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Cosmological model with gravitational and scalar waves

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Gowdy's vacuum three-torus universe, which was recently the subject of a detailed investigation by Berger and Misner, is generalized by incorporating a minimally coupled massless scalar field. We construct the general solution and consider some particular cases, one of which reduces to the Taub-Tabenski solution. The asymptotic behavior is found (i) in the vicinity of the initial singularity (ii) in the high-frequency limit. It is shown that in both asymptotic cases the behavior of the present solution is significantly different from that of the vacuum solution. The high-frequency limit is shown to be a new exact solution of Einstein's equations with the energy-momentum tensor corresponding to a spatially homogeneous massless scalar field plus a null fluid. Finally, the full model is used to generate a new solution of coupled gravitational-conformal scalar field equations.

I. INTRODUCTION

Present-day cosmology is based on Friedmann's solutions of the Einstein equations, which describe a completely uniform and isotropic universe. As stated by Lifshitz, Khalatnikov, and Belinskii,[1] it is now safe to assume that the present state of the evolution of the universe is adequately described by homogeneous isotropic cosmological models. There is, however, no reason to assume that such regular expansion is also suitable for description of the early stages of the development of the universe. On the contrary, it would be more natural to assume that the early universe was characterized by irregular expansion, and only in the process of evolution were the initial inhomogeneities and anisotropies damped. These arguments have led to rapidly increasing interest in studies of the so-called "irregular" cosmological models,[2-9] that is, models which are (i) anisotropic and homogeneous and (ii) anisotropic and inhomogeneous.

Lifshitz, Khalatnikov, and Belinksii have considered in a series of papers[1] the approach to the big-band singularity by a very general class of cosmological models having a perfect fluid as a source. In the homogeneous models of the Bianchi types I–VII the initial stages of evolution were found to behave like the Kasner vacuum solution.[10] For the models of types VIII and IX the initial regime is much more complicated. It can be decomposed into "eras"; during each era there exist oscillations of expansion along two spatial axes and monotonic expansion along the third one. These results were generalized to the case of the inhomogeneous models, in which the metric has a nontrivial dependence on the space coordinates. Lifshitz *et al.* have shown that each era in these models can be approximately described by a generalized wave solution of the Einstein-Rosen type, with the spatial singularity replaced by the initial singularity at the time $t=0$. These studies indicate the important role played by the gravitational waves in the early universe. Particular exact solutions representing universes with gravitational waves were later on considered by Gowdy,[11] Berger,[12,13] and Misner.[14] They thoroughly investigated the three-torus vacuum model, which is the simplest example of the closed universe filled with linearly polarized gravitational waves. The quantum version of this solution was also studied.[13,14] It was found that, at least in this case, quantum effects cannot prevent the initial singularity.

In a more recent paper, Belinskii and Khalatnikov[15] presented a general discussion of the influence of the scalar and vector fields on the early stages of the evolution of the universe. In the case of a massless scalar field, minimally coupled to gravity, a significant phenomenon was found. Instead of a Kasner-type behavior, one obtains a more general regime, which admits the expansion along all of the spatial axes. Another interesting solution with conformally invariant massless scal ar field was constructed by Bekenstein[16] for the homogeneous and isotropic case. It is a Robertson-Walker cosmology, which, unlike the Friedmann models, is free of singularity. Some other solutions of the coupled Einstein–massless-scalar field equations have been recently discussed.[17-19]

Such specific features, found in the cosmological models with massless scalar field, naturally require further research in this area. Of special interest is a problem of constructing cosmological models filled with both gravitational and scalar waves. Such solutions should provide exactly solvable models for the study of the interactions between the scalar particles and the gravitational field in the early universe.

In the present paper we construct a new spatially inhomogeneous exact cosmological solution of Einstein's equations with minimally coupled massless scalar field source. This solution can be considered as a generalization of Berger's work[12] to the case of coupled gravitational and massless scalar fields. It also generalizes the results of Belinskii and Khalatnikov to the spatially inhomogeneous case with inhomogeneities caused by standing scalar and linearly polarized gravitational waves. The present model includes the Taub-Tabenski metric[20] as a particular case. The whole dynamics of this solution is completely defined by two decoupled variables representing the scalar field and the transverse part of the gravitational field. The interaction between the scalar and the gravitational fields is thus reduced to the interaction of the scalar waves with the longitudinal part of gravity only. It is shown that the presence of the scalar field leads to a significant change in the behavior of the metric in the neighborhood of the big-bang singularity. Furthermore, it is shown that in the high-frequency limit the present model asymptotically evolves into an anisotropic, but spatially homogeneous universe filled with the spatially homogeneous scalar field and the null fluid. This fluid consists of gravitons and scalar particles with zero rest mass. Further expansion of the model is dominated by the energy-momentum tensor of the null fluid and, hence, finally the model asymptotically goes into the Doroshkevich, Zeldovich, and Novikov universe.[3]

It is shown that the high-frequency limit of the present solution is by itself an exact solution of the Einstein equations with the energy-momentum tensor corresponding to a spatially homogeneous massless scalar field plus null fluid. Under some special conditions, imposed on the scalar field, the behavior of such a universe is similar to that of the Doroshkevich *et al.* solution during the whole process of the cosmological expansion. Hence, like the latter solution, it can be used in order to describe some particular stages of the "mixmaster" universe.

As indicated in the last section, an application of Bekenstein's transformation[21] to the present model leads to an inhomogeneous cosmological solution of Einstein's equations with conformal scalar field source. This model will be the subject of a subsequent paper.

II. FIELD EQUATIONS

The following conventions are used in this paper: The metric signature is $+2$, the Greek indices range from 0 to 3, and the comma and the semicolon denote partial and covariant differentiation, respectively. The Riemann tensor is defined by $\zeta_{\lambda;\nu;\mu} - \zeta_{\lambda;\mu;\nu} = \zeta_\sigma R^\sigma_{\lambda\nu\mu}$, where ζ_λ is an arbitrary covariant tensor, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, and the constants c and G are set equal to unity. The system of Einstein's equations for a minimally coupled massless scalar field can be written as

$$R^\mu_\nu = 8\pi(T^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu T), \qquad \varphi_{,\mu}{}^{;\mu} = 0, \tag{1}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\varphi_{,\alpha}\varphi^{,\alpha}. \tag{2}$$

The generic metric is taken in the form

$$\frac{1}{L^2} ds^2 = e^f(dz^2 - d\xi^2) + \gamma_{ab}dx^a dx^b . \tag{3}$$

Here the Latin indices a, b denote the spatial coordinates x and y, and L is a constant length, which is set equal to unity. Both f and γ_{ab} are supposed to be functions of ξ and z only.

Introducing the notations

$$|\gamma_{ab}| = \gamma, \quad \partial\gamma_{ab}/\partial\xi = \kappa_{ab}, \quad \partial\gamma_{ab}/\partial z = \lambda_{ab}, \tag{4}$$

along with

$$\kappa = \gamma^{ab}\kappa_{ab}, \quad \lambda = \gamma^{ab}\lambda_{ab}, \tag{5}$$

the field equations (1) can be written as

$$\frac{\partial}{\partial\xi}\left(\sqrt{\gamma}\,\frac{\partial\varphi}{\partial\xi}\right) - \frac{\partial}{\partial z}\left(\sqrt{\gamma}\,\frac{\partial\varphi}{\partial z}\right) = 0, \tag{6}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial\xi}(\sqrt{\gamma}\,\kappa^b_a) - \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial z}(\sqrt{\gamma}\,\lambda^b_a) = 16\pi e^f(T^a_b - \tfrac{1}{2}\delta^a_b T), \tag{7}$$

$$-\frac{\kappa}{2}\frac{\partial f}{\partial z} - \frac{\lambda}{2}\frac{\partial f}{\partial\xi} + \frac{\partial\kappa}{\partial z} + \tfrac{1}{2}\kappa^a_b\lambda^b_a = 16\pi e^f T^\xi_z, \tag{8}$$

$$-\lambda\frac{\partial f}{\partial z} - \kappa\frac{\partial f}{\partial\xi} + \frac{\partial\kappa}{\partial\xi} + \frac{\partial\lambda}{\partial z} + \tfrac{1}{2}\kappa^a_b\kappa^b_a + \tfrac{1}{2}\lambda^b_a\lambda^a_b = 16\pi e^f(T^\xi_\xi - T^z_z), \tag{9}$$

$$2\left(\frac{\partial^2 f}{\partial\xi^2} - \frac{\partial^2 f}{\partial z^2}\right) + \frac{\partial\kappa}{\partial\xi} + \tfrac{1}{2}\kappa^a_b\kappa^b_a - \tfrac{1}{2}\lambda^b_a\lambda^a_b = 16\pi e^f(T^\xi_\xi + T^z_z - T). \tag{10}$$

Further simplification occurs by setting the two-dimensional metric γ_{ab} equal to

$$\gamma_{ab} = \mathrm{diag}(\xi e^p, \xi e^{-p}), \tag{11}$$

where the metric variable p is taken to be a function of both ξ and z. Then the field equations can be reduced to the following form:

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\varphi}{\partial\xi} - \frac{\partial^2\varphi}{\partial z^2} = 0, \tag{12}$$

$$\frac{\partial^2 p}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial p}{\partial\xi} - \frac{\partial^2 p}{\partial z^2} = 0, \tag{13}$$

$$\frac{\partial f}{\partial\xi} = -\frac{1}{2\xi} + \xi\left\{8\pi\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\partial p}{\partial\xi}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\right]\right\}, \tag{14}$$

$$\frac{\partial f}{\partial z} = \xi\left(16\pi\,\frac{\partial\varphi}{\partial\xi}\,\frac{\partial\varphi}{\partial z} + \frac{\partial p}{\partial\xi}\,\frac{\partial p}{\partial z}\right). \tag{15}$$

Equations (12)–(15) have some interesting features. First of all, the scalar field φ and the transverse part p of the gravitational field are completely decoupled and satisfy the same wave equation. Second, both radiation fields create the "background" (longitudinal) part f of the geometry exactly in the same manner (up to the factor 16π). Finally, the back reaction of the longitudinal part of the gravitational field leads to the decrease of the radiation amplitudes.

III. THE GENERAL SOLUTION

Following Gowdy and Berger[11,12] the three-torus topology is imposed by closing the spatial variables $0 \leq z \leq 2\pi$, $\oint dx\,dy = 16\pi$, and requiring that the metric variables f and p, as well as the scalar field φ, be periodic in z. Then integration of Eqs. (12) and (13) along with the boundary condition, stated above, leads to the standing-wave form of both p and φ:

$$p = p_0 + \alpha_0 \ln\xi + \sum_{n=1}^{\infty} \cos[n(z - z_n)][A_n J_0(n\xi) + B_n N_0(n\xi)], \tag{16}$$

$$\varphi = \varphi_0 + \beta_0 \ln\xi + \sum_{n=1}^{\infty} \cos[n(z - z_n)][C_n J_0(n\xi) + D_n N_0(n\xi)]. \tag{17}$$

Here p_0, φ_0, α_0, β_0 are constants of integration, and $J_n(x), N_n(x)$ denote the Bessel and Neumann functions of order n. Integration of Eqs. (14)–(15) gives, after some straightforward calculations, the function f:

$$f = f_0 - \tfrac{1}{2}\ln\xi + f_{\mathrm{GW}} + f_S\,, \tag{18}$$

where f_0 is an integration constant and the explicit form of f_{GW} is given by

$$\begin{aligned} f_{\mathrm{GW}} = {} & \frac{\alpha_0^2}{2}\ln\xi + \alpha_0 \sum_{n=1}^{\infty} \cos[n(z - z_n)][A_n J_0(n\xi) + B_n N_0(n\xi)] \\ & + \frac{\xi^2}{4} \sum_{n=1}^{\infty} n^2 \{[A_n J_0(n\xi) + B_n N_0(n\xi)]^2 + [A_n J_1(n\xi) + B_n N_1(n\xi)]^2\} \\ & - \frac{\xi}{2} \sum_{n=1}^{\infty} n \cos^2[n(z - z_n)]\{A_n^2 J_0(n\xi) J_1(n\xi) + A_n B_n [N_0(n\xi) J_1(n\xi) + J_0(n\xi) N_1(n\xi)] + B_n^2 N_0(n\xi) N_1(n\xi)\} \\ & + \frac{\xi}{2} \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \sum_{m=1}^{\infty} \frac{nm}{n^2 - m^2} \{\sin[n(z - z_n)] \sin[m(z - z_m)][n U_{nm}^{(0)}(\xi) - m U_{nm}^{(1)}(\xi)] \\ & \qquad + \cos[n(z - z_n)] \cos[m(z - z_m)][m U_{nm}^{(0)}(\xi) - n U_{nm}^{(1)}(\xi)]\}, \end{aligned} \tag{19a}$$

and the functions $U_{nm}^{(0)}, U_{nm}^{(1)}$ are defined by

$$U_{nm}^{(0)}(\xi) = A_n A_m J_1(n\xi) J_0(m\xi) + B_n B_m N_0(m\xi) N_1(n\xi) + 2A_n B_m J_1(n\xi) N_0(m\xi), \tag{19b}$$

$$U_{nm}^{(1)}(\xi) = A_n A_m J_0(n\xi) J_1(m\xi) + B_n B_m N_0(n\xi) N_1(m\xi) + 2A_n B_m J_0(n\xi) N_1(m\xi). \tag{19c}$$

The function f_S is obtained from Eq. (19) by replacing $\alpha_0 \to \beta_0$ and multiplying all the terms by factor 16π.

The function f given by Eq. (18) represents the longitudinal part of the gravitational field, and includes all the nonlinearities of the model. The first two terms are generated by the spatially homogeneous source-free gravitational field, which, as shown in the next section, corresponds to the Kasner solution. The function f_{GW} includes all the nonlinear effects, caused by the transverse part of the gravitational field, whereas the massless scalar field contribution is given by the function f_S. As can be seen from Eq. (19) both f_{GW} and f_S can be further decomposed into the spatially homogeneous and inhomogeneous parts. The $\ln\xi$ term in Eqs. (16) and (17) is responsible for the homogeneous part, and the wave-type terms contribute to both the homogeneous and inhomogeneous parts.

IV. PARTICULAR SOLUTIONS

(A) When both the scalar field φ and the transverse part of the gravitational field p are excluded, one obtains

$$f = -\tfrac{1}{2}\ln\xi, \tag{20}$$

which is easily recognized as the Kasner $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ metric.

(B) Taking only the spatially homogeneous solutions of Eqs. (16) and (17)

$$p = \alpha_0 \ln\xi + p_0 , \tag{21}$$

$$\varphi = \beta_0 \ln\xi + \varphi_0 \tag{22}$$

leads to the following expression for the function f:

$$f = (-\tfrac{1}{2} + \tfrac{1}{2}\alpha_0{}^2 + 8\pi\beta_0{}^2) \ln\xi + f_0. \tag{23}$$

Transforming to the synchronous frame, one obtains the Belinskii-Khalatnikov solution

$$ds^2 = -dt^2 + t^{2p_1}dz^2 + t^{2p_2}dx^2 + t^{2p_3}dy^2 \tag{24}$$

along with

$$\varphi = q \ln t + \varphi_0 , \tag{25}$$

where p_1, p_2, p_3, and q are constant parameters which can be expressed in terms of α_0, β_0, φ_0, and f_0. It is important to note that the exponents p_1, p_2, and p_3 are related by

$$\begin{aligned} &p_1 + p_2 + p_3 = 1, \\ &p_1{}^2 + p_2{}^2 + p_3{}^2 = 1 - 8\pi q^2. \end{aligned} \tag{26}$$

The last expression leads to a possibility of expansion along all the spatial axes in contrast with the vacuum Kasner case. This solution includes also the isotropic Robertson-Walker-type cosmology as a particular case corresponding to $p_1 = p_2 = p_3 = \frac{1}{3}$.

(C) and (D) In the case of pure scalar radiation ($p = 0$), Eqs. (16)–(19) correspond to the Taub-Tabensky solution, whereas in the vacuum case ($\varphi = 0$) they represent the Gowdy-Berger model.

V. ASYMPTOTIC BEHAVIOR OF THE SOLUTION

First, the asymptotic behavior of the solution will be studied in the vicinity of the initial singularity, i.e., in the region where $n\xi$ is so small that the Neumann functions can be represented approximately as

$$N_0(n\xi) \simeq \frac{2}{\pi} \ln\xi, \tag{27}$$

$$N_1(n\xi) \simeq -\frac{2}{\pi n\xi} . \tag{28}$$

This leads to the following asymptotic form of radiation fields:

$$p \simeq \alpha(z) \ln\xi, \tag{29}$$

$$\varphi \simeq \beta(z) \ln\xi , \tag{30}$$

whereas the longitudinal part of the gravitational field f is given (up to the unimportant time-independent term) by

$$f \simeq \left[-\tfrac{1}{2} + \frac{\alpha^2(z)}{2} + 8\pi\beta^2(z) \right] \ln\xi, \tag{31}$$

along with

$$\alpha(z) = \alpha_0 + \frac{2}{\pi} \sum_{n=1}^{\infty} B_n \cos[n(z - z_n)], \tag{32}$$

$$\beta(z) = \beta_0 + \frac{2}{\pi} \sum_{n=1}^{\infty} D_n \cos[n(z - z_n)]. \tag{33}$$

Comparing Eqs. (29)–(31) with Eqs. (21)–(23) one finds that at each value of z the present model behaves like a particular Belinskii-Khalatnikov solution.

The high-frequency limit of the solution will be considered now. Such a regime corresponds to coordinate time ξ large compared with the period of the lowest mode in Eqs. (16) and (17), namely $n\xi \gg 1$ for each value of n. Using the large argument expansions of the Bessel and Neumann functions, one can find the asymptotic expressions for the functions p, φ, and f:

$$p \simeq \alpha_0 \ln\xi + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n\xi} \right)^{1/2} \cos[n(z - z_n)] \times \left[A_n \cos\left(n\xi - \frac{\pi}{4} \right) + B_n \sin\left(n\xi - \frac{\pi}{4} \right) \right], \tag{34}$$

$$\varphi \simeq \beta_0 \ln\xi + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n\xi} \right)^{1/2} \cos[n(z - z_n)] \times \left[C_n \cos\left(n\xi - \frac{\pi}{4} \right) + D_n \sin\left(n\xi - \frac{\pi}{4} \right) \right], \tag{35}$$

and

$$f \simeq \left(\frac{\alpha_0{}^2 - 1}{2} + 8\pi\beta_0{}^2 \right) \ln\xi + \frac{\xi}{2\pi} \sum_{n=1}^{\infty} (A_n{}^2 + B_n{}^2) n + 8\xi \sum_{n=1}^{\infty} n (C_n{}^2 + D_n{}^2). \tag{36}$$

Then the metric tensor $g_{\mu\nu}$ can be decomposed into the "background" part $\eta_{\mu\nu}$ and the wave part $h_{\mu\nu}$ as

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} , \tag{37}$$

where

$$\eta_{\mu\nu} = \mathrm{diag}(-e^f, e^f, \xi^{\alpha_0+1}, \xi^{-\alpha_0+1}), \tag{38}$$

$$h_{\mu\nu} = \mathrm{diag}(0, 0, \xi^{\alpha_0+1}\bar{p}, -\xi^{-\alpha_0+1}\bar{p}), \tag{39}$$

$$\bar{p} = p - \alpha_0 \ln\xi. \tag{40}$$

We will now prove that the background geometry is created partly by the gravitational waves and partly by the scalar field. Since, however, the Isaacson procedure[22] is not valid for a matter-filled universe, a different approach, based strictly upon the Einstein equations, will be used.

Substitution of the background metric $\eta_{\mu\nu}$ into Eqs. (7)–(10) gives

$$T^x_x = T^y_y = \tfrac{1}{2} T, \tag{41}$$

$$T^t_z = 0, \tag{42}$$

$$16\pi e^{f}(T^{\xi}_{\xi}-T^{z}_{z})=-\frac{2}{\xi}\frac{\partial f}{\partial \xi}-\frac{2}{\xi^2}+\frac{1+\alpha_0{}^2}{\xi^2}\,, \tag{43}$$

$$16\pi e^{f}(T^{\xi}_{\xi}+T^{z}_{z}-T)=2\frac{\partial^2 f}{\partial \xi^2}-\frac{2}{\xi^2}+\frac{1+\alpha_0{}^2}{\xi^2}\,. \tag{44}$$

From Eq. (41) it follows then that

$$T^{\xi}_{\xi}=-T^{z}_{z}\,. \tag{45}$$

Substituting Eqs. (36) and (45) into Eqs. (43) and (44), one obtains

$$T=\beta_0{}^2\xi^{-2}e^{-f} \tag{46}$$

and

$$T^{\xi}_{\xi}=-\frac{e^{-f}}{2\pi\xi}\sum_{n=1}^{\infty} n\left[\frac{1}{16\pi}(A_n{}^2+B_n{}^2)+C_n{}^2+D_n{}^2\right]-\tfrac{1}{2}T. \tag{47}$$

It is convenient to decompose the energy-momentum tensor, which is given by Eqs. (41), (42), and (45)–(47), as

$$T^{\mu}_{\nu}=T^{(1)\mu}_{\ \ \nu}+T^{(2)\mu}_{\ \ \nu}\,, \tag{48}$$

where $T^{(1)\mu}_{\ \ \nu}$ is a traceless tensor given by

$$T^{(1)\mu}_{\ \ \nu}=+\frac{e^{-f}}{8\pi\xi}K\,\mathrm{diag}(-1,1,0,0) \tag{49}$$

and

$$T^{(2)\mu}_{\ \ \nu}=\tfrac{1}{2}T\,\mathrm{diag}(-1,1,1,1). \tag{50}$$

Here we defined the parameter K by

$$K=\sum_{n=1}^{\infty} n\left[\frac{1}{4\pi}(A_n{}^2+B_n{}^2)+4(C_n{}^2+D_n{}^2)\right]. \tag{51}$$

The traceless tensor $T^{(1)}_{\mu\nu}$ can be further decomposed into

$$T^{(1)}_{\mu\nu}=T^{\mathrm{SW}}_{\mu\nu}+T^{\mathrm{GW}}_{\mu\nu}\,, \tag{52}$$

where the energy-momentum tensor of the gravitational waves $T^{\mathrm{GW}}_{\mu\nu}$ is given by

$$T^{\mathrm{GW}}_{\mu\nu}=\frac{1}{64\pi}\sum_{n=-\infty}^{+\infty}\frac{1}{|n|}(A_n{}^2+B_n{}^2)k_\mu k_\nu\,, \tag{53}$$

and the scalar radiation contribution to the $T^{(1)}_{\mu\nu}$ is

$$T^{\mathrm{SW}}_{\mu\nu}=\tfrac{1}{4}\sum_{n=-\infty}^{+\infty}\frac{1}{|n|}(C_n{}^2+D_n{}^2)k_\mu k_\nu. \tag{54}$$

Here the null vector k_μ is defined by

$$k_\mu=\frac{1}{\sqrt{\pi\xi}}(|n|,n,0,0). \tag{55}$$

Then it follows that the traceless part $T^{(1)}_{\mu\nu}$ of the energy-momentum tensor corresponds to the null fluid, consisting of collisionless flows of scalar massless particles and "gravitons."

Next we give the physical interpretation of the energy-momentum tensor $T^{(2)}_{\mu\nu}$. Rewritten with the help of the metric $\eta_{\mu\nu}$, the scalar field equation (6) is reduced to

$$\frac{d^2\varphi}{d\xi^2}+\frac{1}{\xi}\frac{d\varphi}{d\xi}=0. \tag{56}$$

Integrating Eq. (56) one obtains

$$\varphi=\varphi_0+\beta_0\ln\xi. \tag{57}$$

Calculation of the energy-momentum tensor of this field yields $T^{(2)\mu}_{\ \ \nu}$ exactly. Therefore, we conclude that the exact solution, which is defined by Eqs. (16)–(19), asymptotically evolves into a spatially homogeneous anisotropic universe, given by Eqs. (36), (38), and (56). This universe is by itself an exact solution of Einstein equations with the energy-momentum tensor defined by Eqs. (46), and (48)–(55). In the vicinity of the initial singularity the metric $\eta_{\mu\nu}$ behaves like the Belinskii-Khalatnikov solution, discussed in Sec. IV. In the special case $\beta_0=0$ (no spatially homogeneous scalar field) Eqs. (36) and (38) define a cosmological model, which is essentially the Doroshkevich, Zeldovich, and Novikov[3] universe, filled with collisionless null fluid. The latter solution was originally written in synchronous coordinates. The synchronous form of the metric $\eta_{\mu\nu}$ and the corresponding energy-momentum tensor are given in the Appendix.

In the limit $\xi\gg 1$ the null fluid contribution $T^{(1)\nu}_{\ \ \mu}$ is dominant and the metric $\eta_{\mu\nu}$ asymptotically evolves into the Doroshkevich *et al.* solution.

It is interesting to note that the null fluid-type asymptotic behavior of a scalar field source can be obtained by a different approach. Rewriting Eq. (2) explicitly with the help of the exact metric $g_{\mu\nu}$, given by Eqs. (3), (11), and (16)–(19), one finds

$$T_{\xi\xi}=T_{zz}=\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right], \tag{58}$$

$$T_{\xi z}=\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi}{\partial z}\,, \tag{59}$$

$$T_{xx}=\tfrac{1}{2}\xi e^{-f+p}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2-\left(\frac{\partial\varphi}{\partial z}\right)^2\right], \tag{60}$$

$$T_{yy}=\tfrac{1}{2}\xi e^{-f-p}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2-\left(\frac{\partial\varphi}{\partial z}\right)^2\right], \tag{61}$$

where p, φ, and f are defined by Eqs. (16)–(19). Then up to the order ξ^{-1}, $T_{\xi\xi}$ has the following asymptotic form:

$$T_{\xi\xi}=\frac{1}{\pi\xi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{nm}\left\{\cos[n(z-z_n)]\cos[m(z-z_n)]\left[C_n\sin\left(n\xi-\frac{\pi}{4}\right)+D_n\cos\left(n\xi-\frac{\pi}{4}\right)\right]\right.$$
$$\times\left[C_m\sin\left(m\xi-\frac{\pi}{4}\right)+D_m\cos\left(m-\frac{\pi}{4}\right)\right]$$
$$+\sin[n(z-z_n)]\sin[m(z-z_m)]\left[C_n\cos\left(n\xi-\frac{\pi}{4}\right)+D_n\sin\left(n\xi-\frac{\pi}{4}\right)\right]$$
$$\left.\times\left[C_m\cos\left(m\xi-\frac{\pi}{4}\right)+D_m\sin\left(m\xi-\frac{\pi}{4}\right)\right]\right\}. \tag{62}$$

Averaging Eq. (62) over phase leads to

$$\langle T_{\xi\xi}\rangle=\frac{1}{2\pi\xi}\sum_{n=1}^{\infty}n(C_n^2+D_n^2). \tag{63}$$

This is essentially the $T_{\xi\xi}$ component of the energy-momentum tensor $T_{\mu\nu}^{\text{SW}}$ given by Eq. (54). Similarly it can be shown that in the same approximation

$$\langle T\rangle=\langle T_x^x\rangle=\langle T_y^y\rangle=0. \tag{64}$$

Hence, in both methods the energy-momentum tensor of the massless scalar field can be reduced for $\xi\gg 1$ to the null fluid form (up to the order ξ^{-1}).

Finally we note that the presence of the minimally coupled massless scalar field does not suppress the increase of the spacelike volume of the universe, which is defined by

$$V_{\text{univ}}=\oint e^{f/2}\xi\,dx\,dy\,dz. \tag{65}$$

A straightforward application of Eqs. (31) and (36) shows that there is an increase of V_{univ} from zero at $\xi=0$ to infinity at $\xi=\infty$.

VI. DISCUSSION

To the best of our knowledge the cosmological model, given by Eqs. (3), (11), and (16)–(19) is the first inhomogeneous model with inhomogeneities due both to gravitational and scalar waves. An important feature of the present solution is related to the total decoupling of the dynamical variables φ and p, which represent the scalar field and the transverse part of the gravitational field, respectively. Nevertheless the scalar source contribution to the longitudinal part of the gravitational field leads to a significant change in the nature of the initial singularity as compared to the vacuum model. The Kasner-type asymptotics is replaced by the more general Belinskii-Khaltnikov-type behavior. For large times ($\xi\to\infty$), or equivalently, in the high-frequency limit, the present solution evolves into an anisotropic, but spatially homogeneous model filled with a spatially homogeneous scalar field and collsionless flows of massless scalar particles and gravitons. Both fluids propagate along the null geodesics. Further expansion is dominated by these null fluids and the final state of evolution is described by Doroshkevich, Zeldovich, and Novikov's spatially homogeneous anisotropic universe.

It is still not clear if the present solution can be used in order to describe some particular eras in more complicated inhomogeneous "mixmaster" universe. In this context it is interesting to consider the simpler case of the spatially homogeneous anisotropic universe filled with ultrarelativistic matter. The initial stages of such a universe can be represented by the Kasner solution with contraction along the z axis and expansion along the x and y axes. During this stage the components T_ξ^ξ and T_z^z of the energy-momentum tensor rapidly increase, and later they dominate the further evolution of the universe. The contraction along the z axis is then replaced by a rapid expansion (much more rapid than the expansion along the x and y axes). During this process T_ξ^ξ and T_z^z rapidly decrease and finally T_x^x and T_y^y become important. The above stages of the cosmological expansion are adequately described by the Doroshkevich *et al.* solution. If we include in this picture also the massless scalar field, it can lead to a significant change in the behavior of the universe. The initial contraction along the z axis occurs only for some particular values of parameter β_0, which characterize the homogeneous part of the scalar field. For these values of β_0 the general features of the cosmological expansion are qualitatively the same as those without the scalar field, and later stages of expansion can be taken to be the same as those described by Doroshkevich *et al.* This is due to a very rapid decrease of the energy-momentum tensor of the homogeneous part of the scalar field $T_{\mu\nu}^{(2)}$. It is worthwhile to note that the inclusion of a massless scalar field which causes expansion on all three axes initially might lead to complete removal of the mixmaster behavior in models which are mixmaster like without the scalar field.

The present model has been used to construct a new solution, which describes a conformally invariant massless scalar field, coupled to the gravitational field. Such a solution was found by performing Bekenstein's transformation[21]

$$\bar{g}_{\mu\nu} = (\cosh\beta\varphi)^2 g_{\mu\nu}, \quad \psi = \beta^{-1}\tanh\beta\varphi, \tag{66}$$

where $\bar{g}_{\mu\nu}$ is the metric tensor of a new solution, $\beta = (4\pi/3)^{1/2}$, and ψ is a massless scalar field, which satisfies the conformally invariant wave equation

$$\psi_{,\alpha}{}^{;\alpha} - \tfrac{1}{6}R\psi = 0. \tag{67}$$

A full presentation of this model will be given in a subsequent paper.

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APPENDIX

Transforming the metric $\eta_{\mu\nu}$, which is defined by Eqs. (36) and (38) one obtains

$$ds^2 = -dt^2 + a_1^2(t)dz^2 + a_2^2(t)dx^2 + a_3^2(t)dy^2, \tag{A1}$$

where

$$a_1(t) = K^{-S+1/4}e^{r(t)}r^{S-1/4},$$
$$a_2(t) = K^{-(\alpha_0+1)/2}r(t)^{(\alpha_0+1)/2}, \tag{A2}$$
$$a_3(t) = K^{(\alpha_0-1)/2}r^{-(\alpha_0-1)/2}.$$

The function $r(t)$ is defined by

$$t = \tau_0 \int r^{S-1/4}e^r dr \tag{A3}$$

along with

$$S = 4\pi\beta_0^2 + \tfrac{1}{4}\alpha_0^2, \quad \tau_0 = K^{-S-3/4}. \tag{A4}$$

The energy-momentum tensor given by Eqs. (41), (42), and (45)–(47) has the following components:

$$T^z_z = -T^t_t = \frac{1}{8\pi}\tau_0^{-2}r(t)^{-1/2-2S}e^{-2r(t)} \times [1 + 4\pi\beta_0^2 r(t)^{-1}], \tag{A5}$$

$$T^x_x = T^y_y = \tfrac{1}{2}\tau_0^{-2}r(t)^{-3/2-2S}e^{-2r(t)}. \tag{A6}$$

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