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Possibility of Landau damping of gravitational waves

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There is considerable uncertainty in the literature concerning whether or not transverse traceless gravitational waves can Landau damp. Physically, the issue is whether particles of nonzero mass can comove with surfaces of constant wave phase, and therefore, loosely, whether gravitational waves can have phase speeds less than that of light. We approach the question of Landau damping in various ways. We consider first the propagation of small-amplitude gravitational waves in an ideal fluid-filled Robertson-Walker universe of zero spatial curvature. We argue that the principle of equivalence requires those modes to be lightlike. We show that a freely moving particle interacting only with the collective fields cannot comove with such waves if it has nonzero mass. The equation for gravitational waves in collisionless kinetic gases differs from that for fluid media only by terms so small that deviations from lightlike propagation are unmeasurable. Thus, we conclude that Landau damping of small-amplitude, transverse traceless gravitational waves is not possible.

I. INTRODUCTION

There has been considerable recent interest in the generation and propagation of gravitational waves, which extends to the interaction of gravitational waves with material media—elastic, fluid, or kinetic. This paper is concerned with the propagation of transverse-traceless gravitational waves in a collisionless kinetic gas of point masses and ultimately, whether such waves can Landau damp, as plasma waves do. Resolution of this issue of principle actually has a general interest, because if Landau damping exists, experience with plasma physics then suggests that nonthermal velocity distributions could lead to resonant instabilities of gravitational waves.

Whether or not resonant Landau damping can occur reduces ultimately to the question of whether gravitational waves have a phase velocity greater or less than the "speed of light"—a concept which has precise meaning only in a local Lorentz frame. If the phase velocity is less than that of light, a particle can remain in a time-stationary frame with respect to the wave, and absorb energy from it. While there is a consensus in the literature concerning the propagation of gravitational waves in a vacuum,[1-4] the literature offers inconsistent answers concerning the speed of wave propagation in kinetic media and whether particles may resonate with the wave. We will show that some of the differences in results can be ascribed to the differences in the various authors' assumptions concerning the background geometry. For example, Chesters[5] considered the gravitational wave equation derived from linearized general relativity. Using the flat-space Vlasov equation with acceleration terms calculated from the coefficients of the affine connection of a transverse traceless (TT) metric perturbation, Chesters found a dispersion relation for sinusoidal small-amplitude waves, with a pressure correction to the vacuum solution. His phase speed was subluminous (numerical value $<c$ in appropriate units), and consequently wave-particle interaction was deemed possible. When particle beams were present, the waves could become superluminous. Polnarev[6] also used a flat-space background metric, but added a phenomenological collision term, which does not conserve proper number density, to the Vlasov equation. In the collisionless limit, Polnarev found a superluminous phase velocity. As the collision frequency approached infinity, the phase velocity approached the speed of light, in presumable agreement with fluid theory. While still retaining a flat *isotropic* background geometry, Polnarev argued that *anisotropic* distributions could produce either subluminous or superluminous waves. Ignatev[7] noted that the group velocity derivable from Chesters' dispersion relation exceeded the speed of light, which is unphysical. Ignatev derived a dispersion relation by perturbing around a solution of the full field equation, with background curvature due to the medium; the full Vlasov equation was perturbed in the curved space-time as well. His dispersion relation for small-amplitude sinusoidal waves has both density and pressure corrections leading to a superluminous phase velocity. Even if the temperature were zero, his solution would indicate a superluminous phase velocity.

In the treatment closest to the one we will pursue here, Asseo *et al.*[8] perturb around the specific Robertson-Walker (RW) background geometry, but allow the amplitude $h_{\mu\nu}$ and vector k to vary slowly in space-time. Their derived phase velocity is superluminous, the group velocity subluminous, and their dispersion relation has only a pressure correction.

We may characterize the literature dealing with transverse-traceless gravitational waves (TTGW's) in kinetic theory by a few statements. First, the cited authors considered only small-amplitude waves. Secondly, the fact that all authors obtain algebraic dispersion relations indicates that a short-wavelength local approximation has been made implicitly or explicitly, and that the waves have been assumed sinusoidal. Thirdly, although some papers consider a curved background, none found completely self-consistent modes. "Completely self-consistent" means not only using a background geometry consistent with the unperturbed matter distribution, but correctly ordering the magnitudes of all terms in the analysis. As a result of this, there have been papers reporting both subluminous and superluminous phase and group velocities, and resonant and nonresonant behavior. Finally, whatever their sign and provenance, the "corrections" to the dispersion relation have all been exceedingly small, a fact which will play an important role in our physical arguments to come.

Our strategy will be to approach the question of Landau damping from a variety of routes. After defining terms and notation in Sec. II, we turn in Secs. III and IV to the simpler problem of TTGW's in a perfect fluid medium, since the fluid analysis will shed light on the kinetic analysis to come. We also consider the simplest possible geometry, a RW background with zero spatial curvature, as this is the closest analog in general relativity to the classical infinite homogeneous analyses carried out in plasma physics. We will comment on the generalization to other geometries later in Sec. IX.

In the derivation of the wave equation, there is an explicit cancellation of some "matter" and "curvature" terms which lead to a wave equation in the fluid identical to that for a TTGW in a vacuum background space with curvature due to distant sources. This equation has previously been derived for the vacuum by Isaacson.[4]

We develop a WKB solution for short-wavelength solutions to this differential equation. While it is tempting to identify the instantaneous time derivative of the phase as the frequency, we will argue in Sec. V that the deviations from lightlike propagation suggested by the time derivative of the phase are not measurable according to the principle of equivalence. Since in the absence of a material medium, one would expect GW's to propagate in a lightlike manner, the fact that TTGW's in an ideal fluid are described by the same equation as TTGW's in a vacuum suggests that TTGW's propagate at the "speed of light." We derive the full differential equation for electromagnetic waves in the same RW geometry; while electromagnetic waves and GW's have different nonlocal equations, we show that they have the same geometric optics limit, as they must according to the principle of equivalence.

In Sec. V we check for the possibility of Landau resonance when the matter propagates waves like a fluid. Weinberg[2] has produced exact solutions to the full differential wave equation in the limits of a cold and relativistically hot RW fluid-filled universe. We consider a freely streaming massive particle which interacts only with the self-consistent gravitational fields. We compute the relative phase between the wave field, given by the exact solution, and the freely streaming particle to show that the relative phase cannot be time stationary. Thus there can be no resonant wave-particle interaction in the Landau sense.

The kinetic response to a TTGW perturbation differs from that of an ideal fluid whose pressure always remains isotropic in the proper frame. The cancellation of terms in the derivation of the fluid wave equation indicates that moments different from pressure are the only possible ones which can affect the propagation of the wave. In Sec. VI, we derive an integrodifferential equation for propagation of small-amplitude nonlocal TTGW's in a flat RW universe, making no approximations other than linearization. While we do not solve it in general, we can derive from it an ordered geometric-optics solution. The "kinetic corrections" to the dispersion relation are again small, and by the arguments of Sec. V are not measurable by a local inertial observer. This leaves open the question of the Landau damping of nonlocal modes with wavelength or order $c\times$(Hubble period) and periods of order the age of the universe. According to the classical plasma analysis of Landau, one must wait several wave periods after the initial perturbation for a nearly monochromatic wave to organize itself and for the conditions for Landau wave-particle resonance to be set up. Thus the question of Landau damping seems to have no precise meaning for nonlocal modes.

Since we have used the RW geometry as the simplest one to illustrate our arguments, the question naturally arises as to how general our conclusions can be. Our calculation of Sec. IX indicates that in the geometric-optics limit, the choice of a mode that can be represented by a purely TT metric perturbation has already built in the requirement that it be lightlike. This argument, which illustrates the claim in Misner, Thorne, and Wheeler[3] (MTW) that only radiative modes can be put into the TT gauge, requires no assumptions about the background geometry or the model of the medium. Thus these results must apply as well to interior Schwarzchild geometry, viscous fluids, and collisional gases. It appears that transverse traceless gravitational modes can have no Landau interactions, and we must turn to vector and scalar modes[9] to search for them.

II. GENERAL APPROACH: CONVENTIONS AND NOTATION

The following examination of the linearized equation for a small perturbation in the metric in the presence of a material background will be done with the choice of the homogeneous, isotropic fluid-filled Robertson-Walker universe with spatial curvature parameter κ set equal to zero. Thus the background metric will be $ds^2=-dt^2+R^2(t)dX^2$ and the values of the density of mass-energy, ρ, and pressure, p, for the background are related to derivatives of the metric coefficient $R^2(t)$. To obtain dispersion relations, we shall assume short wavelengths and use the geometric-optics approximation.

A subscript comma denotes an ordinary partial derivative with respect to x^α, a semicolon denotes the covariant derivative with respect to the full metric $g_{\mu\nu}$, and a vertical bar denotes a covariant derivative using only the background metric. A superscript zero as a prefix denotes unperturbed quantities, and $h_{\mu\nu}$ is the perturbation of the metric.

III. LINEARIZED FIELD EQUATION IN GENERAL RELATIVITY

The Einstein field equation is conveniently written as

$$R_{\mu\nu}=8\pi(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T_\alpha{}^\alpha)\,, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the exact space-time metric tensor, and $T_{\mu\nu}$ is the stress-energy tensor. The field equation itself presumes no form for $T_{\mu\nu}$, but the Bianchi identities imply $T_{\mu\nu}{}^{;\nu}=0$. Since we will deal with both kinetic and fluid pictures for $T_{\mu\nu}$, we derive a general linearized equation for the field first, and specialize to $T_{\mu\nu}$ of a fluid or kinetic nature afterwards.

We ask for the field equation which is a consequence of perturbing the metric of space-time about a pre-existing consistent solution of the field equation. That is, we let

$$T_{\mu\nu}={}^0T_{\mu\nu}+\delta T_{\mu\nu}\,,\quad T\equiv T_\alpha{}^\alpha\,, \tag{2}$$

$$g_{\mu\nu}={}^0g_{\mu\nu}+h_{\mu\nu}\,,\quad h\equiv h_\alpha{}^\alpha\,. \tag{3}$$

The infinitesimal perturbations $\delta T_{\mu\nu}$ and $h_{\mu\nu}$ represent wave quantities; products of infinitesimals are ignored.

It is well known that (3) generates a $\delta R_{\mu\nu}$ given by

$$\delta R_{\mu\nu}=\tfrac{1}{2}(-h_{|\mu\nu}-h_{\mu\nu|\alpha}{}^\alpha+h_{\alpha\mu|\nu}{}^\alpha+h_{\alpha\nu|\mu}{}^\alpha)\,. \tag{4}$$

If we contract (4) with ${}^0g_{\mu\nu}$, change to variables $\bar h_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}{}^0g_{\mu\nu}h$, permute the indices of differentiation, and choose a Lorentz gauge ($\bar h_{\mu\nu}{}^{|\nu}=0$), we obtain from (4)

$$\begin{aligned}-\bar h_{\mu\nu|\alpha}{}^\alpha+{}^0R_{\rho\nu}\bar h^\rho{}_\mu+{}^0R_{\mu\rho}{}^\alpha{}_\nu\bar h_\alpha{}^\rho+{}^0R_{\rho\mu}\bar h^\rho{}_\nu+{}^0R_{\nu\rho}{}^\alpha{}_\mu\bar h_\alpha{}^\rho\\=16\pi\delta(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)\\-\tfrac{1}{2}{}^0g_{\mu\nu}16\pi{}^0g^{\alpha\beta}\delta(T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T)\,.\end{aligned} \tag{5}$$

Noting that $\delta T=\delta(g^{\alpha\beta}T_{\alpha\beta})=-h^{\alpha\beta}T_{\mu\rho}+{}^0g^{\alpha\beta}\delta T_{\alpha\beta}$ and that according to Eq. (1), ${}^0R=-8\pi{}^0T$, we may rewrite Eq. (5) as

$$\begin{aligned}-\bar h_{\mu\nu|\alpha}{}^\alpha+{}^0R_{\rho\nu}\bar h^\rho{}_\mu+{}^0R_{\mu\rho}{}^\alpha{}_\nu\bar h_\alpha{}^\rho+{}^0R_{\rho\mu}\bar h^\rho{}_\nu\\+{}^0R_{\nu\rho}{}^\alpha{}_\mu\bar h_\alpha{}^\rho-{}^0R\bar h_{\mu\nu}+{}^0g_{\mu\nu}\bar h^{\alpha\beta}\,{}^0R_{\alpha\beta}=16\pi\delta T_{\mu\nu}\,.\end{aligned} \tag{6}$$

Equation (6) is our wave equation for the variable $\bar h_{\mu\nu}$, good for an arbitrary background geometry and its consistent matter distribution, in the Lorentz gauge.

IV. GRAVITATIONAL WAVES IN THE FLUID THEORY

Here we consider gravitational radiation in a universe uniformly and isotropically filled with a perfect fluid whose stress tensor is given by

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}\,. \tag{7}$$

Here ρ is the mass-energy density, p the isotropic fluid pressure, and u_μ the fluid's four-velocity. The coefficients for the metric $ds^2=-dt^2+R^2(t)dX^2$ are related to the density and pressure of the fluid.[2] In units where $G=c=1$, the background field equation yields

$$\dot R^2/R^2=\tfrac{8}{3}\pi\rho\,, \tag{8}$$

$$\ddot R/R=-\tfrac{4}{3}\pi(\rho+3p)\,. \tag{9}$$

Thus, with these relations, we may either choose to express the wave equation in terms of $\dot R$ and $\ddot R$, or ρ and p. The important point is that the fluid pressure and density are of the same order of magnitude as the expansion parameters $(\dot R/R)^2$ and $\ddot R/R$.

Evaluating $\delta T_{\mu\nu}$ for the perfect fluid we have

$$\begin{aligned}\delta T_{\mu\nu}=(\rho+p)(\delta u_\mu u_\nu+u_\mu\delta u_\nu)\\+\delta(\rho+p)u_\mu u_\nu+\delta p\,{}^0g_{\mu\nu}+ph_{\mu\nu}\,.\end{aligned} \tag{10}$$

From the unperturbed field equation we find

$$^0R_{\mu\nu}=8\pi\,[(\rho+p)\,u_\mu u_\nu-\tfrac{1}{2}\,{}^0g_{\mu\nu}(\rho-p)]\,,$$
$$^0R=8\pi(\rho-3p)\,.$$

For a mode with $\delta\rho=\delta p=\delta u_j=0$, we find $\delta T_{\mu\nu}=p h_{\mu\nu}$. Furthermore, in RW coordinates $u^\alpha=(u^0,0)$ and Eq. (6) for off-diagonal components reduces to

$$-\bar{h}_{ij|\alpha}{}^{\alpha}+{}^0R_{i\rho}{}^{\alpha}{}_{j}\bar{h}_{\alpha}{}^{\rho}+{}^0R_{j\rho}{}^{\alpha}{}_{i}\bar{h}^{\rho}{}_{\alpha}=0\,. \tag{11}$$

To obtain a similar equation for other components of the metric perturbation requires further gauge constraint to a TT condition. Generally, the imposition of TT constraints is not globally consistent with a Lorentz condition, since the nonvanishing Riemann tensor prevents one from choosing a covariantly constant vector u^α throughout space-time. However, as we show in Appendix A, for RW geometry with zero spatial curvature, one can impose the Lorentz and TT constraints simultaneously and globally. This allows us to arrive at the equation

$$-\bar{h}_{\mu\nu|\alpha}{}^{\alpha}+{}^0R_{\mu\rho}{}^{\alpha}{}_{\nu}\bar{h}_{\alpha}{}^{\rho}+{}^0R_{\nu\rho}{}^{\alpha}{}_{\mu}\bar{h}_{\alpha}{}^{\rho}=0\,. \tag{12}$$

Note that in Eq. (12), all terms in $R_{\mu\nu}$ which represent the local curvature effect of matter have been cancelled by $\delta T_{\mu\nu}$, leaving the equation found for the vacuum by Isaacson.[4] This shows that all deviation from flat-space vacuum propagation in the fluid-filled RW universe comes only from curvature of the background space-time geometry. Further, we learn that if there are to be deviations from vacuum propagation in a kinetic medium, they must come from nonfluid moments of the distribution function.

Expanding the covariant derivatives in Eq. (12) and evaluating the Riemann curvature components gives for space-space components:

$$\left(-\frac{d^2}{dt^2}+\frac{1}{R^2}\frac{d^2}{dx^2}+\frac{\dot{R}}{R}\frac{d}{dt}+2\,\frac{\ddot{R}}{R}\right)\bar{h}_{ij}=0\,, \tag{13}$$

where both i and j are directions perpendicular to the direction of propagation. Weinberg[2] has found exact solutions to (13), for the special cases of cold and radiation-dominated hot universes which are not cast in dispersion-relation form, thus shedding no light on the phase or group velocities.

A. WKB analysis of fluid wave equation

First we transform Eq. (13) to a form amenable to the WKB approximation, by replacing $d^2\bar{h}_{ij}/dx^2$ by $-q^2\bar{h}_{ij}$, following Weinberg in Fourier analyzing in the infinite uniform spatial dimensions. The first-order WKB solution is then

$$\bar{h}_{ij}(t)=\hat{h}_{ij}(q^2/R^2+6\pi p)^{-1/4}\times\exp\left(\pm i\int(q^2/R^2+6\pi p)^{1/2}dt+\tfrac{1}{2}\int(8\pi\rho/3)^{1/2}dt\right), \tag{14}$$

where $\hat{h}_{ij}$ is an appropriate matrix with unit components.

It is tempting to identify the integrand in the phase as the instantaneous wave frequency. However, the WKB assumptions require that $f=(q^2/R^2+6\pi p)$ vary slowly ($\dot{f}/f\ll 1$). The pressure contribution of f is the order of the inverse Hubble time ($\dot{R}/R$), and is small in WKB theory. Thus the (q^2/R^2) term must dominate. It still seems appealing to say we have a pressure "correction" to the frequency, but that it is small. We will see that we must give up this interpretation which seems to tell us that the phase speed is slightly faster than the speed of light (the dispersion relation for which is $\omega^2=q^2/R^2$).

V. PRINCIPLE OF EQUIVALENCE–GEOMETRIC OPTICS

According to the principle of equivalence, physics is locally described by the special theory of relativity. In special relativity, the value c is the speed in a vacuum of propagating null field ($E\cdot B=0=E^2-B^2$) solutions to Maxwell's equations of arbitrary wavelength and frequency. But general relativity introduces scales of space-time curvature so that waves can no longer be considered as objects with a local nature unless their period and/or wavelength are small compared to space-time scales. Indeed, the equivalence principle says that Maxwell's waves travel at the value c only in the limit that the wave 4-vector $k_\alpha\to\infty$ and when measured over local intervals.

For comparison, let us cast Maxwell's equations in a form analogous to Eq. (13) for a GW in a perfect fluid. From Maxwell's equations we may derive a source-free equation[3] for the vector potential A^α in the Lorentz gauge which is analogous to Eq. (12) for a GW in a perfect fluid:

$$-A^{\alpha}{}_{;\beta}{}^{\beta}+R^{\alpha}{}_{\beta}A^{\beta}=0\,. \tag{15}$$

Expanding the covariant derivatives and using the additional gauge condition[10] $A^0=0$, we find that the electromagnetic potential in a RW universe satisfies an equation similar in form to Eq. (13):

$$\ddot{A}^j-\frac{1}{R^2}A^j{}_{,kk}+2(\ddot{R}/R+2\dot{R}^2/R^2)A^j+5(\dot{R}/R)\dot{A}^j=0\,\cdot \tag{16}$$

Equations (16) and (13) present the identical problem of interpretation except that now Eq. (16) describes *light waves*. From this we see that light may have the universal speed c only in the local sense of the equivalence principle, but we need a rigorous ordering scheme to justify dropping those terms which seem to change the propagation speed of the waves despite our physical requirements.

Consider a wave which is as nearly plane, monochromatic, and sinusoidal in space and time as possible. More complicated wave structures may be constructed later by superposition since the equations are linear. In a problem which is infinite and homogeneous, a purely sinusoidal decomposition of the wave is possible, so that the spatial behavior of our wave in RW space-time may be represented as a constant times $e^{iq\cdot x}$. The RW universe is not infinite and homogeneous in time however. We thus must allow our solutions to be modulated-wave trains. Since our solution is not perfectly periodic, there is a limit to how precisely a period can be defined in the wave train. For a wave whose amplitude is a function of time, we can find some average frequency over many "periods," ω_0. Then the wave has the form $\psi(t)e^{-i\omega_0 t}$. Since more than a single frequency is present in this wave, it is not strictly monochromatic, but can be expanded in a Fourier integral of monochromatic waves. On this basis, a well-known argument is Fourier analysis gives an uncertainty relation for modulated wave trains, $\Delta\omega\Delta t \sim 1$. Here $\Delta\omega$ is the uncertainty in frequency, and Δt is the time interval during which the amplitude changes significantly. If we now define a small dimensionless parameter $\epsilon \equiv$ (wave period) $\div$ (space-time scale period) $=(\dot{R}/R)/$(wave frequency), we can conclude that the frequency of the modulated wave is measurable (and definable) to within order ϵ. It certainly is not reasonable to discuss the wave as having a meaningful frequency unless ϵ is small. The size of a frame which can be considered to be Lorentzian is given by distances small compared to

$$|\text{typical value of } R_{\alpha\beta\gamma\delta}|^{-1/2}. \tag{17}$$

It happens that the $(\dot{R}/R)^{-1}$ period is just $|R_{ijkl}|^{-1/2}$. Thus smallness of ϵ implies that many "periods" are contained in a local Lorentz frame of an observer. If many periods fits into a local Lorentz frame, it follows that the period varies slowly compared to wave quantities.

The foregoing physical requirements achieve mathematical representation by discussing solutions with the (asymptotic) form

$$\psi = \sum_{n=0}^{\infty} {}_{(n)}\psi\,\epsilon^n e^{i\varphi/\epsilon}. \tag{18}$$

We define $k_\alpha = \phi_{|\alpha}$. Since the RW universe evolves in time, the Hubble time $\propto(\dot{R}/R)^{-1}$ depends on the age of the universe. Thus in different epochs, the range of wave periods that may be accurately discussed according to the $\Delta\omega\Delta t \sim 1$ criterion increases with the age of the universe. For the purposes of this paper, we pick an epoch by setting approximately $\dot{R}/R$. We then insert the expression (18) into the wave equation, and the Vlasov equation in kinetic theory. Coefficients of like powers of ϵ are then summed to zero. The leading order expression for k_α so found is called the dispersion relation.[12]

To see how this gives the results required by the principle of equivalence, apply Eq. (18) to Eq. (15) with A^α replacing ψ. The $1/\epsilon^2$ terms are leading order and yield $k_\alpha k^\alpha {}_{(0)}A^\mu = 0$. This implies $k_\alpha k^\alpha = 0$ which must be interpreted as showing that light travels on null ray paths. Since $k_\alpha = \phi_{|\alpha}$, we may show these paths are geodesic by computing $(k_\alpha k^\alpha)_{|\sigma} = 0$.

We expand Eq. (12) for gravitational waves in a manner consistent with the above treatment of the electromagnetic wave equation. The results for the first two orders are

$$(1/\epsilon^2):\quad k^\alpha k_\alpha\, {}_{(0)}\bar{h}_{\mu\nu} = 0, \tag{19}$$

$$(1/\epsilon):\quad 2\,{}_{(0)}\bar{h}_{\mu\nu|\alpha}k^\alpha + {}_{(0)}\bar{h}_{\mu\nu}k_\alpha{}^{|\alpha} = 0. \tag{20}$$

Equation (20) gives the area intensity law. For a more elaborate interpretation see Isaacson.[4] Equation (19) for GW must be interpreted just as we previously interpreted the electromagnetic case. For a consistent solution with ϵ ranging over some interval, the coefficients of each order of ϵ must be made to vanish separately. Thus it is not valid to add a higher-order term to $k_\alpha k^\alpha = 0$ and call that the dispersion relation. The ${}_{(0)}\bar{h}_{\mu\nu}$ are referred to as the geometric-optics field. Thus geometric optics sets forward the condition that no terms of the order $\Delta\omega \sim 1/\Delta t$ (or smaller) may consistently appear in the dispersion relation. Indeed, for a wave train whose modulation is governed by the expansion of the universe, Δt is the order of the Hubble time $(\sim\rho^{-1/2})$ and on this scale, the pressure and density ρ and p cannot even be said to be constant; the geometric optics approximation has put into the analysis at the outset, that terms like p and ρ may not appear in an expression for the instantaneous frequency of the gravitational wave in a perfect fluid. Since the matter terms (in the fluid case, these are actually the background curvature terms) are ϵ^2 smaller than the terms from the derivatives of $\bar{h}_{\mu\nu}$ they do not affect the phase velocity of the wave.

VI. DIRECT COMPUTATION OF THE RELATIVE PHASE OF PARTICLE AND WAVE

In addition to mathematical treatments of the Landau damping of plasma oscillations (Landau, 1946), there have been discussions of the physics responsible for Landau damping (Dawson, 1962). These have shown that Landau damping is due to those particles which comove with wave phase surfaces. In linear approximation, these resonant particles exchange energy with the wave in a secular fashion, rather than periodically as do the remaining nonresonant particles. It is therefore interesting to know whether a free-streaming particle can, for a time long compared to a wave period, have stationary phase relative to transverse gravitational waves.

We calculate the position as a function of time of a test particle moving without collisions or drag along a free streaming orbit which is a geodesic of the RW background geometry. We then use exact solutions (Weinberg, 1972) to Eq. (13) for TTGW's in a fluid RW universe to specify the wave-particle phase. Requiring the relative phase to be time-independent gives the criterion for resonance. Our use of fluid theory to specify wave phase is motivated by collisionless plasma kinetic calculations in which nonresonant particles have a fluidlike overall behavior and determine the phase velocity, while the small subset of resonant particles contributes Landau damping. A free-streaming particle obeys the geodesic equation

$$\frac{dp^\mu}{d\tau} = -\frac{1}{m}\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta , \tag{21}$$

where τ is the proper time along the particle's path. For the space components of momentum we have

$$\frac{dp^j}{dt}\frac{dt}{d\tau} = -\frac{1}{m}2\frac{\dot{R}}{R}p^0 p^j . \tag{22}$$

But

$$p^0 = mu^0 = m\,dt/d\tau, \tag{23}$$

so

$$\dot{p}^j = -2\frac{\dot{R}}{R}p^j , \tag{24}$$

which leads to $p^j(t) = p^j(t_1)R^2(t_1)/R^2(t)$, where t_1 is some initial time. We may thus write

$$\frac{dx^j}{dt} = \frac{p^j(t_1)}{p^0}\frac{R^2(t_1)}{R^2(t)} . \tag{25}$$

Upon integration Eq. (25) yields

$$\Delta x^j = p^j(t_1)\int_{t_1}^{t}\frac{1}{p^0}\frac{R^2(t_1)}{R^2(t')}dt' = p^j(t_1)\int_{t_1}^{t}\frac{[R^2(t_1)/R^2(t')]dt'}{t_1[((p^0)^2(t_1)-m^2)R^2(t_1)/R^2(t')+m^2]^{1/2}} . \tag{26}$$

A readily understandable limit is a slow test particle in a hot universe traveling for a short time. In this case

$$\begin{aligned}\Delta x^j &= \frac{p^j(t_1)}{m}\,{}_1\int_{t_1}^{t_2}\frac{dt'}{t'}\\ &= vt_1\ln\left(\frac{t_2}{t_1}\right)\\ &\sim v(t_2-t_1)-\tfrac{1}{2}v\frac{(t_2-t_1)^2}{t_1} .\end{aligned} \tag{27}$$

The particle covers a distance slightly smaller than a Newtonian calculation would indicate.

For $R(t) = R_0 t^n$, Weinberg (1972) presents exact solutions to Eq. (13) assuming $\bar{h}_{ij} = \exp(iq\cdot x)\tilde{h}_{ij}(t)$:

$$\tilde{h}_{ij}(t) = t^{(n+1)/2}J_{\pm\nu}\left(\frac{|q|t}{(1-n)R}\right), \quad \nu = \frac{3n-1}{2-2n} , \tag{28}$$

where $J_{\pm\nu}$ denotes an ordinary Bessel function [we note that a printing error in Weinberg (1972) resulted in the power of t in the secular factor being written as $1/2(n+1)$]. The cases $n=\frac{1}{2}$ and $\frac{2}{3}$ correspond to hot and cold universes, respectively. For brevity, we treat the hot case here; the treatment of the cold universe parallels that for the hot. Setting $n=\frac{1}{2}$, expressing half-integer Bessel functions in terms of trigonometric functions, and superposing solutions to obtain a propagating wave, we obtain

$$\begin{aligned}h_{ij\,(\mathrm{hot})} &\propto \left(\frac{R_0 t}{\pi q}\right)^{1/2}\exp\left[i\left(q\cdot x+\frac{2q}{R_0}t^{1/2}\right)\right]\\ &= \left(\frac{R_0 t}{\pi q}\right)^{1/2}\exp\left[i\left(q\cdot x+q\int_{t_1}^{t}\frac{dt'}{R(t')}\right)\right] .\end{aligned} \tag{29}$$

Equation (29) describes a manifestly nonlocal representation of the wave since the phase advances as $t^{1/2}$. The geometric-optics limit emerges upon neglecting the secular time factor of (29), and defining the local frequency ω as the partial time derivative of the phase, whereupon $\omega = q/R$, the condition for lightlike propagation.

Equation (26) in a hot universe gives

$$\Delta x^j(t)_{\mathrm{hot}} = p^j(t_1)\int_{t_1}^{t}\frac{(t_1/t')dt'}{[((p^0)^2(t_1)-m^2)t_1/t'+m^2]^{1/2}} . \tag{30}$$

Combining Eq. (29) and Eq. (30) for a particle traveling with a wave in the x^3 direction, we obtain for the relative wave-particle phase

$$\phi = q\left[p^3(t_1)\int_{t_1}^{t}\frac{(t_1/t')dt'}{[((p^0)^2(t_1)-m^2)t_1/t'+m^2]^{1/2}} + \frac{1}{R_0}\int_{t_1}^{t}(t')^{-1/2}dt' + \text{const}\right]. \qquad (31a)$$

The condition of stationary phase, $d\phi/dt=0$,

$$\left[p^3(t_1)\frac{t_1/t'}{[((p^0)^2(t_1)-m^2)t_1/t'+m^2]^{1/2}} \pm \frac{(t')^{-1/2}}{R_0}\right]=0 \qquad (31b)$$

can be satisfied for a finite interval of time only if $m=0$. Thus, no massive particle may comove with this wave. The condition $m=0$ leaves $p^3/p_0 = 1/R(t)$, a lightlike condition for propagation of the particle.

With the insight from the fluid analysis, let us turn to the kinetic description.

VII. COLLISIONLESS KINETIC THEORY OF TRANSVERSE TRACELESS GRAVITATIONAL WAVES

The kinetic description replaces the fluid stress tensor Eq. (7) by

$$T_{\mu\nu} = \int p_\mu p_\nu \frac{F}{p^0}\sqrt{-g}\,d^3p\,, \qquad (32)$$

where the distribution function F obeys the Vlasov equation[13]

$$p^\alpha F_{,\alpha} - \Gamma^j_{\alpha\beta}p^\alpha p^\beta \frac{\partial F}{dp^j} = 0\,. \qquad (33)$$

By letting $F = f_0 + \delta f$, we arrive at the linearized Vlasov equation:

$$p^\alpha \delta f_{,\alpha} - \delta\Gamma^j_{\alpha\beta}p^\alpha p^\beta \frac{\partial f_0}{dp^j} - {}^0\Gamma^j_{\alpha\beta}p^\alpha p^\beta \frac{\partial\,\delta f}{dp^j} = 0\,. \qquad (34)$$

Variation of Eq. (32) (see Appendix B) yields

$$\delta T_{\mu\nu} = \int p_\mu p_\nu R^3 \frac{df}{{}^0p^0}d^3p - \int p_\mu p_\nu R^3 \frac{f_0}{2} h \frac{d^3p}{({}^0p^0)^2} - \int p_\mu p_\nu R^3 \frac{f_0}{2} h_{\alpha\beta}p^\alpha p^\beta \frac{d^3p}{({}^0p^0)^3}\,. \qquad (35)$$

Since Ehlers[13] showed that $T_{\mu\nu}$ for the unperturbed RW universe has the same form as in a perfect fluid, we may rewrite the linearized field equation so that its left-hand side contains the same operator as in the fluid and vacuum cases. To do this we must subtract from both sides of Eq. (6) a term which is an integral over momentum space of the unperturbed distribution. A space-space component of the field equation will then read

$$\hat{I}_{12}(t)(h_{12}) = h_{12|\alpha}{}^{\alpha} + 2R_{1212}h^{12} = 16\pi\left(\delta T_{12} - \frac{h_{12}}{R^2}\int (p_1)^2\frac{f_0}{p^0}d^3p\right), \qquad (36)$$

where $\hat{I}_{12}(t)$ is the 1, 2 component of the Isaacson operator in RW space-time. $\hat{I}_{12}$ and δT_{12} are to be computed at the same time t in Eq. (36). Our requirement of TT gauge makes $\delta T_{\mu\nu}$ also subject to the same conditions $\delta T_{\mu 0} = \delta T_\mu{}^\mu = 0$ (see Appendix A). According to Ehlers,[14] the Robertson-Walker parameter R is related to the density moment of the unperturbed distribution function by

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi}{3}\int p^0 f_0\sqrt{-{}^0g}\,d^3p = \frac{8\pi}{3}4\pi\int_m^\infty f_0(p^0)^2[(p^0)^2-m^2]\,dp^0 = \frac{8\pi}{3}\rho\,. \qquad (37)$$

Equation (37), together with the unperturbed equation of motion, $T_{\mu\nu|}{}^\nu = 0$, determines $R(t)$. These equations yield

$$\frac{d}{dR}(\rho R^3) = -3pR^2\,, \qquad (38)$$

where the pressure p equals $(4\pi/3)\int_m^\infty f_0[(p^0)^2 - m^2]^{3/2}dp^0$.

Equation (38) gives an expression for $\rho(R)$ in the cold- and hot-gas limits which allows Eq. (37) to be integrated to yield $R \propto t^{2/3}$ for cold matter and $R \propto t^{1/2}$ for hot matter. Ehlers[14] shows that solutions to the unperturbed Vlasov equation in RW geometry are of the form $f_0(p^0)$. The time dependence of $R(t)$ is not sensitive to the precise form of $f_0(p^0)$, but rather to whether the pressure is negligible or relativistic.

Approximate solutions in the high-frequency limit of the system (34), (35), (36) have been published by Asseo *et al.*[8] and Ignatev.[7] Flat-space solutions substantially similar to curved-space solutions in the WKB limit have been obtained by Chesters[5] and Polnarev.[6]

We will formulate the exact linearized problem which, in principle, also describes low-frequency solutions. We then obtain the WKB limit to leading order. The equations contain a potentially resonant term which we approach by seeing whether the real part of the frequency, when solved self-consistently, leads to a phase speed which permits resonance. This turns out not to be the case.

We rewrite Eq. (34) in Lagrangian form:

$$\left.\frac{D\delta f}{d\tau}\right|_{\substack{\text{unperturbed}\\ \text{orbit}}} = \frac{1}{m}\delta\Gamma^j_{\alpha\beta}p^\alpha p^\beta\frac{\partial f_0}{\partial p^j}\,.$$

Using $dt/d\tau = p^0/m$, we may integrate along unperturbed orbits to give

$$\delta f(t,p^j)=\int_{t_1}^{t} dt'\,\frac{1}{p^0(t')}\delta\Gamma'^j_{\alpha\beta}(t')p'^\alpha(t')p'^\beta(t')\,\frac{\partial f_0(t',p^j)}{\partial p^j}+\delta f(t')_{\rm fs}\,, \tag{39}$$

where we choose t_1 to be late enough in cosmological history for a collisionless assumption to be reasonable and the geometry isotropic at zeroth order. The term $\delta f(t)_{\rm fs}$ (where the subscript fs means free-stream) is an arbitrary function whose time derivative along particle orbits is zero. For a TT wave characterized by h_{12}, Eq. (39) becomes

$$\delta f(t,p^j)=\int_{t_1}^{t} dt'\,\frac{1}{[p^0(t')]^2}\,\frac{\partial f_0}{\partial p^0}\left[2p^0p^1p^2\,\frac{d}{dt'}\left(\frac{h_{12}}{R^2}\right)R^2+p^1p^2p^3h_{12,3}\right]+\delta f_{\rm fs}(t)\,. \tag{40}$$

For free-streaming particles, p^ν may be obtained using the argument leading to Eq. (25):

$$p^j(t')=p^j(t_1)\frac{R^2(t_1)}{R^2(t')},\quad p^0(t')=\left(\frac{R^2(t_1)}{R^2(t')}\{[p^0(t_1)]^2-m^2\}+m^2\right)^{1/2}. \tag{41}$$

Substituting the expressions (41) into (40) gives

$$\delta f(t,p^j)=p^1(t)p^2(t)R^4(t)\int_{t_1}^{t} dt'2\,\frac{\partial f_0}{\partial p^0}\,\frac{1}{p^0}\,\frac{1}{R^2}\,\frac{d}{dt}\left(\frac{h_{12}}{R^2}\right)$$
$$+p^1(t)p^2(t)p^3(t)R^6(t)\int_{t_1}^{t} dt'\,\frac{\partial f_0}{\partial p_0}\,\frac{1}{(p^0)^2}\,\frac{h_{12,3}}{R^6}+\delta f_{\rm fs}(t)\,.$$

Inserting this into Eq. (35), we obtain

$$\delta T_{12}=R^7(t)R^4(t)\int d^3p\,\frac{[p^1(t)]^2[p^2(t)]^2}{p^0(t)}\int_{t_1}^{t} dt'2\,\frac{\partial f_0}{\partial p^0}\,\frac{1}{p^0(t')}\,\frac{1}{R^2(t')}\,\frac{d}{dt'}\left(\frac{h_{12}}{R^2}\right)$$
$$+R^6(t)R^7(t)\int d^3p\,\frac{[p^1(t)]^2[p^2(t)]^2p^3(t)}{p^0(t)}\int_{t_1}^{t} dt'\,\frac{\partial f_0}{\partial p^0}\,\frac{1}{[p^0(t)]^2}\,\frac{h_{12,3}}{R^6(t')}$$
$$+R^7(t)\int d^3p\,\frac{p^1(t)p^2(t)}{p^0(t)}\delta f_{\rm fs}(t)-R^7(t)h_{12}\int[p^1(t)]^2[p^2(t)]^2\frac{f_0}{(p^0)^3}\,d^3p$$
$$=I_1+I_2+I_3+I_4\,. \tag{42a}$$

We note that the second term in Eq. (35) vanishes for TT waves. Substituting Eq. (42a) into Eq. (35) leads to an integrodifferential equation for TT waves in collisionless kinetic theory which we abbreviate

$$\hat{I}_{12}(t)h_{12}=16\pi\left(I_1+I_2+I_3+I_4-\frac{h_{12}}{R^2}\int p_1{}^2f_0\,\frac{d^3p}{p^0}\right). \tag{42b}$$

Although we will not present a general solution of (42b) we will consider its short-wavelength "WKB" limit, for we know from our analysis in Sec. V of the fluid case that the equation $I_{12}h_{12}=0$ gives waves which propagate on null geodesics. That experience motivates us to ask whether the terms on the right-hand side of Eq. (42b) can be of sufficiently high order in the expansion parameter ϵ to modify the propagation of the waves. Thus, we must evaluate the *size* of the terms in the right-hand side of (42b). To do so, we must choose a particular unperturbed distribution function, which, for illustrative purposes, we take to be the relativistic Boltzmann distribution.

$$f_0=\frac{\bar{n}\exp[-p^0/(kT)]}{4\pi m^2kTK_2(m/kT)}\,, \tag{43}$$

where kT is the particle thermal energy, m the rest mass, and $\bar{n}$ the average proper number density, and $K_2(m/kT)$ is a modified Bessel's function. We will then take the appropriate moments of f_0 to construct δT_{12}. We will evaluate these in the limits of small and relativistic gas pressures. Since moments appear, we do not expect the size of the integrals to depend sensitively upon details of the unperturbed distribution functions, and so the relativistic Boltzmann distribution is a choice which leads to representative answers.

We will look for the leading order contributions to the right-hand side of (42b); thus we ignore the variation of background quantities and substitute $h_{12}=\hat{h}_{12}e^{ik_\alpha x^\alpha}$ into the integrands, where $\hat{h}_{12}$ is a constant. Let us first consider the sum I_1+I_2. In evaluating the integrals, we use the fact that owing to the invariance of the scalar product, k_0p^0 in the particle frame equals $k_\alpha p^\alpha$ in any Lorentz frame. I_1+I_2 thus reduces to

$$I_1+I_2=R^7\int \frac{(p^1p^2)^2}{p^0}h_{12}\frac{\partial f_0}{\partial p^0}\left(1+\frac{k_0p^0}{k_\alpha p^\alpha}\right)d^3p$$
$$=I_{1A}+I_{1B}\,. \qquad (43)$$

The integral I_{1A} is nonresonant and I_{1B} is potentially resonant. I_{1A} and I_{1B} are calculated in Ref. 11, and their real (principal) parts are displayed in Table I together with those of I_4.

Reference 11 shows that the resonant denominator of Eq. (43) may be written as

$$\left(x-\frac{p^0\omega R/k}{[(p^0)^2-m^2]^{1/2}}\right).$$

Since the integration is over the range $-1\leqslant x\leqslant 1$, this term may vanish only if $p^0\gg m$ and $\omega R/k\leqslant 1$. Since the integrand is not singular when $\omega R/k=1$, wave-particle resonance is strictly possible only if $\omega R/k<1$, which corresponds to the expected condition that the phase speed be less than that of light, which obeys $\omega R/k=1$ in the geometrical-optics limit.

To decide whether or not $\omega R/k$ can be less than 1, we must check the magnitude of the terms appearing in Table I, all of which are proportional to the pressure

$$p=(4\pi R^3/3)\int |p^ip^i|(f_0/p_0)d^3p=\bar{n}kT\,.$$

The pressure is strictly less than the mass energy of the gas

$$\rho=4\pi R^3\int p^0f_0d^3p\ ,$$

which in turn is of order $(\dot{R}/R)^2\sim\epsilon^2$ according to the unperturbed field equation (37). Only entries I_{1B} in the hot approximation and I_4 in the cold approximation require further comment. At first sight, I_{1B} could be large if $\omega R/k$ were large; however, this does not lead to a consistent solution of the WKB dispersion relation. Thus $\omega R/k$ must be ~1 and $\omega R/k$ must therefore *be* unity according to the dispersion relation. Since I_4 for a cold gas is proportional to $(kT)^2$, it is even smaller than the I_{1A} and I_{1B} in the limit $T\to 0$.

We now turn to the integral of the free-steaming term I_3. As it appeared in the solution by integration over unperturbed orbits, δf_{fs} was an arbitrary function whose time derivative along the orbits vanished. But $|\delta T_{12}|$ must be small compared to $|{}^0T_{\mu\nu}|$ for consistency with our choice of background and perturbed field equations. The perturbation equation is essentially the linear term in an expansion in terms of the small parameters $|h_{12}/g_{\mu\nu}|$ and $|\delta f/f_0|$, where we assume that these two parameters are comparable. We then perform a high-frequency expansion of the perturbation equation. The task at hand is to compare the order of magnitudes $\hat{I}_{12}$ and δT_{12} in this latter expansion of the field equation. The magnitude of δf_{fs} is not completely arbitrary; rather it must be consistent with the assumption of the first expansion.

We note that for a wide class of δf_{fs}, the integral I_3 vanishes. For δf_{fs} a function of initial position only, or any even function of p^1p^2, I_3 is trivially zero. This suggests that for δf_{fs} in this class, no TT wave will result from the initial perturbation if an external wave source is also excluded.

We must ask about the class of δf_{fs} for which I_3 is not trivially zero. For perturbations which have a spatial dependence of the form e^{iqx}, the initial perturbation will be carried forward in time by the freely streaming particles, so that δf_{fs} will have the spatial dependence $\exp(|iq\int_{t_1}^{t}v\,dt'|)$, where v is the free-streaming velocity. This term may be multiplied by any consistently normalized function of initial momenta. As t increases, δf_{fs} becomes highly oscillatory in momentum space. The decay of I_3 in time that results is known in plasma physics as phase mixing. Such arguments are characteristic of the Landau problem. Landau damping only emerges in the limit of long-elapsed time after the initial perturbation. In the intervening period, the initial free-streaming "transients" die down by phase mixing, and the collective response of the gas sorts itself out into a single

TABLE I. Kinetic integrals in hot and cold limits for solutions of the Vlasov equation in the geometric-optics limit.

	Hot ($m/kT\ll 1$)	Cold ($m/kT\gg 1$)
I_{1A}	$\bar{n}kT_{(0)}h_{12}(-\frac{8}{5})$	$\bar{n}kT_{(0)}h_{12}\left(-2\left(\frac{2}{\pi}\right)^{1/2}\right)$
I_{1B}	$+\bar{n}kT_{(0)}\left\{\frac{3}{4}\left(\frac{\omega R}{k}\right)2\left(\frac{\omega R}{k}\right)^4\left(\frac{\omega^2R^2}{k^2}-\frac{5}{3}\right)+\left(1-\frac{\omega^2R^2}{k^2}\right)^2\ln\left\lvert\frac{1-\omega R/k}{1+\omega R/k}\right\rvert\right\}$	$+\bar{n}kT_{(0)}h_{12}\left(-2\left(\frac{2}{\pi}\right)^{1/2}\right)$
I_4	$\bar{n}kT_{(0)}h_{12}(-\frac{2}{5})$	$\bar{n}kT_{(0)}h_{12}\left(-2\left(\frac{2}{\pi}\right)^{1/2}\frac{kT}{m}\right)$

propagating wave. Thus the problem of Landau damping of GW's is well posed only in the WKB limit since many wave periods are required to span the time between the initial time and the time when the damping is to be measured.

We would know how to handle I_3 if we knew how large the term could be at the initial time. At a late time compared to the initial time, I_3 would then be even smaller. By the requirement that δf be of similar order of magnitude as h_{12}, the largest δf could be is something roughly of order $h_{12} f_0$. Thus, for a reasonably smooth initial perturbation which satisfies TT constraints, the moment I_3 will have at most the same general magnitude as I_1, I_2, and I_4.

If the relativistic Boltzmann distribution yields reasonable estimates of the integrals I_1, I_2, I_3, I_4, the kinetic medium makes contributions to the propagation equation (42b) no larger than the background terms for the fluid medium, despite the different responses of the two media to TTGW perturbations. The fluid response must always be isotropic, whereas the kinetic medium can have a "tensor" response. In the special case of zero temperature, the two models are equivalent. For finite temperature the sign and magnitude of the $\delta T_{\mu\nu}$ can differ in the two models; however, in order of magnitude, the matter influence in either model is of relative order ϵ^2 and being this small, must be dropped from the dispersion relation. We thus conclude that TTGW travel with the speed of light in either medium and that TTGW can have no Landau damping.

VIII. COMPARISON WITH PREVIOUS WORK

Let us compare our argument and results with those of previous authors on the subject. The authors cited who dealt with curvature at all approached the problem by noting that ${}^0\Gamma^0_{jk} = \dot{R}R\delta_{jk}$ and ${}^0\Gamma^j_{0k} = (\dot{R}/R)\,\delta^j_k$, and all other connections vanish.

From this observation, it is argued that the last term of Eq. (34) is to be ignored. This, however, has been shown[11] to assume away the resonance being investigated because this term contains a resonant denominator. Although this assumption is formally inconsistent, with the choice of sinusoidal perturbations it leads to the same answer for the part of $\delta T_{\mu\nu}$ due to δf as we have obtained at leading order. This result occurs because both arguments employ $D\delta f/d\tau = \delta\Gamma^i_{\alpha\beta}(p^\alpha p^\beta/m)\partial f_0/\partial p^j$. The full calculation shows that the dropped term, in the limit of high frequency, is truly small so that the agreement is explained. We note also that our calculation uses a different expression, Eq. (35), for $\delta T_{\mu\nu}$ than any previous paper. Although

TABLE II. Some dispersion relations found for GW's with a kinetic matter model.

Author(s)	Dispersive term
Chesters (1973)	$-28.8\pi p$
Asseo *et al.* (1976)	$64\pi p$
Polnarev (1972)	$8\pi p$
Ignatev (1974)	$\frac{16}{3}\pi(\rho + 3p)$

each of our terms has appeared separately in previously published calculations, this is the only case where all the terms were included. None of these details is responsible for our general result, however, because their contributions are too small.

At this point, let us compare the results of the current work with that of previous authors. Previous work has consistently led to dispersion relations of the form $\omega^2 = k^2 +$ (dispersive term). Expressing the dispersive term in terms of the pressure p we summarize the results in Table II. The present paper has also found a term $\simeq 51\pi p$. It is our contention, however, that terms of this order of magnitude do not modify the measurable propagation speed of the waves.

IX. TT WAVES NECESSARILY HAVE PHASE SPEED EQUAL TO *c*

The fact that in the WKB limit we obtained resonance for neither fluid nor kinetic model can be shown to result from the transverse traceless property of the waves. Our argument follows these lines. If an observer sees a TT wave, then we can show (Appendix C) that the observer will still see a TT wave after a Lorentz boost entirely in the direction of the wave. This assures us that potentially resonant particles moving relative to the frame of the medium still see a TT wave, although the TT property was only established in the proper frame of the medium. We then must show that TT and time-independent $h_{\mu\nu}$ are incompatible conditions for GW in the WKB limit. Since a comoving observer would see the wave as time independent, this would show that no particles subject to Lorentz transformations (i.e., massive particles) may comove with the wave.

Let us look at the advancing phase of the wave in the local Lorentz frame of a potentially resonant test particle. If the phase advances slower than the speed of light, there is a frame in which the wave can be made time independent. Similarly if the phase advances faster than the speed of light, there is a frame in which the wave can be made spatially independent. We shall show that the frames which make the wave temporally or spatially independent move on the light cone. We shall use the terms of $R_{\alpha\beta\gamma\delta} - {}^0R_{\alpha\beta\gamma\delta}$ that are linear in $h_{\mu\nu}$ denoting them by

$$\delta R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(h_{\alpha\delta\,|\,\beta\gamma} + h_{\beta\gamma\,|\,\alpha\delta} - h_{\beta\delta\,|\,\alpha\gamma} - h_{\alpha\gamma\,|\,\beta\delta} + {}^0R_{\alpha\rho\gamma\delta}\,h^\rho{}_\beta + {}^0R_{\beta\rho\gamma\delta}\,h^\rho{}_\alpha)\,.$$

Isaacson[4] showed that in the geometric-optics limit, the corrections in a gauge transformation to $\delta R_{\alpha\beta\gamma\delta}$ are ϵ^2 smaller than $h_{\alpha\beta,\gamma\delta}$. Thus to a high degree of accuracy we may say that $\delta R_{\alpha\beta\gamma\delta}$ is gauge invariant in the WKB limit. For a mode characterized by the amplitude h_{12}^{TT} propagating in the $\hat{x}^3$ direction we have, using Eq. (18) with h_{12}^{TT} replacing ψ, the nonvanishing leading order (in ϵ) terms of $\delta R_{\alpha\beta\gamma\delta}$

$$\delta R_{1020} = -\tfrac{1}{2}\,{}_{(0)}h_{12,00}^{\mathrm{TT}}\,,$$
$$\delta R_{1323} = -\tfrac{1}{2}\,{}_{(0)}h_{12,33}^{\mathrm{TT}}\,,$$
$$\delta R_{1023} = -\tfrac{1}{2}\,{}_{(0)}h_{12,03}^{\mathrm{TT}} = \delta R_{1203}\,.$$

For a local Lorentz observer $\delta R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta}$ is an invariant since the perturbation of the curvature is a tensor in the background geometry. We then have

$$\delta R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta} = (-\tfrac{1}{2}h_{12,00}^{\mathrm{TT}})^2 + (-\tfrac{1}{2}h_{12,33}^{\mathrm{TT}})^2 - 2(\tfrac{1}{2}h_{12,03}^{\mathrm{TT}})^2 = C\,, \qquad (49)$$

where we suppress the prefix (0) on h_{12}^{TT}. We now assume all $h_{\mu\nu}$ components to be those of TT gauge. We further use "p" to indicate the proper frame of the medium, and "m" to indicate the moving frame attached to the test particle. From Eq. (49) it follows that if there exists a frame in which $h_{12,0}^m = 0$ then

$$\tfrac{1}{4}(h_{12,00}^p)^2 + \tfrac{1}{4}(h_{12,33}^p)^2 - \tfrac{1}{2}(h_{12,03}^p)^2 = \tfrac{1}{4}(h_{12,33}^m)^2\,. \qquad (50)$$

The Lorentz transformation that makes $h_{12,0}^m$ vanish is determined by

$$\delta R_{\hat{1}\hat{0}\hat{2}\hat{0}}^m = 0 = \Lambda^\alpha{}_{\hat{1}}\Lambda^\beta{}_{\hat{0}}\Lambda^\gamma{}_{\hat{2}}\Lambda^\delta{}_{\hat{0}}\,\delta R_{\alpha\beta\gamma\delta} = \cosh^2\alpha\,\delta R_{1020}^p + 2\cosh\alpha\sinh\alpha\,\delta R_{1320}^p + \sinh^2\alpha\,\delta R_{1323}^p\,. \qquad (51)$$

Substitution of Eq. (50) into Eq. (51) yields $\tanh\alpha = -k_0/k_3$. This is the expected result that to see the wave as time independent one must move at the phase velocity $-k_0/k_3$. In addition to the condition $\tanh\alpha = -k_0/k_3$, $\tanh\alpha = -k_3/k_0$ will also make $h_{12,03}^m = 0$. The general expression for $h_{12,33}^m$ in terms of values in the proper frame is

$$-\tfrac{1}{2}h_{12,33}^m = \delta R_{\hat{1}\hat{3}\hat{2}\hat{3}}^m = \Lambda^\alpha{}_{\hat{1}}\Lambda^\beta{}_{\hat{3}}\Lambda^\gamma{}_{\hat{2}}\Lambda^\delta{}_{\hat{3}}\,\delta R_{\alpha\beta\gamma\delta} = \sinh^2\alpha\,\delta R_{1020}^p + 2\cosh\alpha\sinh\alpha\,\delta R_{1320}^p + \cosh^2\alpha\,\delta R_{1323}^p\,.$$

Equation (50) then becomes

$$k_0^4 + k_3^4 - 2k_0^2k_3^2 = \frac{1}{\cosh^2\alpha\sinh^2\alpha}\left(\tanh\alpha\,k_0^2 + 2k_0k_3 + \frac{k_3^2}{\tanh\alpha}\right). \qquad (52)$$

Using $\tanh\alpha = -k_0/k_3$, manipulation of Eq. (52) shows that no solutions for waves exist unless $k_0^2 = k_3^2$. Thus the only moving frame which gives $h_{12,0}^m = 0$ must move at the speed of light in the proper frame of the medium. Further, the wave itself must have $k_0^2 = k_3^2$ in all Lorentz frames. If we pose the analog of Eq. (49) for the condition that a frame exists in which $h_{12,3}^m = 0$, we obtain the same conditions because of the symmetry of Eq. (49). Thus, TTGW's travel on the light cone in the geometric-optics limit.

X. SUMMARY AND DISCUSSION

Our re-examination of linear gravitational waves in ideal fluid and collisionless kinetic media has allowed us to learn several lessons. First, the importance of using a self-consistent background geometry corresponding to the unperturbed matter distribution is subtle. By virtue of the unperturbed field equation, the terms from the background curvature are the same order of magnitude as the matter influence. In the ideal-fluid case, there is a complete cancellation of the matter terms by some curvature terms, leading to a wave propagation equation equivalent to that for vacuum gravitational waves. We argued by explicit comparison with the propagation equation of electromagnetic waves, that the terms of order ϵ^2 smaller than the leading terms [where $\epsilon = (\dot{R}/R)/\omega_0$ in RW geometry and ω_0 is the wave frequency] cannot be consistently included in the dispersion relation according to the equivalence principle. As is well known, any local observer can only measure a phase and group speed equal to that of light in flat space for electromagnetic waves in curved space-time, when the wavelength is small compared to the radius of curvature. In the case of kinetic media, new terms not given by the ideal fluid do appear in the formal propagation equation. However, for frequencies which exceed the Hubble frequency, the new kinetic terms are also of order ϵ^2. Since the dispersion relation ignores terms of order $(\Delta\omega/\omega)^2$ which are also $O(\epsilon^2)$, to be consistent, the new kinetic contributions cannot modify the vacuum dispersion relation.

For the RW case, these results can be further illuminated by a form of an "uncertainty principle." The formal deviations from lightlike propagation implied by our propagation equation, or for that matter by any of the results in the literature heretofore, yield "corrections" of order $(\dot{R}/R)^2/\omega_0^2$ to the basic dispersion relation. To measure the frequency with sufficient precision to resolve the pressure corrections, an observer would have to accumulate his measurements for a time comparable with the Hubble time. During such an integration time, the pressure will have varied and so is not unambiguously defined. In this sense, the pressure is not comeasurable with the frequency to $O(\epsilon)$. They should not appear in the same dispersion relation.

Our conclusion is that TTGW's do not Landau damp, albeit for a reason different from those other authors who have reached this negative conclusion. Since the phase velocity equals that of light, particles with nonzero rest mass cannot comove with the wave. We have generalized this argument from RW geometry to any in which TT polarization and geometrical optics apply. For logical completeness, it was necessary to establish the validity of the global gauge constraint for the particle-phase calculation of Sec. V using the nonlocal equation for the fluid case. Our general argument, however, is local in nature and does not depend on globally applied gauge constraints.

The above arguments leave open the question of long-period modes for which ϵ is not small. In the standard Landau problem, several wave periods must elapse before initial transients phase mix away and before a nearly monochromatic wave emerges. Thus the admissible range of frequencies for which the concept of Landau damping is most clearly defined is limited to those exceeding several times the Hubble frequency, which are then amenable to the above geometrical-optics arguments. A geometrical reason to consider the Landau interaction only in the short-wavelength limit is that, according to the $\Delta\omega\Delta t \sim 1$ limitation, long-wave nonlocal modes may have a single "phase surface" fill an entire local Lorentz frame of a test particle. Thus the particle will not have a unique interaction with the wave since there is no invariant way to separate wave effects from background effects when the wave amplitude is infinitesimal and has the same space-time scale as the background.

One further comment may be relevant to future searches for instabilities. Considerable care must be devoted to the questions of self-consistency between matter and geometry terms. For example, if one seeks instabilities due to an anisotropic distribution, one should also choose an anisotropic-background metric consistent with it. Moreover, one must be careful to ensure that any matter dispersion terms are sufficiently large to affect the propagation in a measurable way.

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APPENDIX A: JUSTIFICATION OF SIMULTANEOUS GLOBAL LORENTZ AND TT GAUGE CONSTRAINTS

For a general infinitesimal coordinate transformation of the form

$$x'^{\mu}=x^{\mu}+\xi^{\mu},$$

the perturbations of the metric are transformed as

$$h'_{\mu\nu}=h_{\mu\nu}-\xi_{\nu|\mu}-\xi_{\mu|\nu},$$

where ξ_μ is the generator of the transformation and has small covariant derivative. This equation for barred quantities (those of our wave equation) is

$$\bar{h}'_{\mu\nu}=\bar{h}_{\mu\nu}-\xi_{\nu|\mu}-\xi_{\mu|\nu}+{}^0g_{\mu\nu}\xi_{\alpha}{}^{|\alpha} \tag{A1}$$

where ${}^0g_{\mu\nu}$ is the background metric. We put all variation into $h_{\mu\nu}$ so ${}^0g_{\mu\nu}$ is the same in both coordinates.

First we assume a solution in an arbitrary gauge (no constraints yet applied). We now impose the Lorentz condition. If $\bar{h}_{\mu\nu}{}^{|\nu}\neq 0$, find $\bar{h}'_{\mu\nu}$ such that $\bar{h}'_{\mu\nu}{}^{|\nu}=0$. We must require

$$\bar{h}'_{\mu\nu}{}^{|\nu}=0=\bar{h}_{\mu\nu}{}^{|\nu}-\xi_{\nu|\mu}{}^{\nu}-\xi_{\mu|\nu}{}^{\nu}+g_{\mu\nu}\xi_{\alpha}{}^{|\alpha\nu}. \tag{A2}$$

The rule for commutation of covariant derivatives yields

$$\xi_{\nu|\alpha}{}^{\nu}=\xi_{\nu}{}^{|\nu}{}_{\alpha}+{}^0R_{\alpha}{}^{\nu}\xi_{\nu}.$$

Thus Eq. (A2) becomes

$$\bar{h}'_{\mu\nu}{}^{|\nu}=0$$
$$=\bar{h}_{\mu\nu}{}^{|\nu}-(\xi_{\nu}{}^{|\nu}{}_{\mu}+{}^0R_{\mu}{}^{\nu}\xi_{\nu})-\xi_{\mu|\nu}{}^{\nu}+\xi_{\nu}{}^{|\nu}{}_{\mu}$$

or

$$0=\bar{h}_{\mu\nu}{}^{|\nu}-\xi_{\mu|\nu}{}^{\nu}-R_{\mu}{}^{\nu}\xi_{\nu}. \tag{A3}$$

Equation (A3) is a differential equation for ξ_μ, so that the resulting $\bar{h}'_{\mu\nu}$ satisfies $\bar{h}'_{\mu\nu}{}^{|\nu}=0$.

Now assume we have already imposed Lorentz gauge constraints, so that $\bar{h}'_{\mu\nu}{}^{|\nu}=0$. Then Eq. (A3) becomes

$$0=\xi_{\mu|\nu}{}^{\nu}+R_{\mu}{}^{\nu}\xi_{\nu}. \tag{A4}$$

Equation (A4) gives a condition that further gauge changes preserve the Lorentz gauge.

First we note that contracting Eq. (A1) yields a condition that $\bar{h}'^{\mu}_{\mu}=0$, which is

$$\xi_{\alpha}{}^{|\alpha}=\tfrac{1}{2}\bar{h}_{\mu}{}^{\mu}.$$

We may establish the consistency of $\bar{h}_{\mu}{}^{\mu}=0$ with the Lorentz gauge and the wave equation (this establishes globality) by contracting the Lorentz gauge wave equation.

$$-\bar{h}_{\mu}{}^{\mu}{}_{|\alpha}{}^{\alpha}-2\,{}^{0}R_{\rho}{}^{\alpha}\bar{h}_{\alpha}{}^{\rho}+{}^{0}R_{\rho\nu}\bar{h}^{\nu\rho}+{}^{0}R_{\rho\nu}\bar{h}^{\rho\nu}-\bar{h}_{\mu}{}^{\mu}\,{}^{0}R+4\bar{h}^{\alpha\beta}\,{}^{0}R_{\alpha\beta}=16\pi\delta T_{\mu}{}^{\mu}.$$

For a wave with no fluid quantities perturbed $\delta T_{\mu}{}^{\mu}$ is proportional to $\bar{h}_{\mu}{}^{\mu}$. Then

$$-\bar{h}_{\mu}{}^{\mu}{}_{|\alpha}{}^{\alpha}-\bar{h}_{\mu}{}^{\mu}({}^{0}R+16\pi p)+4\bar{h}^{\alpha\beta}\,{}^{0}R_{\alpha\beta}=0. \qquad (A5)$$

For RW geometry, ${}^{0}R_{\alpha\beta}$ is diagonal and $R_{11}=R_{22}=R_{33}$. Then

$$-\bar{h}_{\mu}{}^{\mu}{}_{|\alpha}{}^{\alpha}-\bar{h}_{\mu}{}^{\mu}({}^{0}R+16\pi p)+4\bar{h}^{00}R_{00}+4\bar{h}^{j}{}_{j}R_{11}=0.$$

Thus the traceless condition can be enforced provided $\bar{h}^{00}=0$. We would like to enforce $\bar{h}_{\mu 0}=0$, so this condition is no problem, if $\bar{h}_{\mu 0}=0$ can also be required with no further assumptions than $\bar{h}_{\mu}{}^{\mu}=0$.

Turn attention now to the gauge condition that

$$\bar{h}'_{\mu 0}=0=\bar{h}_{\mu 0}-\xi_{0|\mu}-\xi_{\mu|0}+{}^{0}g_{\mu 0}\,\xi_{\alpha}{}^{|\alpha}. \qquad (A6)$$

The divergence of the left-hand side of (A6) is

$$\bar{h}'_{\nu 0}{}^{,\nu}+3\frac{\dot{R}}{R}\bar{h}'^{0}_{0}-\frac{\dot{R}}{R}\bar{h}'^{j}_{j}. \qquad (A7)$$

By condition (A6) we then have that $\bar{h}'_{0\nu}{}^{|\nu}=\bar{h}'^{j}_{j}(-R/R)$. Thus we may take $\bar{h}'_{0\nu}{}^{|\nu}=0$ simultaneously with $\bar{h}'_{0\mu}=0$ only if $\bar{h}'^{j}_{j}=0$. Taking this latter condition the divergence of (A6) then gives

$$\bar{h}'_{\mu 0}{}^{|\mu}=\bar{h}_{\mu 0}{}^{|\mu}-\xi_{0|\mu}{}^{\mu}-\xi_{\mu|0}{}^{\mu}+{}^{0}g_{\mu 0}\,\xi_{\alpha}{}^{|\alpha\mu}=0.$$

Then since $\bar{h}_{\mu 0}{}^{|\mu}=0$ from the Lorentz condition, we have

$$\xi_{\alpha}{}^{|\alpha}{}_{0}=\xi_{0|\mu}{}^{\mu}+\xi_{\mu|0}{}^{\mu}$$

or

$$\xi_{0|\mu}{}^{\mu}=-{}^{0}R_{0}{}^{\mu}\xi_{\mu},$$

which is in agreement with (A4), showing that the conditions $\bar{h}'_{0\mu}=0$, $\bar{h}'_{0\mu}{}^{|\mu}=0$, $\bar{h}'^{j}_{j}=0$ may be consistent. Final proof of the consistency of these equations with the wave equation in a Robertson-Walker background is obtained by examination of the Lorentz gauge version of the perturbed field equation. Focus on the μ arbitrary, $\nu=0$ equations:

$$-\bar{h}_{\mu 0|\alpha}{}^{\alpha}+2\,{}^{0}R_{0\rho}{}^{\alpha}{}_{\mu}\bar{h}_{\alpha}{}^{\rho}+{}^{0}R_{\rho 0}\bar{h}_{\mu}{}^{\rho}+{}^{0}R_{\rho\mu}\bar{h}^{\rho}{}_{0}-\bar{h}_{\mu 0}\,{}^{0}R+{}^{0}g_{\mu 0}\bar{h}^{\alpha\beta}\,{}^{0}R_{\alpha\rho}=16\pi\delta T_{\mu 0}. \qquad (A8)$$

The condition $\bar{h}_{\mu 0}=0$ then implies

$$2\,{}^{0}R_{0j}{}^{0}{}_{\mu}\bar{h}_{0}{}^{j}+2\,{}^{0}R_{0l}{}^{j}{}_{\mu}\bar{h}_{j}{}^{l}+{}^{0}R_{00}\bar{h}_{\mu}{}^{0}+{}^{0}R_{j0}\bar{h}_{\mu}{}^{j}+{}^{0}R_{0\mu}\bar{h}^{0}{}_{0}+{}^{0}R_{j\mu}\bar{h}^{j}{}_{0}+{}^{0}g_{\mu 0}(\bar{h}^{j0}\,{}^{0}R_{j0}+\bar{h}^{jl}\,{}^{0}R_{jl})=16\pi\delta T_{\mu 0}. \qquad (A9)$$

This condition cannot in general be satisfied because it ties $h_{\mu\nu}$ to the background curvature. For diagonal metrics, we observe that (A9) becomes

$$2R_{0l}{}^{j}{}_{\mu}\bar{h}_{j}{}^{l}+R_{j0}\bar{h}_{\mu}{}^{j}+{}^{0}g_{\mu 0}\bar{h}^{jl}R_{jl}=16\pi\delta T_{\mu 0}. \qquad (A10)$$

For RW geometry with zero spatial curvature, the first term in Eq. (A10) is proportional to $\delta_{\mu 0}\bar{h}_{j}{}^{j}$, the second vanishes, the third is proportional to $\delta_{\mu 0}\bar{h}_{j}{}^{j}$, and the right-hand side is arbitrary. Thus for a traceless mode with $\bar{h}_{\mu 0}=0$ we have with this special choice of background

$$0=16\pi\delta T_{\mu 0}. \qquad (A11)$$

For $\delta T_{\mu 0}\propto\bar{h}_{\mu 0}$, as in the case where no fluid perturbations exist, this condition is consistent. Thus, the desired gauge constraints may be applied both consistently and globally for the fluid mode discussed in this paper. For TT polarizations in kinetic theory we need conditions such that $\delta T_{\mu 0}$ is also zero.

APPENDIX B: THE PERTURBED STRESS TENSOR

The definition of $T_{\mu\nu}$ in kinetic theory is[12]

$$T_{\mu\nu}=\int p_{\mu}p_{\nu}\frac{F}{p^{0}}\sqrt{-g}\,d^{3}p,\quad g\equiv\det\|g_{\mu\nu}\| \qquad (B1)$$

so that with $F=f_{0}+\delta f$

$$\delta T_{\mu\nu}=\delta\int p_{\mu}p_{\nu}\frac{F}{p^{0}}\sqrt{-g}\,d^{3}p=\int p_{\mu}p_{\nu}\left[\delta f\,\frac{\sqrt{-{}^{0}g}}{{}^{0}p^{0}}+f_{0}\,\delta\left(\frac{\sqrt{-g}}{p^{0}}\right)\right]d^{3}p. \qquad (B2)$$

We note that a perturbation in the metric perturbs the invariant volume element as well as the momentum normalization and the distribution function. The first term in the square brackets of (B2) is just $\delta f R^{3}(m^{2}+\bar{p}^{2}R^{2})^{-1/2}$. The second term is

$$f_{0}[\delta(\sqrt{-g})/p_{0}-\sqrt{-{}^{0}g}\,\delta(p^{0})(p^{0})^{2}]\equiv X.$$

We evaluate

$$\delta\sqrt{-g}=\tfrac{1}{2}(-g)^{-1/2}\,\delta(-g)=-\tfrac{1}{2}R^{3}h_{\alpha}{}^{\alpha}.$$

The variation in p^{0} due to a variation of the metric with p^{i} held fixed is found by using

$$-m^{2}=g_{\alpha\beta}p^{\alpha}p^{\beta}$$

or

$$0=h_{\alpha\beta}p^{\alpha}p^{\beta}+{}^{0}g_{\alpha\beta}(p^{\alpha}\delta p^{\beta}+p^{\beta}\delta p^{\alpha}).$$

Using the RW metric, we find $\delta p^{0}=\frac{1}{2}h_{\alpha\beta}p^{\alpha}p^{\beta}/p^{0}$.

Thus we obtain

$$X = f_0\left(-\frac{1}{2}\frac{R^3}{{}^0p^0}h_\alpha{}^\alpha - \tfrac{1}{2}R^3 h_{\alpha\beta}p^\alpha p^\beta/({}^0p^0)^3\right),$$

where we emphasize the unperturbed energy with the notation ${}^0p^0$. It then follows that

$$\delta T_{\mu\nu} = \int p_\mu p_\nu R^3 \frac{\delta f}{{}^0p^0} d^3p - \int p_\mu p_\nu f_0 \frac{R^3}{2}\frac{h_\alpha{}^\alpha}{(p^0)^2} d^3p - \int p_\mu p_\nu \frac{R^3}{({}^0p^0)^3} f_{0}\tfrac{1}{2}h_{\alpha\beta|}p^\alpha p^\beta d^3p \,.$$

APPENDIX C: LORENTZ TRANSFORMATION OF WAVE COMPONENTS

The transformation to a local Lorentz frame is given by[2]

$$x'^{\alpha} = b^\alpha{}_\mu x^\mu + \tfrac{1}{2} b^\alpha{}_\lambda \Gamma^\lambda_{\mu\nu} x^\mu x^\nu \,. \tag{C1}$$

For our needs, x'^α represent local Lorentz coordinates and x^μ and $\Gamma^\lambda_{\mu\nu}$ are given by RW geometry. From (C1) we see that at the origin of the local frame

$$\frac{\partial x'^j}{\partial x^k} = \frac{1}{R^2}\delta^j{}_k \quad \text{and} \quad \frac{\partial x'^0}{\partial x^0} = 1\,.$$

All other transformation coefficients vanish. Thus the $h_{\mu\nu}$ of the local frame are merely those of the RW frame rescaled by the factor $1/R^2$. The components in a local Lorentz frame moving with respect to the first local frame are given by the Lorentz transformation $\Lambda^{\hat\alpha}_{\hat\beta}$ so that

$$h_{\mu\nu\,(\text{particle frame})} = \Lambda^{\hat\alpha}_{\mu}\Lambda^{\hat\beta}_{\nu} h_{\hat\alpha\hat\beta(\text{TT fluid-frame components})}\,.$$

Thus

$$h_{\mu 0} = \Lambda^{\hat 1}_{\mu}\Lambda^{\hat 1}_{0} h_{\hat 1\hat 1} + \Lambda^{\hat 2}_{\mu}\Lambda^{\hat 1}_{0} h_{\hat 2\hat 1} + \Lambda^{\hat 1}_{\mu}\Lambda^{\hat 2}_{0} h_{\hat 1\hat 2} + \Lambda^{\hat 2}_{\mu}\Lambda^{\hat 2}_{0} h_{\hat 2\hat 2}\,.$$

For a particle moving parallel to a TT wave propagating in the $x^{\hat 3}$ direction we have

$$_3\Lambda^{\hat\alpha}_{\hat\beta} = \begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}.$$

Thus $h_{\mu 0}$, $h_{\mu 3}$, and the trace $h_\mu{}^\mu$ are still zero in the frame of the moving particle. The transformed wave is transverse traceless.

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