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Inconsistency between the Boltzmann distribution for relativistic free particles and the Rayleigh-Jeans law for thermal radiation within classical electrodynamics

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The classical electrodynamic model introduced by Einstein and Hopf of a massive free particle containing an electric dipole oscillator in interaction with random classical radiation is extended to relativistic particle velocities. The equilibrium distribution of classical particle velocities enforced by the Rayleigh-Jeans law of thermal radiation is found to be different from the Boltzmann distribution for relativistic free particles at the same temperature. Thus apparently there is an inconsistency within classical theory between the relativistic Boltzmann particle distribution and the Rayleigh-Jeans law for thermal radiation.

INTRODUCTION

The use of the Boltzmann factor for the distribution of mechanical systems on phase space represents one of the fundamental aspects of classical statistical mechanics. At the same time the idea that classical electromagnetism must involve the Rayleigh-Jeans law for thermal radiation is a tenet of every physics textbook writer and virtually every physicist. In this article we point out that these two fundamental ideas are apparently in conflict within classical electrodynamics involving relativistic classical particles.

It is a familiar fact that for nonrelativistic mechanical systems the two ideas are often joined successfully. For example, in Born's *Atomic Physics* we find[1] the average energy $\overline{E}$ of a charged nonrelativistic oscillator of natural frequency ω_0 is related to the spectral energy density $\rho(\omega)$ of the random classical radiation field as

$$\overline{E}=\frac{\pi^2c^3}{\omega_0^{\,2}}\,\rho(\omega_0)\ , \tag{1}$$

where c is the speed of light. The average energy for the oscillator found from the Boltzmann distribution on phase space,

$$P(x,p)=\text{const}\times\exp(-E/kT)\ , \tag{2}$$

for the nonrelativistic energy

$$E=p^2/2m+\tfrac{1}{2}m\omega_0^{\,2}x^2 \tag{3}$$

leads to energy equipartition for the oscillator

$$\overline{E}=kT\ , \tag{4}$$

and hence through the result (1) to the Rayleigh-Jeans radiation spectrum

$$\rho(\omega)=\frac{\omega^2}{\pi^2c^3}\,kT\ . \tag{5}$$

This indeed seems to establish that the Boltzmann distribution for a charged mechanical system is connected to the Rayleigh-Jeans law for thermal radiation.

Now in the past there has been interest[2] in extending the analysis of classical radiation equilibrium beyond the interaction with nonrelativistic charged mechanical systems, where indeed the Rayleigh-Jeans law seems to be appropriate,[2] over to the interaction of radiation with relativistic charged mechanical systems. Apparently this relativistic problem has never been treated within classical electrodynamics.

Rather than work with the difficult calculational problem of a relativistic particle in a potential, we turn here instead to free particles. An effort to check the agreement between the Boltzmann distribution for nonrelativistic free particles and the Rayleigh-Jeans law for thermal radiation appeared in 1910 and 1926 in the work of Einstein and Hopf,[3] and of Milne.[4] The model involved a massive particle with an electric dipole oscillator mounted in the particle. The dipole oscillator interacted with the thermal radiation receiving random impulses which caused the entire particle to move in a form of Brownian motion. It was found again that the Maxwell-Boltzmann distribution for nonrelativistic particles and the Rayleigh-Jeans law for thermal radiation gave equilibrium between the particle and radiation energies.

Now this model used by Einstein and Hopf can be extended to the case of relativistic free particles so that we find a relativistic Brownian motion enforced on the particle by the random radiation. The equilibrium distribution of particle momenta can be obtained from a Fokker-Planck equation and can then be compared with the Boltzmann distribution for relativistic free particles first given by Jüttner[5] in 1911. We find a discrepancy. The Rayleigh-Jeans law for thermal radiation does not lead to the Boltzmann distribution for relativistic free particles.

THE MODEL OF EINSTEIN AND HOPF

We consider the model of Einstein and Hopf[3] as described more recently[6] in 1969. A particle of large mass M is constrained to move in one dimension along the x axis. The particle interacts with radiation by means of an electric dipole oscillator mounted rigidly in the particle and oriented along the z direction. The dipole oscillator has a natural frequency ω_0, and may be pictured crudely as a particle of charge e and mass m on the end of a spring. The damping constant Γ of the oscillator (where $\Gamma=\frac{2}{3}e^2/mc^3$ for a particle of charge e and mass m) is taken very small, and indeed can be taken to zero as equilibrium is approached for the particle distribution.

In the work of Einstein and Hopf[3] and in our work[6] of 1969 the analysis was limited to nonrelativistic velocities for the massive particle M. Here we lift that restriction. In the rest frame of the massive particle, the dipole oscillator can always be treated accurately with nonrelativistic mechanics. In order to treat the entire system relativistically we carry out a Lorentz transformation from the laboratory frame, where the radiation distribution is the isotropic Rayleigh-Jeans law, over to the instantaneous rest frame of the particle where the radiation distribution is not isotropic. Then in the rest frame of the particle we find the electric dipole $\vec{p}'$ is driven into oscillation by the random radiation and experiences random forces on the dipole

$$\vec{\mathfrak{F}}'=(\vec{p}'\cdot\nabla')\vec{E}'+c^{-1}\dot{\vec{p}}'\times\vec{B}', \qquad (6)$$

including components $\mathfrak{F}_x'$ in the x direction which will lead to motion of the entire particle. Next we Lorentz transform these forces $\mathfrak{F}_x'$ on the particle back to the laboratory frame and so have a random force $\mathfrak{F}_x(\beta)$ on the particle moving with velocity $v=\beta c$ in the x direction.

The equilibrium distribution $P(p)$ in particle momentum p may be obtained from a Fokker-Planck equation for the momentum

$$-P(p)F_x(p)\tau+\frac{1}{2}\frac{\partial}{\partial p}[P(p)\langle\Delta^2(p)\rangle]=0\ , \qquad (7)$$

where F_x is the average force on the particle during time τ,

$$F_x(p)=\langle\mathfrak{F}_x(\beta)\rangle\ , \qquad (8)$$

with

$$p=Mc\gamma\beta\ , \qquad (9)$$

$$\gamma=(1-\beta^2)^{-1/2}\ , \qquad (10)$$

and $\langle\Delta^2(p)\rangle$ is the average impulse squared delivered to the particle during time τ,

$$\langle\Delta^2(p)\rangle=\left\langle\left(\int_{t=0}^{t=\tau}\mathfrak{F}_x(\beta)dt\right)^2\right\rangle . \qquad (11)$$

The time τ is chosen sufficiently short so that the momentum does not change significantly over the time interval and sufficiently long so that many oscillations of the dipole will have occurred. The averaging is over the phases of the random radiation, or equivalently, over the starting times used in calculating the fluctuating force.

RESULTS FOR $F_x(p)$ and $\langle\Delta^2(p)\rangle$

The accurate calculation of the average force and the mean-square impulse are crucial to obtaining the correct particle distributions in random radiation, and hence crucial to the thesis of this paper. The impatient reader can calculate these parameters starting from the explanation given here and the calculations at nonrelativistic velocities given earlier.[6] We will present our detailed calculations in a subsequent article dealing with the full question of the equilibrium of relativistic free particles in random classical radiation. Here we merely cite our results

$$F_x(p)=\frac{3}{8}\frac{\Gamma\omega_0^3}{c}\int_{x=-1}^{x=1}dx\,x(1+x^2)\left(\frac{g(\omega_0\gamma(1+\beta x))}{\omega_0\gamma(1+\beta x)}\right) \qquad (12)$$

and

$$\langle\Delta^2(p)\rangle=\frac{9}{32}\frac{\Gamma\omega_0^4}{c^2}\tau\gamma\int_{x=-1}^{x=1}dx(1+x^2)\left(\frac{g(\omega_0\gamma(1+\beta x))}{\omega_0\gamma(1+\beta x)}\right)\int_{y=-1}^{y=1}dy\,y^2(1+y^2)\left(\frac{g(\omega_0\gamma(1+\beta y))}{\omega_0\gamma(1+\beta y)}\right) \qquad (13)$$

where as before $p=Mc\gamma\beta$, $\gamma=(1-\beta^2)^{-1/2}$ and $g(\omega)$ is the energy per normal mode in the random radiation field at frequency ω, with $g(\omega)=kT$ for the Rayleigh-Jeans law.

TWO CHECKS ON THE RESULTS FOR F_x and $\langle\Delta^2\rangle$

It is easy to check the results given for $F_x(p)$ and $\langle\Delta^2(p)\rangle$ in two limits. In the nonrelativistic limit in which we retain terms only through first order in β, we should obtain exactly the nonrelativistic results given previously.[6] Indeed we find from Taylor series expansion,

$$F_x(p) \simeq \frac{3}{8}\frac{\Gamma\omega_0^3}{c}\int_{x=-1}^{x=1} dx\frac{x(1+x^2)}{\omega_0}\left[g(\omega_0)(1-\beta x) + \omega_0\beta x\frac{dg(\omega_0)}{d\omega_0}\right]$$

$$\simeq -\frac{2}{5}\frac{\Gamma\omega_0^2}{c}\beta\left[g(\omega_0)-\frac{dg}{d\omega_0}\right], \qquad (14)$$

$$\langle\Delta^2(p)\rangle \simeq \langle\Delta^2(0)\rangle = \frac{9}{32}\frac{\Gamma\omega_0^4}{c^2}\tau\int_{x=-1}^{x=1} dx(1+x^2)\frac{g(\omega_0)}{\omega_0}\times\int_{y=-1}^{y=1} dy\, y^2(1+y^2)\frac{g(\omega_0)}{\omega_0}$$

$$= \frac{4}{5}\frac{\Gamma\omega_0^2}{c^2}\tau g^2(\omega_0). \qquad (15)$$

Recalling the connection between the radiation spectrum $\rho(\omega)$ and the energy per normal mode $g(\omega)$,

$$\rho(\omega) = (\omega^2/\pi^2c^3)g(\omega), \qquad (16)$$

we see that these results (14) and (15) are in exact agreement with the earlier nonrelativistic expressions.[7]

The second check we can make involves the case of zero-point radiation. It has been shown[8] that up to a multiplicative constant, the zero-point radiation spectrum is the unique Lorentz-invariant spectrum of random classical radiation. Since the zero-point radiation is Lorentz-invariant and has no preferred frame of reference, we expect physically that the same will be true for any spectrum of particles in equilibrium with the radiation. Thus we expect

$$P(p)dp = \text{const}\times\frac{dp}{E} \qquad (17)$$

for our one-dimensional case. Now if we substitute the zero-point radiation spectrum

$$g(\omega) = \tfrac{1}{2}\hbar\omega \qquad (18)$$

into (12) and (13), we find $F_x(p)=0$ corresponding to the expected absence of any velocity-dependent forces, and

$$\langle\Delta^2(p)\rangle = \text{const}\times\gamma = \text{const}\times E. \qquad (19)$$

Substituting these values for $F_x(p)$ and $\langle\Delta^2(p)\rangle$ into the Fokker-Planck equation (7) for the equilibrium particle distribution we indeed find the expected Lorentz-invariant solution (17).

Equilibrium Particle Distribution for the Rayleigh-Jeans Law

We now turn to our primary concern, the determination of the equilibrium particle distribution in the presence of the Rayleigh-Jeans law for classical thermal radiation. We treat first the nonrelativistic case. For the Rayleigh-Jeans law $g(\omega)=kT$ and for the nonrelativistic expressions (14) and (15) substituted into (7), we have the Fokker-Planck equation for the distribution of particle momenta.

$$P(p)\frac{2}{5}\frac{\Gamma\omega_0^2}{c}\frac{p}{Mc}kT\tau + \frac{1}{2}\frac{\partial}{\partial p}\left(P(p)\frac{4}{5}\frac{\Gamma\omega_0^2}{c^2}\tau(kT)^2\right) = 0, \qquad (20)$$

where the nonrelativistic momentum is

$$p = Mv = Mc\beta. \qquad (21)$$

The solution of this equation is

$$P(p) = \text{const}\times\exp(-p^2/2MkT), \qquad (22)$$

just the expected Boltzmann distribution for nonrelativistic free particles at temperature T. The Rayleigh-Jeans law is consistent with the nonrelativistic Boltzmann particle distribution.

The same procedure can be applied for the relativistic case. Now we must use the full relativistic expressions (12) and (13) with the radiation energy per normal mode for the Rayleigh-Jeans law, $g(\omega)=kT$. The integrals for $F_x(p)$ and $\langle\Delta^2(p)\rangle$ can be evaluated in closed form by expansion in partial fractions

$$\frac{1+x^2}{1+\beta x} = \frac{x}{\beta} - \frac{1}{\beta^2} + \left(1+\frac{1}{\beta^2}\right)\frac{1}{1+\beta x},$$

$$\frac{x(1+x^2)}{1+\beta x} = \frac{x^2}{\beta} - \frac{x}{\beta^2} + \frac{1}{\beta}\left(1+\frac{1}{\beta^2}\right) - \frac{1}{\beta}\left(1+\frac{1}{\beta^2}\right)\frac{1}{1+\beta x},$$

$$\frac{x^2(1+x^2)}{1+\beta x} = \frac{x^3}{\beta} - \frac{x^2}{\beta^2} + \frac{1}{\beta}\left(1+\frac{1}{\beta^2}\right)x - \frac{1}{\beta^2}\left(1+\frac{1}{\beta^2}\right) + \frac{1}{\beta^2}\left(1+\frac{1}{\beta^2}\right)\frac{1}{1+\beta x}.$$

Upon integration we obtain

$$F_x(p) = \frac{3}{8}\frac{\Gamma\omega_0^2}{c\gamma}kT\left[\frac{2}{3\beta} + \frac{2}{\beta}\left(1+\frac{1}{\beta^2}\right) - \frac{1}{\beta^2}\left(1+\frac{1}{\beta^2}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right], \qquad (23)$$

$$\langle\Delta^2(p)\rangle = \frac{9}{32}\frac{\Gamma\omega_0^2}{c^2\gamma}\tau(kT)^2\left[-\frac{2}{\beta^2}+\frac{1}{\beta}\left(1+\frac{1}{\beta^2}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right] \times\left[-\frac{2}{3\beta^2}-\frac{2}{\beta^2}\left(1+\frac{1}{\beta^2}\right) +\frac{1}{\beta^3}\left(1+\frac{1}{\beta^2}\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right], \quad (24)$$

where $\beta = pc/E$, $E=(p^2c^2+M^2c^4)^{1/2}$.

These results (23) and (24) are now substituted into the Fokker-Planck equation (7) to obtain the equilibrium distribution for particle momenta in the Rayleigh-Jeans spectrum of radiation. The solution for $P(p)$ seems rather complicated. However, one thing is sure. The Boltzmann distribution for relativistic free particles,

$$P(p) = \text{const}\times\exp[-(p^2c^2+M^2c^4)^{1/2}/kT], \quad (25)$$

is not a solution of the equation. This is apparent even by considering the number of factors of kT. Thus for the Boltzmann distribution (25), the term $\frac{1}{2}P(p)\partial\langle\Delta^2(p)\rangle/\partial p$ will involve $(kT)^2$ whereas the other terms, $-P(p)F_x(p)\tau$ and $\frac{1}{2}\langle\Delta^2(p)\rangle\partial P(p)/\partial p$, will involve only kT. The Boltzmann distribution for relativistic free particles is inconsistent with the Rayleigh-Jeans law of thermal radiation within classical electrodynamics.

CONCLUDING SUMMARY

Classical physics leads inevitably to the Rayleigh-Jeans law for thermal radiation. This idea is endemic in the physics literature and it is borne out repeatedly in calculations involving nonrelativistic classical mechanical systems interacting with random classical electromagnetic radiation. However, classical electrodynamics should involve relativistic particle mechanics so as to be consistent with the relativistic content of Maxwell's equations. And the interaction of relativistic classical particles with random classical radiation has never been solved.

In the present paper we show that relativity does indeed interject a new aspect into classical radiation equilibrium. We reconsider the old model of Einstein and Hopf for the interaction of a free particle and radiation, now extended to allow the possibility of relativistic particle velocities. We find an inconsistency in the usual ideas of classical theory. The Rayleigh-Jeans law for thermal radiation does not lead to the Boltzmann distribution for relativistic free particles.

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