PHYSICAL REVIEW D VOLUME 19, NUMBER 4 15 FEBRUARY 1979

Conformal anomalies for interacting scalar fields in curved spacetime

I. T. Drummond and G. M. Shore
D.A.M.T.P., University of Cambridge, Cambridge, England
(Received 1 December 1977)

The occurrence of anomalies in the trace of the energy-momentum tensor for scalar field theories in curved space-time is discussed. For the special case of spherical space-time, an $O(n+1)$-covariant formalism is used to rederive the standard free-field anomaly in four dimensions, and to calculate the anomaly in six dimensions. It is then shown that for an interacting scalar field theory there is a further contribution to the trace anomaly proportional to the renormalization group β function. This assertion is then checked by explicit calculations in ϕ^4 theory in four dimensions and ϕ^3 theory in six dimensions and values for the anomaly found to fourth order in the renormalized coupling constants λ and g. Finally, these results are generalized to the case of an arbitrary background space-time, where it is shown that the introduction of a position-dependent coupling constant $\lambda(x)$ enables the relation between the trace anomaly and the β function to be expressed in the form $T^{\mu}_{I\mu} = -\beta(\lambda)\delta W_I/\delta\lambda(x)|_{\lambda(x)=\lambda}$ where W_I is the sum over vacuum bubble diagrams with interactions.

I. INTRODUCTION

There has been a great deal of interest recently[1-10] in the vacuum expectation value of the energy-momentum tensor for quantum field theories in a curved space-time background. The result of those investigations is the discovery that the curvature induces anomalous contributions to the trace of the energy-momentum tensor.

So far only noninteracting theories have been considered. In this paper we extend the discussion to the case of interacting fields,[11,12] namely ϕ^4 theory in four and ϕ^3 theory in six dimensions.

Briefly, our result is that in addition to the trace anomaly previously encountered, there is a further anomalous term due to the self-interaction of the fields. This additional term is proportional to $\beta(\lambda)$, the coefficient of $\partial/\partial\lambda$ in the renormalization-group equation, where λ is the renormalized coupling constant. Such a result is consistent with previous investigations of the trace of the "improved" version of the energy-momentum tensor in a flat space-time background.[13-19] In particular we note that the extra trace contribution vanishes for values λ which give rise to a scale-invariant theory. The original anomaly remains, however. If mass terms are added to the Lagrangian, then the trace acquires further contributions of a more conventional kind.

We believe our result to be perfectly general, although our derivation of it is not. We proceed in two steps. First, we analyze in detail the special case of a background of constant curvature. Then we show, on the basis of assumptions suggested by this case, how the argument may be generalized.

Two artificial features of our work should be mentioned. For technical conveience we work with a Riemannian rather than a Minkowskian manifold. Furthermore, we assume that the manifold is compact. For the special case of constant curvature, then, our manifold is a sphere.[1,2,20] Neither of these assumptions we feel is crucial for our final result.

It is worth emphasizing that we work with a *conformal* massless theory. Previous investigations[11,12] of such a theory on a spherical manifold suggest that the field and coupling-constant renormalizations are exactly the same as in flat space. Moreover it seems that the curvature does not give rise to an anomalous mass renormalization. In other words, the conformal massless theory remains massless after renormalization. This is what we will assume for most of the discussion. However, it has been verified only up to the three-loop level in perturbation theory.

Collins[13] has pointed out that his analysis of the energy-momentum tensor (in flat space) implies that this assumption ought to break down at the four-loop level. The implication of his work is that in ϕ^4 theory a term proportional to $(n-4)^3\phi^2 R$ (R is the curvature scalar) must be added to the action density in order to complete the renormalization process in a curved manifold. Because the calculation is a difficult one, we have not been able to check this directly on the sphere. For simplicity of exposition we postpone a discussion of this complication to the end of the paper. It does not change our general conclusion.

II. EFFECTIVE ACTION AND TRACE ANOMALIES FOR FREE FIELDS

Our renormalization procedure[11,12] makes use of dimensional regularization,[21-23] so it is convenient to regard the manifold as an n sphere embedded in an $(n+1)$-dimensional Euclidean space. Points in this space are denoted by $(n+1)$-vectors $\eta = (\eta_1, \ldots, \eta_{n+1})$ and those on the sphere satisfy $\eta^2 = a^2$. The volume element on the sphere is $d\sigma = a^n d\Omega_{n+1}(\hat{\eta})$.

The operator which plays the role of the d'Alembertian is

$$M = \frac{1}{a^2}\left[\tfrac{1}{2}L_{ab}L_{ab} - \frac{n}{2}\left(\frac{n}{2}-1\right)\right], \tag{2.1}$$

where

$$L_{ab} = \eta_a \partial_b - \eta_b \partial_a . \tag{2.2}$$

The presence of the "extra" term

$$-\frac{n}{2}\left(\frac{n}{2}-1\right)\Big/a^2$$

in M reflects the fact that we are dealing with a conformal scalar field. Note however that our method of dimensional regularization gives the curvature scalar an explicit n dependence. This is somewhat different from the regularization technique of Ref. 5 where such a dependence is absent. We feel that this difference is not important but the point requires further investigation.

The free-field propagator is

$$D(\eta, \eta') = \frac{\Gamma(\frac{1}{2}n-1)}{4\pi^{n/2}} \frac{1}{|\eta - \eta'|^{n-2}} \tag{2.3}$$

and satisfies

$$MD(\eta, \eta') = -\delta(\eta, \eta'). \tag{2.4}$$

The generating functional $W[J]$ for connected Green's functions is given by

$$W[J] = \ln\left\{ \int \mathfrak{D}\Phi \, \exp\left[\int d\sigma \left(\tfrac{1}{2}\Phi M \Phi - \frac{\lambda_0}{4!}\Phi^4 + J\Phi \right)\right]\right\} \tag{2.5}$$

where $\Phi(\eta)$ is the scalar field on the sphere. The effective action which determines the back reaction of the matter field on the metric is $W = W[0]$. More precisely, the energy-momentum tensor in the "vacuum" state is

$$T^{\mu\nu}(x) = \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g_{\mu\nu}(x)}, \tag{2.6}$$

where $g_{\mu\nu}$ is the metric for the coordinates $\{x\}$. For the case of the sphere, $T^{\mu\nu}$ is determined entirely by its trace, which is constant over the sphere, so it is sufficient to calculate

$$\int d\sigma T^{\mu}{}_{\mu} = a \frac{\partial W}{\partial a}. \tag{2.7}$$

Now W can be computed as a sum over vacuum bubbles, the first few of which, for ϕ^4 theory, are shown in Fig. 1. The free-field contribution from Fig. 1(a),

$$W_0 = -\tfrac{1}{2}\,\mathrm{Tr}\ln(-M) = \tfrac{1}{2}\,\mathrm{Tr}\ln D, \tag{2.8}$$

is different in character from the rest of the series, being indpendent of a for all n and having a pole at $n=4$. In the Appendix we give a derivation which conveniently combines our dimensional regularization procedure with the ζ-function method[10] of the standard result[1-10] that for $n \simeq 4$

$$W_0 = \frac{1}{90}\frac{1}{n-4} + O((n-4)^0). \tag{2.9}$$

If we require that this infinity is removed by the addition of a counterterm to the action which is local in the metric field and conformally invariant in four dimensions then on the sphere W will acquire a further contribution

$$-\frac{1}{90}\frac{1}{n-4}(\mu a)^{n-4}, \tag{2.10}$$

where μ is an arbitrary mass parameter introduced to give the term the correct dimensional character. When $n=4$ the resultant free-field contribution becomes, up to an irrelevant constant,

$$W_0^1 = -\tfrac{1}{90}\ln \mu a. \tag{2.11}$$

In turn, this leads via Eq. (2.7) to the well-known anomalous trace contribution

$$T^{\mu}{}_{\mu} = -\frac{1}{90}\frac{1}{a^4\Omega_5}. \tag{2.12}$$

The method explained in the Appendix allows us to calculate W_0 near $n=6$, which is relevant to the discussion of ϕ^3 theory. We find

$$W_0 = -\frac{1}{756}\frac{1}{n-6} + O((n-6)^0). \tag{2.13}$$

This infinity can be removed by adding to the action a local gravitational counterterm which is conformally invariant in six dimensions. Whatever the detailed form of this term, it must give rise to a contribution in the present case of

$$\frac{1}{756}\frac{1}{n-6}(\mu a)^{n-6}. \tag{2.14}$$

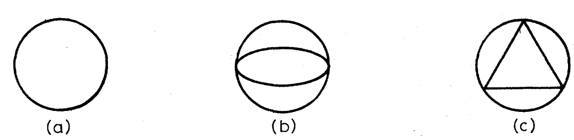

FIG. 1. Low-order vacuum bubbles in Φ^4 theory which do not vanish.

The trace anomaly in six dimensions is then

$$T^{\mu}{}_{\mu} = \frac{1}{756}\ \frac{1}{a^6\Omega_7}\ . \tag{2.15}$$

As we shall see, neither of these two free-field anomalies has any obvious connection with the additional anomalous terms due to the self-interaction of the field.

III. TRACE ANOMALY FOR INTERACTING FIELDS

The crucial difference between the vacuum bubbles with interaction and the free-field term W_0 is that they are not primitively divergent. At first sight this is a paradoxical result since the corresponding Feynman integrals are certainly divergent. The paradox is resolved by the method of dimensional regularization. When continued in the dimension variable n, the interacting bubbles are essentially finite at $n=4$ for ϕ^4 theory or $n=6$ for ϕ^3 theory. Of course they do have primitive poles for other lower values of n.

This is well illustrated by the lowest-order term from Fig. 1(b)

$$W_2 = \frac{\lambda_0^2}{48} \int d\sigma_1 \int d\sigma_2 [D(\eta_1, \eta_2)]^4 . \tag{3.1}$$

Using Eq. (2.3) for $D(\eta_1, \eta_2)$ we readily evaluate the right-hand side of Eq. (3.1) as

$$W_2 = \frac{\lambda_0^2}{48}\ \frac{\Omega_{n+1}(a^2)^{4-n}[\Gamma(\frac{1}{2}n-1)]^4\Gamma(4-\frac{3}{2}n)}{(4\pi)^{3n/2}\Gamma(4-n)}\ . \tag{3.2}$$

Apart from singularities hidden in λ_0, W_2 is finite at $n=4$. The divergence of the original integral in Eq. (3.1) corresponds to the fact that the lowest pole in $\Gamma(4-\frac{3}{2}n)$ occurs below $n=4$ at $n=\frac{8}{3}$.

The third-order term from Fig. 1(c) does have a singularity, a single pole at $n=4$. However, this is not a primitive divergence and is due to the divergence of a subintegration. As we shall check in the next section, it is removed by the renormalization procedure.

The absence of a primitive divergence from any of the interacting vacuum bubbles can be established quite generally. The proof depends on a representation of the Feynman integrals derived in a previous paper[12] in which the mechanism giving rise to the primitive divergence can be clearly isolated.

A general vacuum bubble in ϕ^4 theory with p vertices yields a contribution to the effective action W of the form

$$W_p = \frac{(-\lambda_0)^p}{S} \int d\sigma_1 \cdots \int d\sigma_p \prod_{1\le i<j\le p} [D(\eta_i, \eta_j)]^{\lambda_{ij}}, \tag{3.3}$$

where the possible values of λ_{ij} are 0, 1, 2, 3, and S is the appropriate symmetry factor. This may be written[12] as

$$W_p = \frac{(-\lambda_0)^p}{S}\ \frac{\Gamma(\frac{1}{2}n-1)}{4\pi^{n/2}}^{2p} \Omega_{n+1}\, a^{p(4-n)} I(\beta_{ij}), \tag{3.4}$$

where

$$I(\beta_{ij}) = \int \prod_i{}' d\sigma_i \prod_{i<j} |\eta_i - \eta_j|^{-\beta_{ij}} . \tag{3.5}$$

In this equation, $\beta_{ij} = \lambda_{ij}(n-2)$, η_i has been scaled down onto the unit sphere, and the prime on the product of integration measures indicates that one of them (any one) has been left out.

The function $I(\beta_{ij})$ was analyzed in Sec. III of Ref. 12. It is shown there that $I(\beta_{ij})$ can be given the representation

$$I(\beta_{ij}) = \int \frac{1}{dC}\ \frac{d^{n+1}z}{|z-\bar{z}|^{n+1}} \prod_{i=1}^{p} d^n x_i \prod_{i<j} |x_i - x_j|^{-\beta_{ij}} |z-\bar{z}|^{np-B} \prod_i |z - x_i|^{B_i - 2n}, \tag{3.6}$$

where the $\{x_i\}$ are n-dimensional vectors and z is an $(n+1)$-dimensional vector. If $z=\{y, z_{n+1}\}$, then $\bar{z}=\{y, -z_{n+1}\}$. The integration range is restricted only by the condition $z_{n+1} \geq 0$.

The measure dC which is divided out of the integrand may be chosen to be

$$dC = \frac{1}{2\Omega_n}\ \frac{d^n x_\alpha d^n x_\beta d^n x_\gamma}{|x_\alpha - x_\beta|^n |x_\beta - x_\gamma|^n |x_\gamma - x_\alpha|^n}\ , \tag{3.7}$$

and the three points x_α, x_β, x_γ may be set at any convenient values. The necessity of dividing out this conformally invariant measure is explained in Ref. 12. Also,

$$B = \sum_{i<j} \beta_{ij} \tag{3.8}$$

and

$$B_i = \sum_{i\neq j} \beta_{ij} . \tag{3.9}$$

It is easy to see since the diagram has $2p$ lines and four lines enter each vertex that

$$B = 2p(n-2) \tag{3.10}$$

and

$$B_i = 4(n-2)\,. \tag{3.11}$$

We have then

$$I(\beta_{ij}) = \int \frac{1}{dC} \prod_i d^n x_i \prod_{i<j} |x_i - x_j|^{-\beta_{ij}} \times \int d^n y 2^{p(4-n)-n-1} \times \int_0^\infty dz_{n+1} (z_{n+1})^{p(4-n)-n-1} \times [f(z_{n+1}, y, x)]^{n-4}\,, \tag{3.12}$$

where

$$f(z_{n+1}, y, x) = \prod_i |z - x_i|^2 = \prod_i (z_{n+1}{}^2 + |y - x_i|^2)\,. \tag{3.13}$$

Now the primitive divergence of the integral in Eq. (3.5) comes about in a region in which all the points $\{\eta_i\}$ are coincident. A consideration of the scaling properties of the integrand shows that this divergence manifests itself as poles of $I(\beta_{ij})$ in the variable B. In the representation of $I(\beta_{ij})$ in Eq. (3.6), these divergences arise at the end point of the z_{n+1} integration. From Eq. (3.12) we see that these poles occur at points for which

$$p(4-n) - n = -k, \quad k = 0, 1, 2, \ldots\,. \tag{3.14}$$

That is

$$n = \frac{k + 4p}{1 + p}\,. \tag{3.15}$$

When $k=4$, $n=4$, so the ultraviolet pole of interest is never the lowest one. The residue of the pole is proportional to the coefficient of $z_{n+1}{}^4$ in the Taylor expansion of

$$[f(z_{n+1}, y, x)]^{n-4}\,.$$

This coefficient obviously has a factor $(4-n)$ which cancels the primitive pole. Of course the amplitude will still be singular at $n=4$ owing to divergences associated with coincidences of subsets of the $\{\eta_i\}$, but the overall primitive singularity is always removed.

It is worth remarking that this cancellation of the primitive divergence is not an accident. It is due to the vanishing of $B_i - 2n = 2(n-4)$ at $n=4$. This in turn reflects the fact that ϕ^4 theory is *formally* conformally invariant in four dimensions.

Given then that only subdivergences appear in the vacuum bubbles, we expect that they will be canceled by counterterms generated by the renormalization process.[24] In the present case, since vacuum bubbles have no external legs, these counterterms should arise simply from the replacement of the bare coupling constant by its expansion in powers of λ, the renormalized coupling constant That is,

$$\lambda_0 = \mu^{4-n}\left[\lambda + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda)}{(n-4)^\nu}\right], \tag{3.16}$$

where $a_\nu(\lambda)$ is a power series in λ. Here μ is the standard mass parameter which is introduced to make λ dimensionless. We verify this to $O(\lambda^3)$ in this next section.

If account is taken of Collins's result,[13] then additional mass-type insertions will be required at $O(\lambda^4)$ and beyond. However, these are proportional to $(n-4)^3$ and so will not introduce new primitive divergences.

The end result then is that when the sum over interacting vacuum bubbles W_I is reexpressed as a power series in the renormalized coupling constant, it is finite. It follows immediately that W_I satisfies the renormalization-group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda}\right] W_I = 0\,, \tag{3.17}$$

where $\beta(\lambda)$ is the usual coefficient of ϕ^4 theory. Equation (3.17) is a consequence of the general result that in the renormalization process a change in μ can be compensated by a suitable change in λ. It can be regarded as a special case of the standard renormalization-group equation[22,23] for Green's functions in which the number of external legs is zero.

The contribution of the interacting vacuum bubbles to the trace of the energy-momentum tensor is [see Eq. (2.7)]

$$\int d\sigma T^\mu_{I\mu} = a \frac{\partial}{\partial a} W_I\,. \tag{3.18}$$

Now on dimensional grounds we know that W_I depends only on λ and μa. Therefore,

$$a \frac{\partial}{\partial a} W_I = \mu \frac{\partial}{\partial \mu} W_I\,. \tag{3.19}$$

From the renormalization-group equation [Eq. (3.17)] we find then

$$\int d\sigma T^\mu_{I\mu} = -\beta(\lambda) \frac{\partial}{\partial \lambda} W_I\,. \tag{3.20}$$

That is,

$$T^\mu_{I\mu} = -\frac{1}{a^4 \Omega_5} \beta(\lambda) \frac{\partial}{\partial \lambda} W_I\,. \tag{3.21}$$

This is our main result for the spherical case. In the next section we shall verify it explicitly to $O(\lambda^3)$ and use it to obtain $T^\mu_{I\mu}$ to $O(\lambda^4)$.

Finally, we note that the same analysis applied to ϕ^3 bubbles in six dimensions yields the same conclusion, namely, that they too all lack primitive divergences. In that case we expect to obtain the result

$$T^{\mu}_{I\mu} = -\frac{1}{a^6\Omega_7}\,\beta(g)\,\frac{\partial}{\partial g}\,W_I\,, \tag{3.22}$$

where g is the renormalized coupling in ϕ^3 theory.

IV. ϕ^4 THEORY IN FOUR DIMENSIONS

Before passing on to the detailed treatment of the interacting graphs of Fig. 1, it should be noted that certain graphs have been ignored as giving zero contribution to W_I. Such a graph is the first-order Feynman diagram in Fig. 2. Its contribution is proportional to $[D(\eta,\eta)]^2$, and in Ref. 11 it was argued that the diagonal elements of the propagator should be evaluated as zero in the method of dimensional regularization. We omit from consideration then, all graphs which contain a line reentering the vertex from which it emerged.

We have already evaluated the two-vertex graph in Fig. 1(b). If we use the standard result that

$$\lambda_0 = \mu^{4-n}\lambda\left[1 - \frac{3\lambda}{(4\pi)^2}\,\frac{1}{n-4} + O(\lambda^2)\right], \tag{4.1}$$

we find to $O(\lambda^3)$ that

$$W_2 = \frac{1}{864}\,\frac{\lambda^2}{(4\pi)^4} - \frac{1}{144}\,\frac{\lambda^3}{(4\pi)^6}\,\frac{1}{n-4} - \frac{1}{144}\,\frac{\lambda^3}{(4\pi)^6}\,(-\ln 4\pi\mu^2 a^2 + \gamma - \tfrac{43}{12}) + O(n-4)\,. \tag{4.2}$$

The three-vertex diagram in Fig. 1(c) yields a term

$$W_3 = -\frac{\lambda_0^{\,3}}{48}\int d\sigma_1 d\sigma_2 d\sigma_3 [D(\eta_1,\eta_2)]^2 \times [D(\eta_2,\eta_3)]^2 [D(\eta_3,\eta_1)]^2\,. \tag{4.3}$$

This is readily evaluated as

$$W_3 = -\frac{\lambda_0^{\,3}}{48}\,(a^3)^{4-n}\Omega_{n+1}\left[\frac{\Gamma(\frac{1}{2}n-1)}{4\pi^{n/2}}\right]^6 \times I(2n-4,2n-4,2n-4)\,, \tag{4.4}$$

where the three-point integral $I(\beta_{12},\beta_{23},\beta_{31})$ is evaluated in Ref. 12. For the values of β_{ij} in Eq. (4.4) we find

$$I(2n-4,2n-4,2n-4) = 2^{11-4n}\pi^{n/2}\Omega_n\Gamma(6-2n)\left[\frac{\Gamma(2-\frac{1}{2}n)}{\Gamma(4-n)}\right]^3\,. \tag{4.5}$$

Clearly, I, and therefore W_3, has the single pole

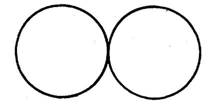

FIG. 2. Vanishing vacuum bubble.

at $n=4$ referred to in the previous section. To $O(\lambda^3)$ we find

$$W_3 = \frac{1}{144}\,\frac{\lambda^3}{(4\pi)^6}\,\frac{1}{n-4} + \frac{1}{96}\,\frac{\lambda^3}{(4\pi)^6}\,(-\ln 4\pi\mu^2 a^2 + \gamma - \tfrac{29}{9}) + O(n-4)\,. \tag{4.6}$$

On adding W_2 and W_3 we obtain W_I to $O(\lambda^3)$; thus,

$$W_I = \frac{1}{864}\,\frac{\lambda^2}{(4\pi)^4} + \frac{1}{288}\,\frac{\lambda^3}{(4\pi)^6}\,(-\ln 4\pi\mu^2 a^2 + \gamma - \tfrac{5}{2})\,. \tag{4.7}$$

Notice that, as expected, the pole terms have canceled in the sum. This confirms the idea that the pole in W_3 is due entirely to a subdivergence, since it is removed by the counterterm pole in W_2 which appeared as the result of expanding λ_0 according to Eq. (4.1).

The contribution of the bubbles with interaction to the trace anomaly is therefore [cf. Eq. (2.7)]

$$T^{\mu}_{I\mu} = \frac{1}{a^4\Omega_5}\,a\,\frac{\partial}{\partial a}\,W_I = -\frac{\lambda^3}{24a^4(4\pi)^8} + O(\lambda^4)\,. \tag{4.8}$$

From Eq. (4.7) we have

$$\frac{\partial W_I}{\partial\lambda} = \frac{\lambda}{432(4\pi)^4} + \frac{\lambda^3}{96(4\pi)^6}\,(-\ln 4\pi\mu^2 a^2 + \gamma - \tfrac{5}{2}) + O(\lambda^4)\,. \tag{4.9}$$

Now,

$$\beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} - \frac{17}{3}\,\frac{\lambda^3}{(4\pi)^4} + O(\lambda^4)\,, \tag{4.10}$$

so, on substituting these results into Eq. (3.21), we find

$$T^{\mu}_{I\mu} = -\frac{\lambda^3}{24a^4(4\pi)^8} - \frac{3\lambda^4}{16a^4(4\pi)^{10}}\,(-\ln 4\pi\mu^2 a^2 + \gamma - \tfrac{473}{162}) + O(\lambda^5)\,. \tag{4.11}$$

It is immediately evident that Eq. (4.11) confirms Eq. (4.8) to $O(\lambda^3)$. Furthermore, our knowledge of $\beta(\lambda)$ enabled us to calculate $T^{\mu}_{I\mu}$ to $O(\lambda^4)$. It is interesting to note that for weak coupling $T^{\mu}_{I\mu}$ has the same sign as the free-field term in Eq. (2.2).

Apart from this there is no obvious relationship between the two contributions.

V. ϕ^3 THEORY IN SIX DIMENSIONS

In Sec. II we computed the free-field anomaly in six dimensions. The simplest diagram which provides a nonvanishing contribution to W_I is the two-point bubble in Fig. 3(a). It yields a term

$$W_2 = \frac{g_0^2}{12} \int d\sigma_1 d\sigma_2 [D(\eta_1, \eta_2)]^3 , \tag{5.1}$$

where g_0 is the bare coupling constant. This is easily computed to give

$$W_2 = \frac{g_0^2}{12} a^{6-n} 2^{6-2n} \pi^{n/2} \Omega_{n+1} \left(\frac{\Gamma(\frac{1}{2}n - 1)}{4\pi^{n/2}} \right)^3 \frac{\Gamma(3-n)}{\Gamma(3-\frac{1}{2}n)} . \tag{5.2}$$

Once again we see that W_2 remains finite at the relevant value of n, apart from the divergences hidden in g_0. If we substitute the standard expansion for g_0 in terms of the renormalized coupling constant g, namely

$$g_0 = \mu^{3-n/2} g \left(1 + \frac{3}{4} \frac{g^2}{(4\pi)^3} \frac{1}{n-6} + \cdots \right) , \tag{5.3}$$

we find

$$W_2 = -\frac{1}{2^6 3^3 5} \frac{g^2}{(4\pi)^3} - \frac{1}{2^7 3^2 5} \frac{g^4}{(4\pi)^6} \frac{1}{n-6} + \cdots , \tag{5.4}$$

where we have omitted a finite term $O(g^4)$. On the basis of the general analysis of Sec. III we expect that the pole on the right-hand side of Eq. (5.4) will be canceled by the pole singularity of the terms of $O(g^4)$ which come from the diagrams in Figs. 3(b) and 3(c). We have verified this cancellation but omit the calculation for the sake of brevity.

Although we were unable to calculate the finite part of the diagrams in Figs. 3(b) and 3(c) we can still calculate the anomaly from Eq. (3.22). We have

$$\beta(g) = -\frac{3}{4} \frac{g^3}{(4\pi)^3} + O(g^4) \tag{5.5}$$

and

$$\frac{\partial W_I}{\partial g} = -\frac{1}{2^5 3^3 5} \frac{g}{(4\pi)^3} + O(g^3) . \tag{5.6}$$

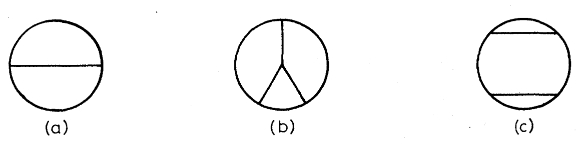

FIG. 3. Low-order vacuum bubbles for Φ^3 theory.

Hence

$$T^{\mu}_{I\mu} = -\beta(g) \frac{\partial}{\partial g} W_I = -\frac{1}{96 a^6} \frac{g^4}{(4\pi)^9} + O(g^5) . \tag{5.7}$$

It is interesting to note that in this case the anomaly has a sign opposite to that of the free-field contribution. However, they do not cancel for small values of g.

VI. EXTENSION TO GENERAL SPACE-TIMES

It seems reasonable to suppose that some of the more basic properties of field theory on a sphere can be extended to more general manifolds with arbitrary metrics. We shall therefore make the following assumptions:

(i) Conformal scalar field theory on an arbitrary manifold can be renormalized by dimensional regularization, and the renormalization constants are identical to those in flat space.

(ii) If the space-time is a compact Riemannian manifold, then the vacuum bubbles with interaction contain no primitive divergences in the sense discussed in Sec. III. It follows that the sum over these amplitudes, when reexpressed as a power series in the renormalized coupling constant, contains only finite terms.

Our preference for a *conformal* scalar field is based on our experience with the spherical case, where the presence of the curvature term in the d'Alembertian is of crucial importance. Our insistence on a compact manifold is intended to allow us to avoid trivial infinities associated with the volume of space-time. The choice of a Riemannian rather then a Minkowskian manifold is made simply on the grounds that compactness seems more natural in the former case. Several interesting noncompact Minkowskian manifolds become compact Riemannian ones after the Wick rotation of an appropriate time variable.

The d'Alembertian for a conformal massless scalar field in n dimensions is

$$K = D_\mu D^\mu - \xi(n) R , \tag{6.1}$$

where D_μ is the covariant derivative, R is the curvature, and

$$\xi(n) = \frac{n-2}{4(n-1)} . \tag{6.2}$$

On the sphere, $R = n(n-1)/a^2$, in which case K becomes identical with M in Eq. (2.1).

The free-field propagator $\Delta(x, x')$ satisfies

$$K \Delta(x, x') = -\delta(x, x') , \tag{6.3}$$

where x and x' represent the coordinates of points on the manifold, and the δ function is normalized so that

$$\int d^n x [g(x)]^{1/2} \delta(x, x') = 1 . \quad (6.4)$$

Here $g(x)$ is the determinant of the metric tensor at $g_{\mu\nu}(x)$.

One of the most important properties of K is the manner in which it transforms under Weyl scaling of the metric. Introduce a new metric

$$g_{\Omega\mu\nu}(x) = [\Omega(x)]^2 g_{\mu\nu}(x) , \quad (6.5)$$

so that

$$\begin{aligned} g_\Omega &= \Omega^{2n} g , \\ \delta_\Omega(x, x') &= \Omega^{-n}(x)\delta(x, x') . \end{aligned} \quad (6.6)$$

If now we denote the new conformal d'Alembertian appropriate to the metric $g_{\Omega\mu\nu}$ by K_Ω, then it is related to K by

$$K_\Omega \Omega^{1-n/2} = \Omega^{-1-n/2} K . \quad (6.7)$$

The new propagator $\Delta_\Omega(x, x')$ satisfies

$$K_\Omega \Delta_\Omega(x, x') = -\delta_\Omega(x, x') . \quad (6.8)$$

It is easy then to verify from the last two equations that

$$\Delta_\Omega(x, x') = [\Omega(x)]^{1-n/2} \Delta(x, x') [\Omega(x')]^{1-n/2} . \quad (6.9)$$

The sum over vacuum bubbles with interaction appropriate to the new metric we denote by $W_I[\Omega]$. It is a functional of $\Omega(x)$ and is calculated using the propagator $\Delta_\Omega(x, x')$. The contribution of a given graph with p vertices to $W_I[\Omega]$ is [cf. Eq. (3.3)]

$$W_p[\Omega] = \frac{(-\lambda_0)^p}{S} \int \prod_i d^n x_i [g_\Omega(x_i)]^{1/2} \times \prod_{1\le i<j\le p} [\Delta_\Omega(x_i, x_j)]^{\lambda_{ij}} , \quad (6.10)$$

where as before λ_{ij} is the number of lines joining x_i to x_j and S is the symmetry factor for the graph. If we assume we are dealing with ϕ^4 theory and take account of the fact that four lines end on each vertex, then we find, after using Eq. (6.9) for Δ_Ω, that

$$W_p[\Omega] = \frac{(-1)^p}{S} \int \prod_i d^n x_i [g(x_i)]^{1/2} \{\lambda_0 [\Omega(x_i)]^{4-n}\} \times \sum_{1\le i<j\le p} [\Delta(x_i, x_j)]^{\lambda_{ij}} . \quad (6.11)$$

This is identical with the vacuum bubble amplitude for the original metric apart from a factor Ω^{4-n} at each vertex.

Now we can use the renormalization group[22,23] to introduce a position-dependent coupling constant in the following way. For convenience we write $\lambda_0 = \mu^{4-n}\hat{\lambda}_0(\lambda)$. We can write then

$$\lambda_0 [\Omega(x)]^{4-n} = [\mu\Omega(x)]^{4-n} \hat{\lambda}_0(\lambda) . \quad (6.12)$$

The renormalization group tells us that we can absorb the scaling parameter into a redefinition of the coupling constant; thus,

$$[\mu\Omega(x)]^{4-n}\hat{\lambda}_0(\lambda) = \mu^{4-n}\hat{\lambda}_0(\lambda(x)) = \lambda_0(\lambda(x)) , \quad (6.13)$$

where $\lambda(x) = f(\lambda, \Omega(x))$ with $f(\lambda, 1) = \lambda$ and

$$\omega \frac{\partial}{\partial \omega} f(\lambda, \omega) = -\beta(f(\lambda, \omega)) . \quad (6.14)$$

If we make the substitution indicated by Eq. (6.12) and (6.13) into Eq. (6.11) for the vacuum bubbles, we see that W_I evaluated for the scaled metric $g_{\Omega\mu\nu}$ can be thought of as a functional of $\lambda(x)$ rather than of $\Omega(x)$.

The trace of the energy-momentum tensor due to the interaction is

$$T^\mu_{I\mu} = \Omega(x) \left. \frac{\delta W_I}{\delta\Omega(x)} \right|_{\Omega=1} . \quad (6.15)$$

Using the ideas indicated above we can rewrite this as

$$T^\mu_{I\mu} = \Omega(x) \frac{d\lambda(x)}{d\Omega(x)} \left. \frac{\delta W_I}{\delta\lambda(x)} \right|_{\lambda(x)=\lambda} . \quad (6.16)$$

But, from Eq. (6.14),

$$\Omega(x) \frac{d\lambda(x)}{d\Omega(x)} = -\beta(\lambda(x)) , \quad (6.17)$$

so we obtain our main result in the form

$$T^\mu_{I\mu} = -\beta(\lambda) \left. \frac{\delta W_I}{\delta\lambda(x)} \right|_{\lambda(x)=\lambda} . \quad (6.18)$$

We can say more than this, however. If we use the notion of a position-dependent bare coupling constant $\lambda_0(x)$, then an examination of the perturbation series shows that

$$\left. \frac{\delta W_I}{\delta\lambda_0(x)} \right|_{\lambda_0(x)=\lambda_0} = \left\langle \frac{\phi^4(x)}{4!} \right\rangle , \quad (6.19)$$

where we use the brackets $\langle\ \rangle$ to indicate vacuum expectation value. The right-hand side of Eq. (6.19) is computed by summing over all vacuum bubbles with one vertex in a general position. The symmetry factor must be adjusted accordingly. Now if we set $\lambda_0(x) = \lambda_0(\lambda(x))$, then

$$\left. \frac{\delta W_I}{\delta\lambda(x)} \right|_{\lambda(x)=\lambda} = \left. \frac{\delta W_I}{\delta\lambda_0(x)} \right|_{\lambda_0(x)=\lambda_0} \times \frac{d\lambda_0}{d\lambda} , \quad (6.20)$$

which leads to

$$T^\mu_{I\mu} = -\beta(\lambda) \frac{d\lambda_0}{d\lambda} \left\langle \frac{\phi^4(x)}{4!} \right\rangle . \quad (6.21)$$

This result is somewhat simpler than the corresponding result in Ref. 13. However, we have so far omitted the graphs corresponding to the insertion

$\propto(n-4)^3R\phi^2$, which Collins suggests is required in order to make the renormalization process work properly in curved space. It is outside the scope of this paper to discuss such contributions in detail. However, we can reasonably assume that they will not upset the basic idea, which is that $W_I[\Omega]$ can be reinterpreted as a functional of the position-dependent coupling $\lambda(x)$. In that case, Eq. (6.18) will still hold. Equation (6.21), however, will be modified by extra terms on the right-hand side.

VII. CONCLUSIONS AND DISCUSSIONS

In this paper we have investigated the anomaly in the trace of the vacuum energy-momentum tensor for both free and interacting fields. Much of our discussion was concerned with spherical space-time, and for this case we were able to confirm the standard free-field anomaly in four dimensions and calculate the corresponding anomaly in six. It turns out that they have opposite signs.

The principal result of this paper is that the trace anomaly due to the interaction is proportional to the renormalization group β function. The derivation of this result hinges on the assumption that vacuum bubble diagrams contain no *primitive* divergences at $n=4$ for ϕ^4 theory and $n=6$ for ϕ^3. This implies that all divergences will be removed when the sum over vacuum bubbles with interaction W_I is expressed as a power series in the renormalized coupling constant. This assumption was justified in detail in the spherical case. It is a consequence of the formal conformal invariance of scalar field theory in the appropriate dimension.

The general arguments were checked in the spherical case by explicit calculation in both ϕ^4 and ϕ^3 theory and the additional trace anomalies calculated to fourth order. In the case of ϕ^4 theory the anomaly due to the interaction has the same sign as the free-field term, while in ϕ^3 theory they have opposite signs.

The assumption that W_I is finite even in a manifold with an arbitrary metric led us to view it as a functional of a position-dependent renormalized coupling constant $\lambda(x)$. Obviously, $\delta W_I/\delta\lambda(x)$ is then a finite quantity. Now, $\lambda(x)$ is related to a Weyl scaling field $\Omega(x)$ by the renormalization-group equation, so

$$\Omega(x)\,\frac{\delta}{\delta\Omega(x)}=-\beta(\lambda(x))\,\frac{\delta}{\delta\lambda(x)}\,. \tag{7.1}$$

This relation is the origin of the factor $\beta(\lambda)$ in Eq. (6.18) for $T^{\mu}_{I\mu}$.

The possibility and the usefulness of introducing $\lambda(x)$ related to $\Omega(x)$ as above depends on the simple scaling properties of the conformal massless scalar field theory we have investigated. Our experience indicates that this is the theory which renormalizes in a clean way in curved space. Collins[13] has suggested in the case of ϕ^4 theory that for higher orders in perturbation theory than we have yet checked, a modification of the n dimensional action is necessary for the successful operation of the renormalization process. However, since this term vanishes in four dimensions, we feel that it does not lead to any divergences which would contradict our hypothesis that W_I is a finite quantity.

Of course it would be of great interest to check these assumptions to higher order in spherical space-time. Indeed it is of considerable importance to extend the analysis of the renormalization process to more general space-time manifolds and check our assumptions by calculation.

Finally we remark that the results of this paper appear to require modification in the case of massless quantum electrodynamics.[25,26] A detailed examination of the trace anomaly for this case on a spherical minifold will be the subject of a separate paper.

ACKNOWLEDGMENT

One of us (G.M.S.) would like to thank the Carnegie Trust for the Universities of Scotland for a research scholarship.

APPENDIX

In this appendix we complete the discussion of Sec. II by presenting the calculation of the pole terms in the effective action W_0 for free scalar fields in four and six dimensions. This derivation conveniently combines the dimensional regularization procedure used throughout the paper with a ζ-function technique.[13]

The effective action W_0 is given by

$$W_0=\tfrac{1}{2}\,\mathrm{Tr}\ln D\,, \tag{2.8}$$

where the propagator may be written in terms of transform space variables as

$$D(\eta,\eta')=\sum_{l,m}\frac{a^2}{(l+\frac{1}{2}n)(l+\frac{1}{2}n-1)}\,y_{lm}(\eta)y_{lm}(\eta')\,. \tag{A1}$$

The label m on the harmonics has a range of $h(l,n)$ values, where

$$h(l,n)=\frac{(2l+n-1)\Gamma(l+n-1)}{\Gamma(n)\Gamma(l+1)}\,. \tag{A2}$$

It follows that

$$W_0=\frac{1}{2}\sum_l h(l,n)\ln\frac{\mu^2a^2}{(l+\frac{1}{2}n)(l+\frac{1}{2}n-1)}\,, \tag{A3}$$

where this μ is a dimensional parameter arising from the functional integral over fields, and was suppressed in the symbolic equation (2.8).

At this stage we introduce the ζ-function technique. Defining

$$\zeta(s,n)=\sum_l h(l,n)\left[\frac{\mu^2a^2}{(l+\frac{1}{2}n)(l+\frac{1}{2}n-1)}\right]^s, \qquad \text{(A4)}$$

it may be seen that

$$W_0=\frac{1}{2}\frac{\partial}{\partial s}\zeta(s,n)\bigg|_{s=0}\equiv\tfrac{1}{2}\zeta'(0,n). \qquad \text{(A5)}$$

Notice that the limit $s\to 0$ is to be taken before the dimension n is set equal to four.

Now using the results

$$y^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty d\alpha\,\alpha^{s-1}e^{-\alpha y} \qquad \text{(A6)}$$

and

$$\sum_l h(l,n)z^l=(1+z)(1-z)^{-n}, \qquad \text{(A7)}$$

it is straightforward to show that

$$\zeta(s,n)=\zeta_1(s,n)+\zeta_2(s,n), \qquad \text{(A8)}$$

where

$$\zeta_1(s,n)=\frac{(\mu^2a^2)^s}{\Gamma(s)^2}\int dx\,x^{s-1}(1-x)^{s-1}\int d\lambda\,\lambda^{2s-1}e^{-\lambda x}e^{-\lambda(n/2-1)}(1-e^{-\lambda})^{-n} \qquad \text{(A9)}$$

and

$$\zeta_2(s,n)=\frac{(\mu^2a^2)^s}{\Gamma(s)^2}\int dx\,x^{s-1}(1-x)^{s-1}\int d\lambda\,\lambda^{2s-1}e^{-\lambda x}e^{-\lambda n/2}(1-e^{-\lambda})^{-n}. \qquad \text{(A10)}$$

Introducing coefficients $a_r(n)$ defined by

$$\left(\frac{1-e^{-\lambda}}{\lambda}\right)^{-n}=\sum_{r=0}^\infty a_r(n)\lambda^r \qquad \text{(A11)}$$

and performing the integrations over λ and x gives

$$\zeta_1(s,n)=(\mu^2a^2)^s\sum_{p=0}^\infty\sum_{r=0}^\infty\frac{(-1)^p\Gamma(s+p)\Gamma(2s+p+r-n)}{\Gamma(p+1)\Gamma(s)\Gamma(2s+p)}a_r(n)(\tfrac{1}{2}n-1)^{n-2s-p-r} \qquad \text{(A12)}$$

and

$$\zeta_2(s,n)=(\mu^2a^2)^s\sum_{p=0}^\infty\sum_{r=0}^\infty\frac{(-1)^p\Gamma(s+p)\Gamma(2s+p+r-n)}{\Gamma(p+1)\Gamma(s)\Gamma(2s+p)}a_r(n)\left(\frac{n}{2}\right)^{n-2s-p-r}. \qquad \text{(A13)}$$

Differentiating with respect to s gives

$$\zeta_1'(s,n)=(\mu^2a^2)^s\sum_{p=0}^\infty\sum_{r=0}^\infty\frac{(-1)^p\Gamma(s+p)\Gamma(2s+p+r-n)}{\Gamma(p+1)\Gamma(s)\Gamma(2s+p)}a_r(n)\left(\frac{n}{2}-1\right)^{n-2s-p-r}$$
$$\times[\ln(\mu^2a^2)-2\ln(\tfrac{1}{2}n-1)+2\psi(2s+p+r-n)-2\psi(2s+p)+\psi(s+p)-\psi(s)], \qquad \text{(A14)}$$

a similar result holding for $\zeta_2'(s,n)$.

However, in the limit $s\to 0$, only the final three terms give nonzero contributions, leaving finally

$$\zeta_1'(0,n)=2\sum_{r=0}^\infty a_r(n)\left(\frac{n}{2}-1\right)^{n-r}\Gamma(r-n)+\sum_{p=1}^\infty\sum_{r=0}^\infty\frac{(-1)^p}{\Gamma(p+1)}a_r(n)\left(\frac{n}{2}-1\right)^{n-p-r}\Gamma(p+r-n) \qquad \text{(A15)}$$

and

$$\zeta_2'(0,n)=2\sum_{r=0}^\infty a_r(n)\left(\frac{n}{2}\right)^{n-r}\Gamma(r-n)+\sum_{p=1}^\infty\sum_{r=0}^\infty\frac{(-1)^p}{\Gamma(p+1)}a_r(n)\left(\frac{n}{2}\right)^{n-p-r}\Gamma(p+r-n). \qquad \text{(A16)}$$

The terms in these summations with $p+r\leq 4$ are divergent at $n=4$. These may be evaluated using the following values for the coefficients $a_r(4)$:

$$a_0(4)=1,\quad a_1(4)=2,\quad a_2(4)=\tfrac{11}{6},\quad a_3(4)=1,\quad a_4(4)=\tfrac{251}{720}. \qquad \text{(A17)}$$

The corresponding results for $n=6$ are

$$a_0(6)=1,\quad a_1(6)=3,\quad a_2(6)=\tfrac{17}{4},\quad a_3(6)=\tfrac{15}{4},\quad a_4(6)=\tfrac{137}{60},\quad a_5(6)=1,\quad a_6(6)=\tfrac{19}{60}-\tfrac{13}{7}(\tfrac{1}{12})^3. \qquad \text{(A18)}$$

Evaluating these, we find for $n=4$

$$\zeta'(0,n)=\tfrac{1}{45}(n-4)+O((n-4)^0)\,, \tag{A19}$$

while for $n=6$

$$\zeta'(0,n)=-\tfrac{1}{378}(n-6)+O((n-6)^0)\,. \tag{A20}$$

In fact, the entire contribution to the pole arises from the single summation ($p=0$) parts of (A15) and (A16). The poles in the double summation add to zero.

Notice also that we do not require the series (A15) and (A16) to be convergent. All that is needed is that the divergence at $n=4$ be correctly given by the sum of terms with $p+r\leqslant 4$.

Finally, using Eq. (A5), we recover the results quoted in the text for the pole parts of the effective action. That is, for $n=4$,

$$W_0=\tfrac{1}{90}(n-4)+O((n-4)^0)\,, \tag{2.9}$$

and for $n=6$

$$W_0=-\tfrac{1}{756}(n-6)+O((n-6)^0)\,. \tag{2.13}$$

---

[1]S. Deser, M. J. Duff, and C. J. Isham, Nucl. Phys. B111, 45 (1976).
[2]J. S. Dowker and R. Critchley, Phys. Rev. D 13, 3224 (1976).
[3]S. M. Christensen and S. A. Fulling, Phys. Rev. D 15, 2088 (1977).
[4]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A356, 569 (1977); A357, 381 (1977)
[5]L. S. Brown, Phys. Rev. D 15, 1469 (1977).
[6]L. S. Brown and J. P. Cassidy, Phys. Rev. D 15, 2810 (1977).
[7]A. Duncan, Phys. Lett. 66B, 170 (1977).
[8]H. S. Tsao, Phys Lett. 68B, 79 (1977).
[9]M. J. Duff, Nucl. Phys. B125, 334 (1977).
[10]S. W. Hawking, Commun. Math. Phys. 55, 133 (1977).
[11]I. T. Drummond, Nucl. Phys. B94, 115 (1975).
[12]I. T. Drummond, preceding paper, Phys. Rev. D 19, 1123 (1979).
[13]J. C. Collins, Phys. Rev. Lett. 36, 1518 (1976); Phys. Rev. D 14, 1965 (1976).
[14]S. L. Adler, J. C. Collins, and A. Duncan, Phys. Rev. D 15, 1712 (1977).
[15]J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. D 16, 438 (1977).
[16]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).
[17]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).
[18]D. Z. Freedman, I. J. Muzinich, and E. J. Weinberg, Ann. Phys. (N.Y.) 87, 95 (1974); 87, 354 (1974).
[19]B. Schroer, Lett. Nuovo Cimento 2, 867 (1971).
[20]S. L. Adler, Phys. Rev. D 6, 3445 (1972); 8, 2400 (1973).
[21]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).
[22]G. 't Hooft, Nucl. Phys. B61, 455 (1974).
[23]J. C. Collins and A. J. MacFarlane, Phys. Rev. D 10, 1213 (1974).
[24]It should be pointed out that even after all counterterms have been included, a Feynman graph may still exhibit a pole of second order or higher. Until this possibility is ruled out our discussion is, of course, a plausibility argument rather than a proof.
[25]I. T. Drummond and G. M. Shore, Ann. Phys. (N.Y.) (to be published).
[26]G. M. Shore, Ann. Phys. (N.Y.) (to be published).