\tau;x,x') = N\int d\Gamma(\tau;x,x')\exp\left\{-\tfrac{1}{2}im^2\tau - i\int_0^\tau d\tau'[\tfrac{1}{2}\dot z^2(\tau') + a(z(\tau'))\cdot\dot z(\tau')]\right\}, \quad (4.8)$$

in which the integration is over all x-space paths $\Gamma(\tau;x,x')$ between the points $x' = z(0)$ and $x = z(\tau)$.

It would be nice to derive an analogous expression for the matrix element

$$V(\tau;x,x') = \langle x|\exp\{i\tau[\not P - \not a(X) - m]\}|x'\rangle \quad (4.9)$$

in the spin-$\frac{1}{2}$ theory. Note that (4.9) is a 4×4 matrix in the spinor space. We begin by using the Trotter[23] formula

$$e^{i(A+B)} = \lim_{N\to\infty}(e^{iA/N}e^{iB/N})^N \quad (4.10)$$

in (4.9) to obtain

$$V(\tau;x,x') = \lim_{N\to\infty}\langle x|[e^{-i(\tau/N)\not a}e^{i(\tau/N)(\not p - m)}]^N|x'\rangle. \quad (4.11)$$

We next insert the completeness relations (4.6) in (4.11) N and $N-1$ times, and use (4.2)–(4.4):

$$V(\tau;x,x')=\lim_{N\to\infty}\int d^4x_1\cdots\int d^4x_{N-1}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,e^{-i(\tau/N)\not a(x)}e^{-ip_N\cdot(x-x_{N-1})}e^{i(\tau/N)(\not p_N-m)}\cdots$$
$$\times e^{-i(\tau/N)\not a(x_1)}e^{-ip_1\cdot(x_1-x')}e^{i(\tau/N)(\not p_1-m)}\,. \tag{4.12}$$

Note that the order of the factors is important here because of the noncommutativity of the exponents. We now write

$$\lim_{N\to\infty}\int\prod_{i=1}^{N-1}d^4x_i\prod_{i=1}^{N}(2\pi)^{-4}d^4p_i=N'\int d\Gamma(\tau;x,x')\int d\Omega\,, \tag{4.13}$$

to cast (4.12) into the form

$$V(\tau;x,x')=N'\int d\Gamma(\tau;x,x')\int d\Omega\exp\left[-i\int_0^\tau d\tau' p(\tau')\cdot\dot z(\tau')-i\tau m\right]T\exp\left\{i\int_0^\tau d\tau'[\not p(\tau')-\not a(z(\tau'))]\right\}, \tag{4.14}$$

in which the integrations are over all x-space paths $\Gamma(\tau;x,x')$ from $x'=z(0)$ to $x=z(\tau)$ and all p-space paths Ω between unrestricted points $p(0)$ and $p(\tau)$, and T denotes τ' ordering.

Let us next change integration variables in (4.14) from p to $\pi=p-a(x)$. We obtain

$$V(\tau;x,x')=\int d\Gamma(\tau;x,x')F(\Gamma(\tau;x,x'))$$
$$\times\exp\left[-i\int_0^\tau d\tau' a(z(\tau'))\cdot\dot z(\tau')-i\tau m\right], \tag{4.15}$$

where the functional

$$F(\Gamma)\equiv N'\int d\pi T\exp\left\{i\int_0^\tau d\tau'[\not\pi(\tau')-\pi(\tau')\cdot\dot z(\tau')]\right\} \tag{4.16}$$

is independent of a and of m. Our final expression (4.15) is formally very similar to the scalar representation (4.8). Unfortunately the expression (4.16) appears to have a purely formal existence.

To illustrate the difficulty with (4.16), we consider the related ordinary integral

$$F(\tau;x)\equiv(2\pi)^{-4}\int d^4p\,e^{i(x\cdot p+\tau\gamma\cdot p)}\,. \tag{4.17}$$

Formally this is $\delta^4(x+\tau\gamma)$, but the nonexistence of (4.17) as an ordinary distribution follows from its formal properties:

$$\left(\frac{\partial}{\partial\tau}-\gamma^\mu\partial_\mu\right)F(\tau,x)=0\,,\quad F(0,x)=\delta^4(x)\,.$$

The quantity F may be expressed in terms of a Lorentz scalar function $C(\tau,x)$

$$F(\tau,x)=\left(\frac{\partial}{\partial\tau}+\gamma^\mu\partial_\mu\right)C(\tau,x)$$

which satisfies

$$\left(\frac{\partial^2}{\partial\tau^2}-\partial^2\right)C(\tau,x)=\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial t^2}+\nabla^2\right)C(\tau,x)=0\,, \tag{4.18a}$$

$$C(0,x)=0\,,\quad \dot C(0,x)=\delta^4(x)\,. \tag{4.18b}$$

This is a hyperbolic partial differential equation in τ and x^μ but the initial surface $\tau=0$ is spacelike instead of timelike (with respect to the hyperbolic operator). In this case no solution as an ordinary distribution exists.[24] A related feature is that usual Feynman path integrals are basically Gaussian and are related to parabolic partial differential equations [typically $(i\partial/\partial\tau+\nabla^2)K=0$] rather than hyperbolic or elliptic ones. It is true, however, that expression (4.17) can be given an unambiguous meaning on a class of testing functions. The power series representation

$$F(\tau;x)=\sum_{j=0}^{\infty}\frac{1}{j!}(\tau\gamma\cdot\partial)^j\delta^4(x)$$
$$=\sum_{j\text{ even}}\frac{1}{j!}\tau^j\Box^{j/2}\delta^4(x)$$
$$+\sum_{j\text{ odd}}\frac{1}{j!}\tau^j\Box^{(j-1)/2}\gamma\cdot\partial\delta^4(x) \tag{4.19}$$

is obviously well defined on polynomial testing functions, and the Fourier transform

$$e^{i\tau\gamma\cdot p}=\cos\tau\sqrt{p^2}+\frac{i\gamma\cdot p}{(p^2)^{1/2}}\sin\tau\sqrt{p^2} \tag{4.20}$$

is well defined on testing functions $f(p)$ which approach zero for $p^2\to-\infty$ faster than $e^{-\tau(-p^2)^{1/2}}$; e.g., on functions of compact support in p^2.

We will not attempt here to show that (4.16) is well defined in the context in which we will use it. We will instead use a different representation for the spin-$\frac{1}{2}$ case and only use (4.15) to illustrate the formal similarity between the spin-$\frac{1}{2}$ and spin-0 theories.

Before proceeding with the spin-$\frac{1}{2}$ case, we consider the general τ-ordered exponential

$$U_{ij}(\tau)=\left[T\exp\left(-i\int_0^\tau d\tau' h(\tau')\right)\right]_{ij},\quad i,j=1,\ldots,M \tag{4.21}$$

where $h(\tau')=h_{ij}(\tau')$ is an arbitrary $M\times M$ matrix function of τ' (not necessarily Hermitian). We wish to exhibit (4.21) as an unordered functional integral. Let $a_i, a_i^\dagger$, $i=1,\ldots,M$ be a set of M Bose or Fermi annihilation and creation operators

$$[a_i, a_j^\dagger]_\pm=\delta_{ij},\quad [a_i,a_j]_\pm=0, \tag{4.22}$$

and let the operator

$$H(\tau)\equiv a_i^\dagger h_{ij}(\tau)a_j \tag{4.23}$$

correspond to the matrix $h_{ij}(\tau)$. The set of one-quantum states

$$\psi_j=a_j^\dagger\Omega,\quad j=1,\ldots,M \tag{4.24}$$

where Ω is the ground state, $a_i\Omega=0$ for $i=1,\ldots,M$, provide a basis for the identity representation since

$$H(\tau)a_j^\dagger\Omega=\sum_i h_{ij}(\tau)a_i^\dagger\Omega. \tag{4.25}$$

Furthermore the desired quantity is

$$U_{ij}(\tau)=(a_i^\dagger\Omega, U(\tau)a_j^\dagger\Omega) \tag{4.26}$$

where

$$U(\tau)=T\exp\left[-i\int_0^\tau d\tau' H(\tau')\right] \tag{4.27}$$

is recognized as the time displacement operator for the quantum-mechanical system with time-dependent Hamiltonian $H(\tau)=a_i^\dagger h_{ij}(\tau)a_j$. The matrix elements in a harmonic-oscillator basis of the time displacement operator can conveniently be expressed as a functional integral using the analytic representation.[25] Let $U(\tau)$ be written as the limit of an ordinary operator product:

$$U(\tau)=\lim_{n\to\infty}\exp[-i\epsilon H(n\epsilon)]\cdots\exp[-i\epsilon H(m\epsilon)] \times\cdots\exp[-i\epsilon H(\epsilon)], \tag{4.28}$$

where $\epsilon=\tau/n$. The generic wave function $\psi=\psi(\bar{s}_i)$ is represented as an analytic function of complex variables $\bar{s}_i=x_i+iy_i$, $i=1,\ldots,M$. (We have placed a bar on the s_i to agree with the convention that creation is done by $a^\dagger\to\bar{s}$ and annihilation by $a\to s$.) The mth infinitesimal time translation acts according to

$$\psi_m(\bar{s}_i{}^m)=\exp[-i\epsilon H(m\epsilon)]\psi_{m-1}(\bar{s}_i{}^m)$$
$$=\int\prod_{i=1}^M\left(\frac{ds_i{}^{m-1}d\bar{s}_i{}^{m-1}}{2i\pi}\right)\exp[\bar{s}_i{}^m s_i{}^{m-1}-i\epsilon\bar{s}_i{}^m h_{ij}(m\epsilon)s_j{}^{m-1}-\bar{s}_i{}^{m-1}s_i{}^{m-1}]\psi_{m-1}(\bar{s}_i{}^{m-1}), \tag{4.29}$$

where $(2\pi i)^{-1}ds_i d\bar{s}_i=\pi^{-1}dx_i dy_i$. The desired functional integral representation follows:

$$U_{ij}(\tau)=\int\prod_{0\le\tau'\le\tau}\prod_{k=1}^M[(2\pi i)^{-1}ds_k(\tau')d\bar{s}_k(\tau')]s_i(\tau)\bar{s}_j(o)$$
$$\times\exp\left\{-\bar{s}_i(\tau)s_i(\tau)+i\int_0^\tau d\tau'[-i\dot{\bar{s}}_i(\tau')s_i(\tau')-\bar{s}_i(\tau')h_{ij}(\tau')s_j(\tau')]\right\}. \tag{4.30}$$

The quantity in the exponent may be symmetrized

$$\exp\left\{-\tfrac{1}{2}[\bar{s}_i(\tau)s_i(\tau)+\bar{s}_i(o)s_i(o)]+i\int_0^\tau d\tau'[(-i/2)(\dot{\bar{s}}_i s_i-\bar{s}_i\dot{s}_i)-\bar{s}_i h_{ij}s_j]\right\}. \tag{4.31}$$

The formulas we have just written apply in the Bose case, but the change to the Fermi case is trivial.[25] The reason either a Bose or a Fermi representation may be used is that only one-quantum states appear in (4.25) and so only one-quantum intermediate states occur in $(a_i^\dagger\Omega, U(\tau)a_j^\dagger\Omega)$ if $U(\tau)$ is written as the operator product (4.28).

We are now in a position to exhibit a path-integral representation for the matrix element

$$W(\tau;x,x')_{ij}=\langle x|\left[\exp(\tfrac{1}{2}i\tau\{[P-a(X)]^2-m^2-\tfrac{1}{2}i\gamma^\mu\gamma^\nu f_{\mu\nu}(X)\})\right]_{ij}|x'\rangle, \tag{4.32}$$

where $a_\mu(x)$ and $f_{\mu\nu}(x)$ are arbitrary functions and i and j are Dirac spinor indices. The procedure used above for (4.7) gives

$$W(\tau;x,x')_{ij}=N\int d\Gamma(\tau;x,x')T\exp\left\{-\tfrac{1}{2}im^2\tau-i\int_0^\tau d\tau'[\tfrac{1}{2}\dot{z}^2(\tau')+\dot{z}(\tau')\cdot a(z(\tau'))+\tfrac{1}{4}i\gamma^\mu\gamma^\nu f_{\mu\nu}(z(\tau'))]\right\}_{ij}. \tag{4.33}$$

Using the representation (4.30) for the ordered exponential of the matrix, this takes the form

$$W(\tau;x,x')_{ij}=\int d\Gamma(\tau;x,x')\exp\left\{-\tfrac{1}{2}im^2\tau-i\int_0^\tau d\tau'[\tfrac{1}{2}\dot{z}^2(\tau')+a(z(\tau'))\cdot\dot{z}(\tau')]\right\}$$
$$\times\int d\Sigma(\tau)s_i(\tau)\bar{s}_j(o)\exp\left[-\frac{i}{4}\int_0^\tau d\tau' f_{\mu\nu}(z(\tau'))\sigma^{\mu\nu}(\tau')\right], \tag{4.34}$$

where

$$d\Sigma(\tau)\equiv\prod_{\tau'}\left(\frac{d\bar{s}\,ds}{2\pi i}\right)\exp\left[-\bar{s}_i(\tau)s_i(\tau)-\int_0^\tau d\tau'\bar{s}_i(\tau')\dot{s}_i(\tau')\right] \tag{4.35}$$

and

$$\sigma^{\mu\nu}(\tau)=\bar{s}(\tau)\frac{i}{2}[\gamma^\mu,\gamma^\nu]s(\tau). \tag{4.36}$$

This representation does not suffer from the formal difficulties of (4.15) and will be used in the next section to prove the Lorentz invariance of the spin-$\frac{1}{2}$ theory.

V. LORENTZ INVARIANCE

The set of Green's functions (3.17) for all local fields θ_i (e.g., $A_\mu, \psi, F_{\mu\nu}, :\bar{\psi}\psi:, :\bar{\psi}\gamma_\mu\psi:$) is equivalent to the full quantum field theory. The Green's functions for gauge-invariant θ_i [e.g., $F_{\mu\nu}, J_\mu, K_\mu, :\bar{\psi}(i\partial_\mu-A_\mu)\psi:$] contain all of the observable information (in particular the S matrix) about the field theory. It is only necessary to establish the Lorentz invariance of these gauge-invariant functions. We will treat in detail only the case when the θ's are the conserved currents J and K. The extension to include the other gauge-invariant operators is straightforward. We thus consider the generating functional

$$W_n(a,b)=\langle 0|T\exp[-i\int d^4x(J\cdot a+K\cdot b)]|0\rangle, \tag{5.1}$$

where $a_\mu(x)$ and $b_\mu(x)$ are arbitrary functions. The current Green's functions are obtained from (5.1) by functional differentiation with respect to a and b followed by setting a and b to zero.

We consider first the spin-0 theory specified by the action $S_n^{(0)}$ of Eqs. (3.8) and (3.10). The functional integral representation of (5.1) is

$$W_n(a,b)=N\int dA\,dB\left(\prod_{i=1}^{l}d\phi_i d\phi_i^\dagger\right)\times\exp\left\{i\left[S_n^{(0)}(A,B,\phi_1,\ldots,\phi_l^\dagger)-\int d^4x(J\cdot a+K\cdot b)\right]\right\}. \tag{5.2}$$

We want to demonstrate the Lorentz-invariance condition

$$W_n(a,b)=W_n(a_\Lambda,b_\Lambda), \tag{5.3}$$

where

$$a_\Lambda(x)\equiv\Lambda^{-1}a(\Lambda x),\quad b_\Lambda(x)\equiv\Lambda^{-1}b(\Lambda x), \tag{5.4}$$

for an arbitrary Lorentz transformation $\Lambda_{\mu\nu}$. Since it is obvious from (3.8)–(3.10) and (5.2) that

$$W_n(a,b)=W_{\Lambda n}(a_\Lambda,b_\Lambda), \tag{5.5}$$

it is sufficient to show that $W_n(a,b)$ is independent of n, i.e.,

$$W_n(a,b)=W_{\Lambda n}(a,b). \tag{5.6}$$

We begin by explicitly performing the Gaussian integrations over the charged scalar fields in (5.2). [In this analysis we omit the quartic interaction term in (3.10). Its effect can be included by introducing a position-dependent mass $\mu_i^2(x)$. Then Eq. (5.2) can be written as

$$W_n(a,b)|_\lambda=\left\{\exp\left(i\sum_{ij}\lambda_{ij}\int d^4x\frac{\delta}{\delta\mu_i^2(x)}\frac{\delta}{\delta\mu_j^2(x)}\right)W_n^{(\mu)}(a,b)\Big|_{\lambda=0}\right\}_{\mu(x)=m}, \tag{5.7}$$

where $W_n^{(\mu)}$ is the generating functional in the position-dependent mass case. The Lorentz invariance of $W_n|_\lambda$ then follows from that of $W_n^{(\mu)}|_{\lambda=0}$. The modifications to our path-integral formula necessary to include a position-dependent mass are trivial.] The well-known result gives

$$W_n(a,b)=N\int dA\,dB\,e^{iS_n}\exp\left((-1)\sum_i \mathrm{Tr}\ln\{[(P-C_i)^2-m^2]/(P^2-m^2)\}\right), \tag{5.8}$$

where we have used the abbreviation

$$C_{i\mu}=e_i(A_\mu+a_\mu)+g_i(B_\mu+b_\mu). \tag{5.9}$$

A convenient representation for the expression in the second exponent in (5.8) is

$$\mathrm{Tr}\ln\{[(P-C)^2-m^2]/(P^2-m^2)\}=-\int_0^\infty\frac{d\tau}{\tau}\int d^4x\langle x|\{e^{(1/2)i\tau[(P-C)^2-m^2]}-e^{(1/2)i\tau[P^2-m^2]}\}|x\rangle. \tag{5.10}$$

Use of the path-integral representation (4.8) now yields

$$\mathrm{Tr}\ln\{[(P-C)^2-m^2]/(P^2-m^2)\}=-\int_0^\infty\frac{d\tau}{\tau}e^{-(1/2)im^2\tau}\int d\Gamma(\tau)f(\Gamma(\tau))\left[\exp\left(-i\int_0^\tau d\tau'\dot{z}\cdot C\right)-1\right], \tag{5.11}$$

where the integration $d\Gamma(\tau)$ is over *all closed* paths $\Gamma(\tau)$ $[z(o)=z(\tau)]$, and

$$f(\Gamma)=\exp\left[-\tfrac{i}{2}\int_0^\tau d\tau'\dot{z}^2(\tau')\right]. \tag{5.12}$$

We can thus cast (5.8) into the form

$$W_n(a,b)=N\int dA\,dB\,e^{iS_n}\exp\left\{\sum_i\int_0^\infty\frac{d\tau}{\tau}e^{-(1/2)m_i^2\tau}\int d\Gamma(\tau)f(\Gamma)\left[-1+\exp\left((-i)\int_0^\tau d\tau'\dot{z}\cdot C_i\right)\right]\right\}. \tag{5.13}$$

We finally expand $\exp(\sum_i)$ in a power series to obtain

$$W_n(a,b)=\sum_{k=0}^\infty W_n^{(k)}(a,b), \tag{5.14}$$

with

$$W_n^{(k)}=\frac{N}{k!}\int dA\,dB\,e^{iS_n}\int\prod_{r=1}^k\left[\frac{d\tau_r}{\tau_r}d\Gamma_r(\tau_r)f(\Gamma_r)\right]$$
$$\times\sum_\rho C_\rho\exp\left(\frac{-i}{2}\sum m_{i_r}{}^2\tau_r\right)\exp\{-i\int d^4x[J_\rho\cdot(A+a)+K_\rho\cdot(B+b)]\}, \tag{5.15}$$

where ρ represents a particular assignment of the indices $i_r=1,\ldots,k$, for $r=1,\ldots,k$, C is a combinatorial factor, and J_ρ and K_ρ are the corresponding *classical* currents associated with the charges e_i and g_i and (closed) trajectories $\Gamma_r(\tau_r)$:

$$J_\rho^\mu(x)=\sum e_{i_r}\oint_{\Gamma_r}dz^\mu\delta^4(x-z),$$
$$K_\rho^\mu(x)=\sum g_{i_r}\oint_{\Gamma_r}dz^\mu\delta^4(x-z). \tag{5.16}$$

We see that, apart from the τ integrations and $f(\Gamma)$ factors, each $W_n^{(k)}$ is just a sum of generating functionals for classical currents, i.e., of functional integrals over classical fields A and B and (closed) trajectories Γ weighted by a classical action $S_n(A,B)-\int(J\cdot A+K\cdot B)$. One way of verifying the n independence of $W_n^{(k)}$, given (1.3), is to exploit the invariance, mod ϵ_{ij}, of this classical action under a combined string rotation and gauge transformation, and the gauge invariance of the functional measure. To see this explicitly, it is most simple to transform back from the (A,B,Γ) variables to the (A,F,Γ) variables to obtain an action of the form of $I_n^{(c)}$ in (2.6). Then all of the n dependence in each term of the ρ sum in (5.15) resides in the $K_\rho\cdot B$ term, and the transformation $n\to\Lambda n$ is equivalent to the gauge transformation [cf. Eq. (2.16)]

$$A\to A'=A+\partial\lambda_\rho,\quad F\to F,\quad \Gamma\to\Gamma, \tag{5.17}$$

with

$$\partial\wedge\partial\lambda_\rho=\{[(\Lambda n\cdot\partial)^{-1}\Lambda n-(n\cdot\partial)^{-1}n]K_\rho\}^d. \tag{5.18}$$

Since K_ρ is a singular classical current, λ_ρ is well defined and has discontinuities $\pm g_j/2$ for some j's. As discussed below Eq. (2.17), this gauge transformation changes $\int J_\rho\cdot A$ by $\sum_{ij}\epsilon_{ij}N_{ij}$ with integers N_{ij} and so, since $dA=dA'$, (5.15) is invariant to $n\to\Lambda n$ if (1.3) is satisfied. This shows that $W_n^{(k)}(a,b)=W_{\Lambda n}^{(k)}(a,b)=W_n^{(k)}(a_\Lambda,b_\Lambda)$, the last equality following from the Lorentz invariance of the measures $dA,dF,d\Gamma$. Thus each term in (5.14) is Lorentz invariant, and so will be the sum if it exists in any sense.

A more direct way to verify the n independence of (5.15) is to explicitly perform the Gaussian functional integrations over A and B. The (threatened) n dependence of (5.15) occurs in

$$\int d\Gamma \int dA\, dB \{\exp[iS_n(A,B)]\}\left\{\exp\left[-i\int (J\cdot A + K\cdot B\right]\right\} \propto \int d\Gamma \exp\left[(-i/2)\int\int (JD_{AA}J + JD_{AB}K + KD_{BA}J + KD_{BB}K)\right], \tag{5.19}$$

where the propagators have been given in momentum space in Eqs. (3.15) and (3.16). By current conservation, only the mixed propagators in (5.19) contribute to the possible n dependence. Using (3.16), the n-dependent part of the exponent in (5.19) is seen to have the form

$$(2\pi)^{-2}\epsilon_{ij}\oint dx_i^\mu \oint dx_j^\nu (n\wedge\partial)^d_{\mu\nu}(n\cdot\partial)^{-1}[(x_i-x_j)^2-i\epsilon]^{-1}. \tag{5.20}$$

Here the closed loop line integrals correspond to the trajectories Γ_i and Γ_j. We apply Stokes's theorem to the first loop integral so that (5.20) becomes

$$(2\pi)^{-2}\epsilon_{ij}\int_{\Sigma_i} dS_i^{d\mu\nu}\oint dx_{j\mu}\{\partial_\nu[(x_i-x_j)^2-i\epsilon]^{-1}+(2\pi)^2 in_\nu(n\cdot\partial)^{-1}\delta^4(x_i-x_j)\}, \tag{5.21}$$

where Σ_i is a two-surface bounded by the first-loop Γ_i. The first term in (5.21) is n independent. Using (2.10), the second term is

$$I_{ij} \equiv i\epsilon_{ij}\int_{\Sigma_i} dS_i^{d\mu\nu}\oint_{\Gamma_j} dx_{j\mu} n_\nu \int_0^\infty d\eta[\delta^4(x_i-x_j-n\eta)-\delta^4(x_i-x_j+n\eta)]. \tag{5.22}$$

Because of the definition of the δ^4 functions, the integrations in (5.22) simply count the number of times the loop Γ_j intersects the oriented three-surface $\Sigma_i \times \pm n\eta$ $(0\le\eta<\infty)$. This is an integer N_{ij} (positive, negative, or zero) for almost all paths. It is only ill defined (e.g., if Γ_j is locally tangent to Σ_i) on a set of measure zero in the space of paths which contribute to (5.19), so that these configurations may be safely ignored.[26] Thus

$$I_{ij} = i\epsilon_{ij}N_{ij}, \tag{5.23}$$

so that the n-dependent factors in (5.15) have the form $\exp(i\epsilon_{ij}N_{ij})$. This shows again that if the quantization condition (1.3) is satisfied, (5.15) will be independent of n. [Integration of $e^{I_{ij}}=1$ over the remaining τ and $\Gamma(\tau)$ variables obviously maintains the n independence.]

It is interesting to note the geometric aspects of the integrations in (5.22). This integral defines a topological integer, analogous to a linking number L_{ij} which counts the number of times the tube $\Gamma_j\times n$ links the surface Σ. It is clear from the definition in (5.22) that L_{ij} is an integer in four dimensions, but because of the limitations of our three-dimensional intuition, the geometrical significance of this number is not too transparent. It may help to consider the three-dimensional analog. The passage from (5.20) to (5.22) has a three-dimensional analog in the identity

$$\begin{aligned} L &= -(4\pi)^{-1}\oint_{\Gamma_1} d\vec{y}\times\oint_{\Gamma_2} d\vec{x}\cdot\vec{\nabla}|\vec{x}-\vec{y}|^{-1} \\ &= \int_{S_1} d\vec{\sigma}\cdot\oint_{\Gamma_2} d\vec{x}\,\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{5.24}$$

Here Γ_1 and Γ_2 are one-dimensional closed loops and S_1 is any surface bounded by loop one, with surface element $d\vec{\sigma}$. It is clear that L is the linking number that counts the number of times loops 1 and 2 link around each other. The above L_{ij} is simply a four-dimensional generalization of L.

The above evaluation of (5.19) can perhaps be rendered more physical by noting that the exponential is

$$\sum_{j>i}\oint_{\Gamma_j} dz_\mu(e_i\mathfrak{A}_j^\mu + g_i\mathfrak{B}_j^\mu), \tag{5.25}$$

where

$$\begin{aligned} \mathfrak{A}_j &\equiv D_{AA}\cdot J_j + D_{AB}\cdot K_j, \\ \mathfrak{B}_j &\equiv D_{BA}\cdot J_j + D_{BB}\cdot K_j, \end{aligned} \tag{5.26}$$

apart from the divergent but n-independent $i=j$

terms. Using Stokes's theorem, (5.25) becomes

$$\sum_{j>i}\int_{S_i} d\sigma_{\mu\nu}\left[(e_i\mathfrak{F}_j^{\mu\nu}+g_i\mathfrak{F}_j^{d\mu\nu}) + \epsilon_{ij}\oint_{\Gamma_j}(dz\wedge n)^{d\mu\nu}(n\cdot\partial)^{-1}(z_i-z_j)\right], \tag{5.27}$$

where the expression

$$\mathfrak{F}_j = D[(\partial\wedge J)_j-(\partial\wedge K)_j^d], \tag{5.28}$$

with D the Feynman propagator, is explicitly seen to be independent of n and to satisfy

$$\partial\cdot\mathfrak{F}_j = J_j, \quad \partial\cdot\mathfrak{F}_j^d = K_j. \tag{5.29}$$

The second term in (5.27) is, mod ϵ_{ij}, also independent of n by our previous discussion.

We proceed now to establish the Lorentz invariance of the spin-$\frac{1}{2}$ theory. The generating functional (5.1) becomes

$$W_n(a,b) = N\int dA\,dB\left(\prod_{i=1}^{l} d\psi_i d\bar\psi_i\right)\exp\{i[S_n(A,B)+S^{(1)}(A+a,B+b,\psi_1,\ldots,\bar\psi_l)]\}, \tag{5.30}$$

where the action is given in Eqs. (3.8), (3.9), and (3.11). The explicit Gaussian spinor integrations now give

$$W_n(a,b) = N\int dA\,dB\, e^{iS_n} \times\exp\left\{\sum_i \mathrm{Tr}\ln[(i\partial\!\!\!/-m_i-C_i)/(i\partial\!\!\!/-m_i)]\right\} \tag{5.31}$$

instead of (5.8), where we again use the abbreviation (5.9).

For a purely formal argument, we may use the representation analogous to (5.10),

$$\mathrm{Tr}\ln[(P\!\!\!/-C\!\!\!/-m)/(P\!\!\!/-m)] = -\int_0^\infty\frac{d\tau}{\tau}\int d^4x\langle x|\mathrm{tr}(e^{i\tau(P\!\!\!/-C\!\!\!/-m)}-e^{i\tau(P\!\!\!/-m)})|x\rangle, \tag{5.32}$$

where the trace "tr" is only over the spinor indices. Substitution of the path-integral representation (4.15) gives

$$\mathrm{Tr}\ln[(P\!\!\!/-C\!\!\!/-m)/(P\!\!\!/-m)] = -\int_0^\infty\frac{d\tau}{\tau}e^{-im\tau}\int d\Gamma(\tau)g(\Gamma(\tau)) \times\left[1-\exp\left(-i\int_0^\tau d\tau'\dot z\cdot C\right)\right], \tag{5.33}$$

where

$$g(\Gamma)\equiv \mathrm{tr}F(\Gamma), \tag{5.34}$$

with $F(\Gamma)$ given by (4.16). This is analogous to (5.11) and (5.12), and substitution into (5.31) gives expressions which differ from (5.13)–(5.16) only by the substitutions $m_i\to\frac{1}{2}m_i^2$ and $f(\Gamma)\to g(\Gamma)$. The proof of Lorentz invariance of (5.30) therefore proceeds exactly as in the spin-0 case. This proof is, however, formal because of the purely formal existence of $F(\Gamma)$ as discussed in Sec. IV. A rigorous proof would have to include a discussion of the existence of this functional on the domain relevant in (5.33). We will instead use a different approach to the spin-$\frac{1}{2}$ theory which avoids this difficult question.

Instead of using (5.32) we will rationalize the Dirac operator. Note that

$$\begin{aligned}\mathrm{Tr}\ln(P\!\!\!/-C\!\!\!/-m+i\epsilon)-\mathrm{Tr}\ln(P\!\!\!/-m+i\epsilon) &= \mathrm{Tr}\int_m^\infty d\mu\left(\frac{1}{P\!\!\!/-C\!\!\!/-\mu+i\epsilon}-\frac{1}{P\!\!\!/-\mu+i\epsilon}\right)\\ &= \mathrm{Tr}\int_m^\infty d\mu\left((P\!\!\!/-C\!\!\!/+\mu)\frac{1}{(P\!\!\!/-C\!\!\!/)^2-\mu^2+i\epsilon}-(P\!\!\!/+\mu)\frac{1}{P\!\!\!/^2-\mu^2+i\epsilon}\right)\\ &= \mathrm{Tr}\int_m^\infty d\mu\,\mu\left(\frac{1}{(P\!\!\!/-C\!\!\!/)^2-\mu^2+i\epsilon}-\frac{1}{P\!\!\!/^2-\mu^2+i\epsilon}\right).\end{aligned}$$

In the last expression we have used the fact that the trace of an odd number of Dirac matrices vanishes. The last expression gives $\frac{1}{2}\mathrm{Tr}\ln[(P\!\!\!/-C\!\!\!/)^2-m^2+i\epsilon]-\frac{1}{2}\mathrm{Tr}\ln(P\!\!\!/^2-m^2+i\epsilon)$. We now proceed as in the scalar case and find

$$-\mathrm{Tr}\ln(\not{P}-\not{C}-m+i\epsilon)+\mathrm{Tr}\ln(\not{P}-m+i\epsilon)=\tfrac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau}\left[\exp\left(\frac{i\tau}{2}[(\not{P}-\not{C})^2-m^2]\right)-\exp\left(\frac{i\tau}{2}(\not{P}^2-m^2)\right)\right]$$

$$=\tfrac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau}\left[\exp\left(\frac{i\tau}{2}[(P-C)^2-\tfrac{1}{2}i\gamma^\mu\gamma^\nu(\partial\wedge C)_{\mu\nu}-m^2]\right)\right.$$

$$\left.-\exp\left(\frac{i\tau}{2}(P^2-m^2)\right)\right]. \tag{5.35}$$

The exponential here is of the form (4.32), and so we can use the representation (4.34) to obtain

$$-\mathrm{Tr}\ln(\not{P}-\not{C}-m)+\mathrm{Tr}\ln(\not{P}-m)=\tfrac{1}{2}\int_0^\infty\frac{d\tau}{\tau}\exp(-\tfrac{1}{2}im^2\tau)\int d\Gamma(\tau)f(\Gamma(\tau))\int d\Sigma(\tau)s_i(\tau)\bar{s}_i(0)$$

$$\times\left[\exp\left(-i\int_0^\tau d\tau'\left[\dot{z}\cdot C+\frac{1}{4}(\partial\wedge C)_{\mu\nu}\sigma^{\mu\nu}\right]\right)-1\right], \tag{5.36}$$

where $f(\Gamma)$ is given in (5.12), and

$$d\Sigma(\tau)=\prod_{0\le\tau'\le\tau}\left(\frac{d\bar{s}\,ds}{2\pi i}\right)\exp\left[-\bar{s}_i(o)s_i(o)-\int_0^\tau d\tau'\bar{s}_i(\tau')\dot{s}_i(\tau')\right] \tag{5.37}$$

as in (4.35), and

$$\sigma^{\mu\nu}(\tau)=\bar{s}(\tau)\frac{i}{2}[\gamma^\mu,\gamma^\nu]s(\tau), \tag{5.38}$$

as in (4.36).

The generating functional (5.31) now reads

$$W_n(a,b)=N\int dA\,dB\,e^{iS_n}\exp\left\{\sum_j\tfrac{1}{2}\int_0^\infty\frac{d\tau}{\tau}\exp(-\tfrac{1}{2}m_j^2\tau)\int d\Gamma(\tau)f(\Gamma)\int d\Sigma(\tau)\bar{s}_j(\tau)s_j(o)\right.$$

$$\left.\times\left[-\exp\left(-i\int_0^\tau d\tau'[\dot{z}\cdot C_j+\tfrac{1}{4}(\partial\wedge C_j)_{\mu\nu}\sigma^{\mu\nu}]\right)+1\right]\right\}, \tag{5.39}$$

which only differs from the spin-0 expression (5.13) by the $\sigma^{\mu\nu}$ contribution to the exponential. We now have in the power series (5.14), (5.15)

$$W_n^{(k)}(a,b)=\frac{N}{k!}\int dA\,dB\,e^{iS_n}\int\prod_{r=1}^k\left[\frac{d\tau_r}{\tau_r}d\Gamma_r(\tau_r)f(\Gamma_r)\right]$$

$$\times\sum_\rho C_\rho\exp\left\{-\frac{i}{2}\sum m_{i_r}{}^2\tau_r-i\int d^4x[J_\rho\cdot(A+a)+K_\rho\cdot(B+b)]\right\}, \tag{5.40}$$

where the currents are

$$\begin{aligned}J_\rho^\mu(x)&=\sum e_{i_r}\int_0^{\tau_r}d\tau_r'[\dot{z}_r^\mu(\tau_r')+\tfrac{1}{2}\sigma_r^{\mu\nu}(\tau_r')\partial_\nu]\delta^4(x-z_r),\\ K_\rho^\mu(x)&=\sum g_{i_r}\int_0^{\tau_r}d\tau_r'[\dot{z}_r^\mu(\tau_r')+\tfrac{1}{2}\sigma_r^{\mu\nu}(\tau_r')\partial_\nu]\delta^4(x-z_r).\end{aligned} \tag{5.41}$$

The new feature here, as compared to the scalar case (5.15) and (5.16), is the existence of first-moment contributions to the classical currents. According to our discussion in Sec. II, following Eqs. (2.18) and (2.19) the presence of these moment contributions does not affect the n independence (mod ϵ_{ij}) of the classical action. We may thus conclude, just as for the spin-0 theory, that (5.40) is indeed Lorentz invariant when the quantization conditions (1.3) are satisfied.

We can again be more direct and explicitly perform the integrations over A and B in (5.40). The result (5.19) is as before, but with the currents (5.41). Write

$$J=J_1+J_2,\quad K=K_1+K_2, \tag{5.42}$$

with J_1,K_1 the charge contributions (5.16) and J_2,K_2 the moment contributions. The charge-charge contributions $J_1\cdot D\cdot K_1$ etc. as before give rise to no n dependence given (1.3). The remaining contributions are explicitly n independent, for any charges. For example, (3.16) and (5.41) give

$$\int\int d^4x d^4y J_2(x) D_{AB}(x-y) K(y)$$
$$=\frac{1}{4}\sum_i \int d\tau\sigma^{\mu\nu}(\tau)\left\{-(n\cdot\partial)^{-1}[n\wedge K(z_i)]_{\mu\nu} + (2\pi)^{-2}\left[\partial\wedge\int d^4y(z_i-y)^{-2} \times K(y)\right]_{\mu\nu}\right\}. \quad (5.43)$$

The second term is n independent and the first term vanishes except when a magnetically charged particle intersects a string. Such a collision corresponds to a set of measure zero in the path integral in (5.40) and so can be ignored. Thus (5.40) and therefore (5.39) are Lorentz invariant.

Although we have explicitly established only the Lorentz invariance of the generating functional (5.1) in the scalar and spinor theories, it should be clear that our analysis can be extended to the Green's functions of arbitrary local gauge-invariant operators in arbitrary field theories.

VI. CONCLUDING REMARKS

The quantum field theories of electric and magnetic charge whose Lorentz invariance we have established were defined in Sec. III by string-dependent actions such as the local action (3.8) or the nonlocal one (3.5). Such string-dependent formalisms for the classical and first-quantized theories were reviewed in Sec. II. For the first-quantized theories, the alternative formalism of Wu and Yang[2] is available. This approach avoids the use of strings and uses instead a topological section formalism. A distinct advantage of this framework is that the absence of unphysical strings makes such properties as Lorentz invariance more manifest. Unfortunately, it appears to be extremely difficult to extend the Wu-Yang program to the full quantum field theory. In fact, the topological formalism even for the classical theory[18] is incomplete at present.

Our proof shows that the quantum field theory of electric and magnetic charge is Lorentz invariant even though the corresponding classical theory of smooth minimally coupled charged fields is not. To accomplish this it was necessary to express the generating functional of the gauge-invariant Green's functions as a path integral over the trajectories of charged classical point particles. Let us review critically some of the assumptions that go into the proof:

(1) It is assumed that the generating functional W may be expressed as a sum (5.14) $W=\sum_{k=\nu}^{\infty} W^{(k)}$ over the number of charged-particle loops. This is a weaker assumption than a perturbative expansion, because each term $W^{(k)}$ contains all orders in the coupling constant and furthermore each term is gauge invariant.

(2) It is assumed that renormalization will not invalidate the conclusion. This is presumably a weakness of present-day renormalization theory rather than our argument, for renormalization theory is inherently perturbative whereas the consistency of monopole theory rests on the Dirac quantization condition $e_i g_j - g_i e_j = 4\pi\times$integer, which is inherently nonperturbative. Our result should (we hope) encourage renormalization theorists to extend their methods beyond individual graphs to $W^{(k)}$ represented as a functional integral over classical particle paths.

(3) The Feynman measure is of course not really a measure and so we are not really justified in claiming that certain configurations—such as intersection of a string and a trajectory—are of measure zero. To be rigorous our argument should be effected in the Euclidean region where the volume in path space is, in fact, a measure.

Our formulation of quantum field theory in terms of integrals over classical particle paths has dynamical aspects that have not been considered here. On the one hand, it suggests new semiclassical approximations around the classical particle solutions which may be thought of as dual to the familiar semiclassical approximation around classical field solutions. Work in this direction has in fact been initiated by Halpern, Senjanovic, and Jevicki,[27] and our results allow systematic inclusion of closed charged-particle loops.

Finally it should be kept in mind that the point nature of electric and magnetic charge, on which we insist, may qualitatively alter the dynamics. This appears particularly relevant for the unobserved magnetic monopole with point Coulombic field of strength $g^2/4\pi\sim 137$, and correspondingly strong pole-antipole attraction. The instability of an external point Coulomb field of strength $Ze^2/4\pi>1$ to pair production is well known.[28,29] The problem is modified for a nucleus of finite extent.[30] But unlike the atomic nucleus, the magnetic monopole is absolutely pointlike. Furthermore there is every reason to believe that the instability of the external Coulomb field belonging to a particle of infinite mass, persists if the mass is finite. This suggests that an isolated magnetic monopole would destabilize the vacuum, and if so, would never be produced even though its occurrence in closed loops made it an essential participant in elementary particle dynamics.

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[1]Our notation suppresses all Lorentz indices. Thus $\partial \cdot F = J$ means $\partial^{\mu} F_{\mu\nu} = J_{\nu}$, etc., F^d is the dual tensor with indices $F^d_{\kappa\lambda} = \frac{1}{2}\epsilon_{\kappa\lambda}{}^{\mu\nu} F_{\mu\nu}$, and $(a \wedge b)$ is the anti-symmetric tensor with indices $a_{\mu} b_{\nu} - a_{\nu} b_{\mu}$.

[2]T. T. Wu and C. N. Yang, Phys. Rev. D 12, 384 (1975).

[3]The potential A actually gives an overdescription of the physics. According to Ref. 2, it is the phase factors $\exp(\int ieA \cdot dx)$ which contains all of the measurable information.

[4]In this paper we choose to consider only monopole theories which are dual invariant, i.e., invariant under the O(2) group which rotates the two-vectors (F, F^d) and (e_i, g_i). Then (1.3) is the relevant dual-invariant quantization condition. If we had chosen otherwise, (1.3) would be replaced by Dirac's original condition $e_i g_j = 2n_{ij}$.

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[26]This statement becomes rigorous in the Euclidean region where the volume in path space is, in fact, a measure. In this way the need for delicate limiting procedures, such as those used by Schwinger in Ref. 12 are avoided by the functional integral over classical particle trajectories.

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[28]M. B. Halpern and W. Siegel, Phys. Rev. D 16, 2486 (1977). These authors also appreciated the importance of the classical particle variables to the magnetic monopole problem.
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