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Improved Weizsäcker-Williams method

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In the spirit of the Weizsäcker-Williams method virtual-photon spectra are derived which have the Weizsäcker-Williams form for low frequencies. It is shown for the cases of pair production of fermions and single-boson production that the photon spectra give remarkably accurate results for one- and two-virtual-photon processes.

The method of equivalent photons as originally developed by Weizsäcker and Williams[1] relates the high-energy photon-induced cross section $d\sigma_{\gamma X\to Y}$ to the corresponding cross section induced by a charged particle $d\sigma_{eX\to Y}$ by the relation

$$d\sigma_{eX\to Y}=\int\frac{d\omega}{\omega}N(\omega)d\sigma_{\gamma X\to Y}\,, \qquad (1)$$

where X is the initial state and Y is the state produced in the collision. The quantity $N(\omega)/\omega$ is the number of equivalent photons with frequency ω in the field of the fast moving charged particle. The corresponding relation for a double-equivalent-photon process, for instance as in an electron-positron collision, is a simple generalization of Eq. (1),

$$d\sigma_{ee\to Y}=\int\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}N(\omega_1)N(\omega_2)d\sigma_{\gamma_1\gamma_2\to Y}\,. \qquad (2)$$

In the early works[1] the spectrum of equivalent photons was obtained by a simple semiclassical picture as

$$N(\omega)=\frac{2\alpha}{\pi}\ln(b_M/b_m)\,,$$

where b_M and b_m are the largest and smallest impact parameters, respectively. b_M is determined by the condition that for impact parameters larger than $b_M=E/\omega m$ the field of the relativistic particle of mass m and energy E varies so rapidly that the contribution to the spectrum is very small, and b_m is in this simple theory taken as the Compton wavelength of the rapidly moving particle $1/m$. The virtual-photon spectrum thus exhibits the logarithmic dependence on the frequency which is characteristic of a Coulomb field,

$$N(\omega)=\frac{2\alpha}{\pi}\ln(E/\omega)\,. \qquad (3)$$

A procedure[2] which appears to be an alternative to the Weizsäcker-Williams method has been developed in connection with attempts to relate electroproduction processes to photoproduction processes. The procedure which starts with the matrix elements for the process and uses essential ingredients of the Weizsäcker-Williams method such as neglecting longitudinal photons and approximating virtual photons by real photons, leads to an equivalent photon spectrum

$$N(\omega)=\frac{\alpha}{\pi}\left[\frac{E^2+E'^2}{E^2}\ln\frac{2EE'}{\omega m}-\frac{(E+E')^2}{2E^2}\ln\frac{E+E'}{\omega}-\frac{E'}{E}\right], \qquad (4)$$

where $E'=E-\omega$. While a first glance at Eq. (4) would seem to indicate that this spectrum has the essential features of the Weizsäcker-Williams spectrum, namely, the logarithmic dependence on ω for $\omega\ll E$, a closer examination of Eq. (4) shows that the factor of $\ln(E/\omega)$ vanishes in the limit of small ω, giving a frequency-independent spectrum[3]

$$N(\omega)=\frac{2\alpha}{\pi}[\ln(E/m)-\tfrac{1}{2}]\quad(\omega\ll E)\,. \qquad (5)$$

The Dalitz-Yennie (DY) spectrum is thus not typical of the Coulomb field and must give results different from the Weizsäcker-Williams (WW) spectrum. In fact it is easy to see that for $ee\to ee\mu^+\mu^-$, Eq. (2) gives in the extreme relativistic limit $\sigma^{\rm WW}_{ee\to ee\mu^+\mu^-}=\frac{2}{3}\sigma^{\rm DY}_{ee\to ee\mu^+\mu^-}$, simply because of the difference between the low-frequency behaviour of the spectra in Eqs. (3) and (5). This difference between the WW and DY spectra was first noticed by Brodsky[4] *et al.* and has been extensively discussed in several later papers.[5] It has even lead to statements[6] that the Weizsäcker-Williams method cannot be used for double-equivalent-photon processes, statements which would be very hard to accept in view of the very simple direct physical basis of the Weizsäcker-Williams method.

In the present paper equivalent-photon spectra are derived, which differ from the Dalitz-Yennie spectrum Eq. (4) and which have the Weizsäcker-Williams asymptotic form for $\omega\ll E$ as given by Eq. (3). The spectra which are explicitly covariant, give automatically the "minimum impact parameter" of the Weizsäcker-Williams theory and at the same time provide an accuracy which goes far beyond that of the Weizsäcker-Williams approximation. Indeed the present work shows that the Weizsäcker-Williams method is more accurate than hitherto believed, provided an appropriate virtual-photon spectrum is used.

It appears that the photon spectra as derived in the present paper depend on the process, and we give below the spectrum for the creation of a pair of fermions, Eq. (9) and for the case of creation of a boson, Eq. (15).

In deriving the spectrum we consider explicitly the photoproduction of a pair of charged particles with mass m_1 and charges e_1^+, e_1^- in the field of a charged particle with mass m and charge e. The cross section integrated over the final-state phase space, as obtained from the leading Feynman diagrams, is given by (see, e.g., Refs. 2 or 4)

$$d\sigma_{k_1 e\to e e_1^+ e_1^-} = \frac{\alpha}{2\pi^2}\int \frac{d^3p'}{EE'}\frac{1}{(k^2)^2}(2p_\mu p_\nu + \tfrac{1}{2}k^2\delta_{\mu\nu}) \times \tfrac{1}{4}M_\mu^\dagger M_\nu\, d\tilde{\Gamma},$$

$$d\tilde{\Gamma} = (2\pi)^4\delta(k + k_1 - p_+ - p_-)d\Gamma,$$

where p and p' are, respectively, the initial- and final-state momenta of the particle of mass m, p_+ and p_- are the momenta of the produced pair, $d\Gamma$ is the invariant phase space of the pair, and M_μ is the matrix element for pair production in the collision of the real photon k_1 with the virtual photon k. When the contribution from the longitudinal/scalar photons is neglected, an approximation which is best justified by the accuracy of the final result, and the azimuthal-angle integration is performed, one finds

$$d\sigma_{k_1 e\to e e_1^+ e_1^-} = \frac{\alpha}{2\pi}\frac{1}{2E|\vec{p}|}\int d\omega \int_{k_-^2}^{k_+^2} dk^2\,\mathfrak{N}(k^2) \times \sum_{i=1}^{2}\tfrac{1}{4}M_i^\dagger M_i\, d\tilde{\Gamma}\,, \quad (6)$$

with $(k^2 = \vec{k}^2 - \omega^2)$

$$\mathfrak{N}(k^2) = 1/k^2 + (1/\vec{k}^2)[2EE'/k^2 - 2\omega^2 m^2/k^4 - \tfrac{1}{2}]\,,$$

and

$$k_\pm^2 = -\omega^2 + (|\vec{p}| \pm |\vec{p}\,'|)^2.$$

We have here neglected the mass m compared to the energies E and E'.

The form of the virtual-photon spectrum obtained from Eq. (6) depends in a crucial way on how the real-photon limiting procedure $k^2 \to 0$ is taken in $\mathfrak{N}(k^2)$ and $M_i(k^2)$. We shall take the limit $k^2 \to 0$ both in $\mathfrak{N}(k^2)$ and $M_i(k^2)$ after first having taken into account the effect of k^2 on the photon and electron propagators in $\mathfrak{N}(k^2)$ and $M_i(k^2)$. At this point our procedure differs from that of Dalitz and Yennie.[2] They keep k^2 in $\mathfrak{N}(k^2)$ but take the real-photon limit in M_i, $k^2 = 0$, which leads to the virtual-photon spectrum Eq. (4). We find

$$\mathfrak{N}(k^2) = \frac{1}{\omega^2}[(E^2 + E'^2)/k^2 - 2m^2\omega^2/k^4 - \tfrac{1}{2}]\,, \quad (7)$$

which, with

$$\sum_{i=1}^{2}\tfrac{1}{4}M_i^\dagger M_i\, d\tilde{\Gamma} \sim 2\omega\, d\sigma_{k_1 k\to e_1^+ e_1^-}\,,$$

immediately shows that the low-frequency behavior of the virtual-photon spectrum would indeed have the Weizsacker-Williams form of Eq. (3). We shall, however, also take into account the effect of k^2 on the fermion propagator in M_i. It is not difficult to show that for high energies and to lowest order in k^2,

$$(t + m^2)/(t_0 + m^2) = (u + m^2)/(u_0 + m^2) \approx 1 + k^2/|s_0|\,,$$

where t, u and t_0, u_0 are the usual t and u values for one real (k_1) and one virtual photon (k) and two real (k_0, k_1) photons, respectively, $u = (k - p_-)^2$, $t = (k - p_+)^2$, $u_0 = (k_0 - p_-)^2$, $t_0 = (k_0 - p_+)^2$, $k_0^2 = 0$, and $s_0 = (k_0 + k_1)^2$. Since M_i is a homogeneous linear function in the two variables $(t + m^2)^{-1}$ and $(u + m^2)^{-1}$ we have

$$M_i = (1 + k^2/|s_0|)^{-1}M_i^0\,,$$

where M_i^0 in our approximation $(k^2 = 0)$ is the matrix element for real photons. Thus we may identify

$$\sum_{i=1}^{2}\tfrac{1}{4}M_i^\dagger M_i\, d\tilde{\Gamma} = 2\omega(1 + k^2/|s_0|)^{-2}d\sigma_{kk_1\to e_1^+ e_1^-}\,. \quad (8)$$

When Eqs. (7) and (8) are introduced into Eq. (6), and the integration over k^2 is performed, the cross section is of the form Eq. (1) with

$$N(\omega) = \frac{\alpha}{2\pi}\left[\frac{E^2 + E'^2}{E^2}\left(\ln\frac{|s_0|}{k_-^2} - 1\right) - 2\frac{E'}{E}\right]\,, \quad (9)$$

where $k_-^2 = \omega^2 m^2/EE'$.

We have here used the fact that the order of magnitude of $|s_0|$ is essentially determined by the photoproduction cross section of Eq. (1), i.e., independent of E and much smaller than E so that we may put $k_-^2 \ll |s_0| \ll E^2$. The spectrum (9) is indeed identical to the Weizsäcker-Williams spectrum $(2\alpha/\pi)\ln(E/\omega)$ in the limit $\omega \ll E$, which guarantees the correctness of the extreme relativistic results for double equivalent photon cross sections. With reference to the Weizsäcker-Williams semiclassical picture, the impact parameter b_M from Eq. (9) is found to be $b_M = (k_-^2)^{-1/2} = E/\omega m$ as in the semiclassical theory, while the smallest impact parameter b_m is given by $b_m^2 = k_{\max}^{-2} = |s_0|^{-1}$. Thus b_m depends on the process. Now clearly $k_{\max}^2 = |s_0|$ is equivalent to $-s_0 - k^2 = -s > 0$. Thus the minimum impact parameter or equivalently the maximum value of k^2 is determined by the condition that the total energy $-s$ of the virtual and real photons is larger than zero, which is physically a quite reasonable condition.

Once the spectral distribution Eq. (9) has been established, it is of interest to see what accuracy may be obtained for single- and double-virtual-photon processes. For the single-virtual-photon process, pair production in the field of a charged particle, one finds by performing the integrations over ω in Eq. (1) with $|s_0| = 4\omega\omega_1 = 4m_1^2/\tau$ and the Dirac pair cross section

$$\sigma_{kk_1\to e_1^+e_1^-} = \frac{\pi\alpha^2}{m_1^2}\tau\left[(2+2\tau-\tau^2)\ln\frac{1+(1-\tau)^{1/2}}{\tau^{1/2}} - (1-\tau)^{1/2}(1+\tau)\right], \qquad (10)$$

the total cross section

$$\sigma_{k_1 e\to ee_1^+e_1^-} = \frac{\alpha^3}{m_1^2}\frac{28}{9}\left[\ln\frac{2\omega_{1l}}{m_1} - \frac{213}{84}\right], \qquad (11)$$

which is remarkably close to the well known exact high-energy pair cross section which may be written in the form

$$\sigma_{k_1 e\to ee_1^+e_1^-} = \frac{\alpha^3}{m_1^2}\frac{28}{9}\left[\ln\frac{2\omega_{1l}}{m_1} - \frac{218}{84}\right].$$

Here $\omega_{1l} = 2E\omega_1/m$ is the photon energy in the system in which initially the particle with mass m is at rest. It should be emphasized that the accuracy of Eq. (11) goes far beyond the Weizsäcker-Williams and the Dalitz-Yennie approximation which give correctly only the logarithmic term.

For the case of the double-equivalent-photon process, Eq. (2), we obtain by integrating Eq. (2) with $M^2 = 4\omega_1\omega_2$ fixed, the cross-section differential in the invariant pair mass M,

$$d\sigma_{ee\to eee_1^+e_1^-} = \sigma\left(\frac{4m_1^2}{M^2}\right) I(M^2)\, dM^2/M^2,$$

where $\sigma(\tau)$ is given by Eq. (10) and

$$I(M^2) = \left(\frac{\alpha}{\pi}\right)^2\left[\frac{2}{3}\left(\ln\frac{s}{M^2}\right)^3 + \ln\frac{s}{M^2}\left(\ln\frac{M^2}{m^2}\right)^2 + 2\ln\frac{M^2}{m^2}\left(\ln\frac{s}{M^2}\right)^2 - \frac{3}{2}\left(\ln\frac{M^2}{m^2}\right)^2 - 7\ln\frac{M^2}{m^2}\ln\frac{s}{M^2} - 4\left(\ln\frac{s}{M^2}\right)^2 + \left(\frac{31}{4} - \frac{\pi^2}{3}\right)\ln\frac{M^2}{m^2} + \left(\frac{13}{2} - \frac{2\pi^2}{3}\right)\ln\frac{s}{M^2} + \frac{7}{6}\pi^2 - 1 + 4\mathcal{L}_3(1)\right], \qquad (13)$$

with

$$s = -(p+p')^2,$$

where $\mathcal{L}_3(z)$ is the Euler trilogarithm. This result reproduces the exact coefficients of all terms cubic and quadratic in the logarithms, for $M^2 \ll s$, and differs from the exact result only for the terms linear in $\ln(M^2/m^2)$ and $\ln(s/M^2)$ and constant terms, i.e., the five last terms are to be compared to the exact result[7]

$$(39/4 - 2\pi^2/3)\ln(M^2/m^2) + (17/2 - 2\pi^2/3)\ln(s/M^2) + \tfrac{7}{6}\pi^2 - \tfrac{7}{2} - 4\mathcal{L}_3(1).$$

As for the single-equivalent-photon approximation, Eq. (11), the double-equivalent-photon approximation of Eq. (13) is remarkably good for high energies. The total cross section, obtained by integrating Eq. (12) over M^2 is given by

$$\sigma_{ee\to eee_1^+e_1^-} = \frac{\alpha^4}{9\pi m_1^2}\left\{14\left[\left(\ln\frac{s}{m^2}\right)^2\ln\frac{s}{m_1^2} - \tfrac{1}{3}\left(\ln\frac{s}{m_1^2}\right)^3\right] - 64\left[\left(\ln\frac{s}{m^2}\right)^2 - \left(\ln\frac{s}{m_1^2}\right)^2\right] - 56\ln\frac{s}{m^2}\ln\frac{s}{m_1^2} + C_1\ln\frac{s}{m^2} + C_2\ln\frac{s}{m_1^2} + C_3\right\}, \qquad (14)$$

where as in Eq. (13) all terms cubic and quadratic[8] in the logarithms are identical to the exact results[7] and where (with exact values in parentheses) $C_1 = 234.4$ (233.3), $C_2 = 100$ (245.0), $C_3 = 409.7$ (993.4). When applied to two photon $\mu^+\mu^-$ production in e^+e^- collisions it is found that the error in Eq. (14) is 3% for 1 GeV total e^+e^- energy and decreases with increasing energy, again a remarkably good result.

Finally we discuss briefly the case of creation of a boson. In a way analogous to the case of pair production discussed above one finds for boson creation the virtual spectrum

$$N(\omega) = \frac{\alpha}{2\pi}\left(\frac{E^2+E'^2}{E^2}\ln\frac{|s_0|}{k_-^2} - 2\frac{E'}{E}\right). \qquad (15)$$

When applied to π^0 production in electron-electron (positron) collision one obtains

$$\sigma_{ee\to ee\pi^0} = \frac{8\alpha^2\Gamma_{\pi^0\to 2\gamma}}{m_\pi^3} I(s), \qquad (16)$$

with $[s=-(p+p')^2]$,

$$I(s)=\frac{2}{3}\left(\ln\frac{s}{m_\pi^2}\right)^3+4\left(\ln\frac{m_\pi}{m}\right)^2\ln\frac{s}{m_\pi^2}+4\ln\frac{m_\pi}{m}\left(\ln\frac{s}{m_\pi^2}\right)^2-6\left(\ln\frac{m_\pi}{m}\right)^2-10\ln\frac{m_\pi}{m}\ln\frac{s}{m_\pi^2}-2\left(\ln\frac{s}{m_\pi^2}\right)^2$$
$$+\left(\frac{1}{2}-\frac{2\pi^2}{3}\right)\ln\frac{s}{m_\pi^2}+\left(\frac{19}{2}-\frac{2\pi^2}{3}\right)\ln\frac{m_\pi}{m}+\frac{21}{4}+\frac{5\pi^2}{6}+4\mathcal{L}_3(1),\qquad(17)$$

where $\Gamma_{\pi^0\to 2\gamma}$ is the decay width of the π^0 decay and m_π is the mass of π^0. We have here for simplicity neglected the effect of the pion form factor. When compared to exact numerical calculation[5] one finds that the error in Eq. (16) is less than 2% for energies larger than 3 GeV and decreases with increasing energy.

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[2] R. H. Dalitz and D. R. Yennie, Phys. Rev. 105, 1598 (1957). See also R. B. Curtis, *ibid.* 104, 211 (1956).

[3] Sometimes in the literature this frequency-independent virtual photon spectrum is referred to as the Weizsäcker-Williams spectrum. This is somewhat misleading and should be avoided since as discussed in the text the properties of the Weizsäcker-Williams spectrum Eq. (3) differ considerably from those of Eq. (4).

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[8] Actually there is a small term $-\frac{10}{3}\ln(s/m^2)\ln(s/m_1^2)$ lacking in Eq. (14), as compared to Eq. (21) of Ref. 7.