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Invariant states and quantized gravitational perturbations

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We study the problem of quantizing the gravitational fluctuations about a symmetric vacuum background spacetime with compact Cauchy surfaces. In the context of lowest-order perturbation theory we show that the allowed physical states must all be invariant under the symmetry transformations of the background spacetime. This constraint does not unduly restrict the range of allowed states and is consistent with temporal evolution (in the presence of a timelike symmetry) or spacial localization (for a spacelike symmetry) provided the evolution or localization is interpreted intrinsically rather than with reference to the background spacetime.

I. INTRODUCTION

Recent work has shown that standard gravitational perturbation theory can sometimes lead to invalid results. If the (vacuum) spacetime which one is perturbing has compact Cauchy surfaces and admits a Killing vector field, then the linearized Einstein equations always admit some spurious, or nonintegrable solutions. A perturbation is said to be nonintegrable if there is no smooth curve of exact solutions to which it is tangent. For each Killing field which occurs there is a second-order condition which must be imposed to exclude the nonintegrable perturbations. The second-order conditions are equivalent to the requirement that the conserved quantity in linearized theory associated with each Killing vector field must be constrained to vanish. Compactness of the Cauchy hypersurfaces is crucial in obtaining this result; no such second-order conditions are implied for asymptotically flat spacetimes.

It is natural to ask whether there is any quantum analog for the classical problem of linearization instabilities described above. One method of quantizing the gravitational field is to choose a classical solution of the Einstein equations to serve as a background spacetime and to quantize the metric fluctuations about this background. At lowest order this procedure reduces to quantizing the linear perturbations of the given background. In this context it seems plausible that there should be a quantum analog to the second-order conditions whenever the background spacetime has compact Cauchy surfaces and Killing symmetries. However, the argument for second-order conditions in the classical theory is usually formulated in terms of the notion of curves of solutions to the exact equations. This idea does not appear to have a natural quantum correspondent.

There is, however, a different way of formulating the classical problem which is free of reference to curves of classical solutions and which admits a quantum interpretation. We shall discuss this formulation in Sec. II and show how it leads to quantum-mechanical second-order conditions which are analogous to the classical ones. The second-order quantum constraints we obtain are that the allowed quantum states must be annihilated by the operator-conserved quantities associated with each Killing field of the background. These conditions supplement the usual (first-order) constraints which imply gauge invariance of the physical states in the linearized theory.

An immediate consequence of the second-order conditions is that the physical states must be invariant under the symmetry transformations of the background spacetime. This conclusion follows from the observation that the conserved quantities are the generators of precisely these symmetry transformations. At first sight these invariance requirements would seem to restrict unduly the range of allowed quantum states. The invariant states cannot describe events which are localized in a spacelike symmetry direction or which evolve in a timelike symmetry direction. We shall argue, however, that the use of such states does not really exclude the description of localized events or of nontrivial evolution. It merely requires that such localization or time evolution be interpreted intrinsically rather than with reference to the background spacetime. We shall discuss several examples in Sec. III which should clarify this idea.

The idea that time evolution or spatial localization might make sense only intrinsically is not at all a new one. It arises rather naturally in Wheeler's geometrodynamical approach to quantum gravity based on the Arnowitt-Deser-Misner (ADM) formalism.[1,2] Our results show that it plays a role in even the more conventional approach to quantization based on a classical background spacetime.

The linearization stability properties of the Einstein equations have been discussed by Brill and Deser,[3] Fischer and Marsden,[4,5] Deser and Choquet-Bruhat,[6] Arms,[7] York and O'Murchadha,[8] and Moncrief.[9,10] Our discussion is based on a simplified and extended form of many of these results given by Fischer, Marsden, and Moncrief.[11]

II. SECOND-ORDER CONSTRAINTS

Let M be a three-dimensional manifold which is compact and without boundary. The phase space for Einstein's equations is a suitable function space of pairs (g,π), where $g=g_{ij}$ is a Riemannian metric and $\pi=\pi^{ij}$ is a tensor density defined over M. (See Ref. 4 for a discussion of Sobolev spaces of Cauchy data.) The constraint subset of phase space is defined by $\Phi(g,\pi)=0$, where Φ is the constraint map

$$\Phi(g,\pi)=(\mathcal{H}(g,\pi),-2\delta\pi) \tag{2.1}$$

with

$$\begin{aligned}\mathcal{H}(g,\pi)&=\frac{1}{\mu_g}\left[\pi^{ij}\pi_{ij}-\tfrac{1}{2}(\mathrm{tr}\,\pi)^2\right]-\mu_g R(g),\\ -2\delta\pi&=-2\pi_{ij}|^{j}\end{aligned} \tag{2.2}$$

in which $\mu_g=(\det g)^{1/2}$, $\mathrm{tr}\,\pi=g_{ij}\pi^{ij}$, $R(g)$ is the curvature scalar, and the vertical bar denotes the covariant derivative of g.

Let (g_0,π_0) be a particular solution of the constraints, $\Phi(g_0,\pi_0)=0$. This will serve as the initial data for the background spacetime. We can approximate the constraint function $\Phi(g,\pi)$ near (g_0,π_0) by its Taylor expansion

$$\begin{aligned}\Phi(g_0+h,\pi_0+\omega)&=D\Phi(g_0,\pi_0)(h,\omega)\\ &\quad+\tfrac{1}{2}D^2\Phi(g_0,\pi_0)((h,\omega),(h,\omega))\\ &\quad+\cdots,\end{aligned} \tag{2.3}$$

where $h=h_{ij}$ and $\omega=\omega^{ij}$ are the perturbations of g and π and where $D\Phi(g,\pi)(h,\omega)$ and $D^2\Phi(g,\pi)((h,\omega),(h,\omega))$ are the first and second derivatives of $\Phi(g,\pi)$ (see Refs. 4, 5, 10, and 11 for the explicit formulas).

Let C be any function and $Y=Y^i$ be any vector field on M and define the projection $\mathcal{P}_{(C,Y)}(\Phi)$ of Φ along (C,Y) by

$$\begin{aligned}\mathcal{P}_{(C,Y)}(\Phi(g,\pi))&=\int_M\langle(C,Y),\Phi(g,\pi)\rangle\,d^3x\\ &=\int_M\left[C\mathcal{H}(g,\pi)+Y^i(-2\pi_{ij}|^{j})\right]d^3x.\end{aligned} \tag{2.4}$$

These integrals are coordinate invariant since $\mathcal{H}(g,\pi)$ and $-2\delta\pi$ are densities. We shall define the first-order approximation to $\mathcal{P}_{(C,Y)}(\Phi(g,\pi))$ near (g_0,π_0) to be its lowest nontrivial contribution from the Taylor expansion of $\Phi(g,\pi)$. Thus, for any choice (C,Y) not identically zero, we approximate $\mathcal{P}_{(C,Y)}(\Phi(g,\pi))$ by

$$\mathcal{P}_{(C,Y)}(\Phi(g,\pi))\approx\int_M\langle(C,Y),D\Phi(g_0,\pi_0)(h,\omega)\rangle\,d^3x \tag{2.5}$$

unless this integral vanishes identically, in which case we approximate this projection by

$$\begin{aligned}\mathcal{P}_{(C,Y)}(\Phi(g,\pi))\approx\tfrac{1}{2}\int_M&\langle(C,Y),D^2\Phi(g_0,\pi_0)\\ &\times((h,\omega),(h,\omega))\rangle\,d^3x,\end{aligned} \tag{2.6}$$

The motivation for considering second-order approximations is that the first-order approximations (2.5) vanish identically if and only if C and Y are the normal and tangential projections on the initial surface of a Killing vector field for the Einstein spacetime which is determined by the initial data (g_0,π_0) (see Refs. 4, 5, 9–11 for details). Thus if Killing fields occur, the Taylor series expansions of the constraints projected along Killing directions begin at second order instead of first order (that the second-order projections are always nontrivial is established in Ref. 11). To avoid an undue truncation of the full set of constraint equations, one must include the second-order projections of $\Phi(g,\pi)$ along any Killing fields of the background as "lowest"-order approximations of the constraints. In the absence of background Killing symmetries, our lowest-order approximation reduces to the conventional linearized theory which we would quantize in the standard way.

Requiring $\mathcal{P}_{(C,Y)}(\Phi(g,\pi))=0$ to lowest order (in the sense we have just defined) for arbitrary (C,Y) thus leads to the first-order conditions

$$D\Phi(g_0,\pi_0)(h,\omega)=0\,, \tag{2.7}$$

which are the usual linearized constraints, and to a second-order condition

$$\int_M\langle(C,Y),D^2\Phi(g_0,\pi_0)((h,\omega),(h,\omega))\rangle\,d^3x=0 \tag{2.8}$$

for each Killing field (C,Y) of the background spacetime. In the usual approach to linearization stability the conditions (2.7) and (2.8) are shown to be necessary conditions to exclude nonintegrable perturbations. The present formulation, though rather intuitive, leads to the same conclusion but avoids reference to curves of solutions of the constraints. The emphasis here is on using a Taylor expansion to approximate the constraint function $\Phi(g,\pi)$ rather than using perturbations (h,ω) to approximate (as tangent vectors) curves of constraint solutions. This change of viewpoint is important for discussing quantization since one quantizes functions on phase space, not curves in phase space.

Let ${}^{(4)}g$ and ${}^{(4)}h$ signify the spacetime metric and metric perturbation determined (with suitable coordinate and gauge conditions) by the initial data (g_0,π_0) and (h,ω). Let ${}^{(4)}X$ be the Killing field with projections (C,Y) on the initial surface. Let Σ be an arbitrary Cauchy surface of the background spacetime and define

$$E_{{}^{(4)}X}({}^{(4)}g,{}^{(4)}h,\Sigma)=\frac{1}{2}\int_\Sigma \langle (C,Y),D^2\Phi(g,\pi)\times((h,\omega)(h,\omega))\rangle\, d^3x\ , \tag{2.9}$$

where now (g,π), (h,ω) and (C,Y) are the data induced on Σ by ${}^{(4)}g$, ${}^{(4)}h$, and ${}^{(4)}X$, respectively. As shown in Refs. 10 and 11, the integral $E_{{}^{(4)}X}({}^{(4)}g,{}^{(4)}h,\Sigma)$ is hypersurface invariant, i.e., it is a conserved quantity for the linearized equations. It is in fact equivalent to the usual conserved quantity one expects on account of the Killing symmetry of the background and can also be written

$$E_{{}^{(4)}X}({}^{(4)}g,{}^{(4)}h,\Sigma)=\int_\Sigma {}^{(4)}X^\alpha[D^2E_{\rm Ein}({}^{(4)}g)({}^{(4)}h,{}^{(4)}h)]_{\alpha\beta}\ {}^{(4)}\eta^\beta\, d^3\Sigma, \tag{2.10}$$

where $D^2E_{\rm Ein}({}^{(4)}g)({}^{(4)}h,{}^{(4)}h)$ is the second derivative of the Einstein tensor $E_{\rm Ein}({}^{(4)}g)$ and where ${}^{(4)}\eta$ is the future-pointing normal field and $d^3\Sigma$ is the volume element of Σ. Thus the second-order constraints, which are equivalent to $E_{{}^{(4)}X}({}^{(4)}g,{}^{(4)}h,\Sigma)=0$ for each Killing field ${}^{(4)}X$, are consistent with the linearized evolution equations since, if imposed on an initial surface, they propagate to any other surface automatically. A simple argument shows that the second-order constraints are necessarily gauge invariant (since otherwise they could not be conserved).

The perturbation equations $DE_{\rm Ein}({}^{(4)}g){}^{(4)}h=0$ can be derived from a variational principle analogous to that of Arnowitt, Deser, and Misner (ADM)[12] for the exact Einstein equations. The variational Lagrangian is

$$L=\int_M d^3x\Big\{\omega^{ij}\frac{\partial}{\partial t}h_{ij}-\tfrac{1}{2}N\,D^2\mathcal{H}(g,\pi)((h,\omega)(h,\omega)) -\tfrac{1}{2}X\cdot[-2D^2\delta\pi((h,\omega),(h,\omega))] -\delta N D\mathcal{H}(g,\pi)(h,\omega)-\delta X\cdot D(-2\delta\pi)(h,\omega)\Big\}\ , \tag{2.11}$$

where (N,X) are the lapse and shift fields of the background spacetime and $(\delta N,\delta X)$ are arbitrary perturbations of the lapse and shift. Varying $(\delta N,\delta X)$ leads to the first-order constraints (2.7) while varying (h,ω) gives the evolution equations for these variables. To take account of linearization instability one must also impose the second-order conditions (2.8) whenever any Killing symmetries occur in the background.

To quantize the above perturbation system one can attempt to define Hermitian field operators $(\hat h,\hat\omega)$ with canonical commutation relations which act on vectors $|\Psi\rangle$ of Hilbert space. One way of implementing the constraints in quantum theory is to construct (suitably ordered) constraint operators and to try to define a physical subspace of Hilbert space by imposing

$$D\Phi(g,\pi)(\hat h,\hat\omega)|\Psi\rangle=0 \tag{2.12}$$

and

$$\int_M d^3x\langle (C,Y),D^2\Phi(g,\pi)((\hat h,\hat\omega),(\hat h,\hat\omega))\rangle|\Psi\rangle=0. \tag{2.13}$$

Since the linearized constraint operators $D\Phi(g,\pi)(\hat h,\hat\omega)$ are the generators of gauge transformations of the linearized theory, Eq. (2.12) requires the physical states to be gauge invariant. However, the inner product in a Hilbert space on which $(\hat h,\hat\omega)$ are Hermitian operators would entail (functional) integration over the gauge variables. Typically this will imply an infinite norm for the invariant states since the "group volume" of gauge transformations would be infinite with respect to the standard type of formal integration measure.

To avoid this problem one can decompose the perturbation $(\hat h,\hat\omega)$ by analogy with the transverse-longitudinal decompositions familiar in electrodynamics. A suitable decomposition which works for an arbitrary vacuum background is given in Ref. 13. One can then define a Hilbert space for the "transverse" variables alone much as one does in electrodynamics. These states are independent of the gauge variables and one thus regards $D\Phi(g,\pi)(\hat h,\hat\omega)$ as the zero operator when it is applied to any of them. We shall consider some explicit examples of this quantization in Sec. III.

The second-order integrals (2.10) are gauge invariant in classical theory. This means that if one substitutes the general transverse-longitudinal decomposition of (h, ω) into the integrals one will find that the longitudinal (i.e., gauge) terms in the decomposition can only occur multiplied by the first-order constraints $D\Phi(g,\pi)(h,\omega)$. In quantizing this system one can always choose the ordering so that the operators $D\Phi(g,\pi)(\hat{h},\hat{\omega})$ stand to the right of the longitudinal terms (and any transverse terms which also occur as coefficients) where they will annihilate the physical states. With this procedure all longitudinal and constraint terms drop out of the second-order conditions which thus reduce to the form

$$\int_M d^3x \langle (C, Y), D^2\Phi(g,\pi)((\hat{h}^T, \hat{\omega}^T), (\hat{h}^T, \hat{\omega}^T))\rangle | \Psi\rangle = 0, \tag{2.14}$$

where $(\hat{h}^T, \hat{\omega}^T)$ are the operators representing the "transverse" summands in the decomposition of Ref. 13.

The classical conserved quantities $E_{{}^{(4)}X}({}^{(4)}g, {}^{(4)}h, \Sigma)$ for a set of Killing vector fields $\{{}^{(4)}X^{(a)}\}$ satisfy Poisson bracket relations of the form

$$\{E_{{}^{(4)}X^{(a)}}, E_{{}^{(4)}X^{(b)}}\} \approx C^c_{ab} E_{{}^{(4)}X^{(c)}} , \tag{2.15}$$

where the C^c_{ab} are constants and where $\approx$ signifies weak equality (i.e., equality up to terms in the linearized constraints). The constants C^c_{ab} are related to the Lie algebra of the Killing fields ${}^{(4)}X^{(a)}$ through

$$[{}^{(4)}X^{(a)}, {}^{(4)}X^{(b)}] = C^c_{ab}\, {}^{(4)}X^{(c)} . \tag{2.16}$$

Equation (2.15) is just an expression of the usual result that conserved quantities are the Hamiltonian generators of the associated symmetry transformations and thus have a Poisson bracket algebra isomorphic to that of the symmetry group.

To formulate the second-order quantum constraints

$$E_{{}^{(4)}X}((g,\pi), (\hat{h}^T, \hat{\omega}^T))|\Psi\rangle = 0 , \tag{2.17}$$

one needs to find an ordering of the operators for which the commutator algebra

$$[\hat{E}_{{}^{(4)}X^{(a)}}, \hat{E}_{{}^{(4)}X^{(b)}}] = i\, C^c_{ab} \hat{E}_{{}^{(4)}X^{(c)}} \tag{2.18}$$

is satisfied. In addition one needs to adopt a "normal" ordering of the field operators to ensure that the $\hat{E}_{{}^{(4)}X^{(a)}}$ are well-defined operators on Hilbert space. Satisfying these simultaneous requirements is usually facilitated by expanding the field operators $(\hat{h}^T, \hat{\omega}^T)$ in a suitable set of tensor harmonics which incorporate the "transversality" conditions automatically.

The identification of the $\hat{E}_{{}^{(4)}X^{(a)}}$ as generators of the symmetry transformations means formally that the second-order conditions $\hat{E}_{{}^{(4)}X^{(a)}}|\Psi\rangle = 0$ demand invariance of the physical states under the full symmetry group of the background spacetime. This conclusion that all physical states should be invariant is in sharp contrast to the usual result in Minkowski-space field theory that only the vacuum state is invariant under the full (Poincaré) symmetry group. The exclusive use of invariant states means that we cannot describe phenomena which are localized, relative to the background spacetime, in a symmetry direction. The actual spatial localization which is allowed by the physical states (even in symmetry directions) must be sought in an intrinsic description rather than as an extrinsic localization on the background spacetime. Several examples discussed in the following section should clarify this idea and make plausible the exclusive use of invariant states.

III. EXAMPLES AND DISCUSSION

An example of linearization instability is provided by the model of Brill and Deser.[3] If M admits a flat metric g_{ij} one can take $\pi^{ij} = 0$ and set $N = 1$ and $X^i = 0$. This choice generates the flat spacetime metric

$$ds^2 = -dt^2 + g_{ij}\, dx^i dx^j , \tag{3.1}$$

for which ${}^{(4)}X = \partial/\partial t$ is a timelike Killing vector field [i.e., $(\partial/\partial t)\, {}^{(4)}g_{\alpha\beta} = 0$]. The decomposition of Brill and Deser (which is modeled on that of ADM and generalized in Ref. 13) gives for the transverse perturbations

$$h^T_{ij} = h^{TT}_{ij} + \tfrac{1}{3}\alpha g_{ij} ,$$
$$\omega^{Tij} = \omega^{TTij} + \beta\mu_g g^{ij} , \tag{3.2}$$

where α and β are spatial constants and where TT signifies transverse and traceless relative to g_{ij}.

The conserved quantity associated with the time-translation symmetry is

$$\tfrac{1}{2}\int_M d^3x \langle (1,0), D^2\Phi(g,0)((h^T,\omega^T),(h^T,\omega^T))\rangle$$
$$= \int_M d^3x \left\{ \frac{1}{\mu_g}\left[\omega^{TTij}\omega^{TT}_{ij} - \tfrac{1}{2}3(\mu_g)^2\beta^2\right] + \frac{\mu_g}{4}\left[h^{TT}_{ij|k} h^{TTij|k}\right]\right\} , \tag{3.3}$$

where for simplicity we have dropped the terms which weakly vanish. To quantize the perturbations one expands the operators $(\hat{h}^{TT}, \hat{\omega}^{TT})$ in a set of transverse traceless tensor harmonics. The coefficients in the expansion become the creation and annihilation operators for transverse quanta. The physical states can be labeled by the eigenvalues $\{\eta_k\}$ of the number operators for the TT modes. In addition one must allow the states to depend upon either the variable α or its conjugate β. The operators $\hat{\alpha}$ and $\hat{\beta}$ satisfy the commutation relations

$$[\hat{\alpha}, \hat{\beta}] = \frac{i}{V}, \quad V = \int_M \mu_g \, d^3x \ , \tag{3.4}$$

To interpret these quantities note that the perturbation of the spatial volume of a t = const hypersurface is

$$\delta V = \tfrac{1}{2} \int_M \mu_g g^{ij} h_{ij} \ , \tag{3.5}$$

Each term in the Brill-Deser decomposition of h_{ij} (including longitudinal and constraint terms) has vanishing integrated trace except the term in α. Thus

$$\delta V = \tfrac{1}{2}\alpha \int_M \mu_g = \tfrac{1}{2}\alpha V \ , \tag{3.6}$$

and classically one has the Hamilton equation

$$\frac{d\alpha}{dt} = -3\beta \ . \tag{3.7}$$

Therefore α measures the perturbed spatial volume of the model and β its expansion or contraction.

Choosing α as the additional parameter for the physical states $|\alpha, \{\eta_k\}\rangle$, we represent $\hat{\beta}$ as

$$\hat{\beta} = \frac{1}{i V} \frac{\partial}{\partial \alpha} \tag{3.8}$$

and write the second-order constraint as

$$\frac{-3}{2V} \frac{\partial^2}{\partial \alpha^2} |\Psi(\alpha)\rangle = : \int_M d^3x \left\{ \frac{1}{\mu_g} [\hat{\omega}^{TT}_{ij} \hat{\omega}^{TTij}] + \frac{\mu_g}{4} [\hat{h}^{TT}_{ij|k} \hat{h}^{TTij|k}] \right\} : |\Psi(\alpha)\rangle \ , \tag{3.9}$$

where : : signifies a suitable normal ordering of the operator enclosed.

The Schrödinger equation is trivial in this case since the Hamiltonian operator obtained from Eq. (2.11) (with $N=1$, $X=0$) reduces to a linear combination of the first- and second-order constraints [Eqs. (2.12) and (3.3), respectively]. Thus the physical states are time translationally invariant (i.e., t independent). To regain a temporal evolution of the physical states requires the identification of an intrinsic time variable. A glance at Eq. (3.9) suggests the use of α, the perturbed volume of the model. This is essentially the choice of time variable made by Misner[4] for quantizing homogeneous cosmological models and by Berger[15] for quantizing some inhomogeneous models.

To achieve a natural probability interpretation we define the inner product of physical states to be just the Fock-space inner product for the TT modes (i.e., we refrain from integrating over the intrinsic time variable α). Furthermore, we adopt the superselection rule of considering only states of either purely positive or purely negative frequencies. These are the states expressible as

$$|\Psi(\alpha)\rangle = \sum_{\{\eta_k\}} C_{\{\eta_k\}} \exp\left[-i \left(\frac{2V}{3}\right)^{1/2} \omega_{\{\eta_k\}} \alpha \right] |\{\eta_k\}\rangle \ , \tag{3.10}$$

where

$$(\omega_{\{\eta_k\}})^2 |\{\eta_k\}\rangle = : \int_M d^3x \left\{ \frac{1}{\mu_g} [\hat{\omega}^{TTij} \hat{\omega}^{TT}_{ij}] + \frac{\mu_g}{4} [\hat{h}^{TTij|k} \hat{h}^{TT}_{ij|k}] \right\} : |\{\eta_k\}\rangle \tag{3.11}$$

and where all of the $\omega_{\{\eta_k\}}$ are taken to be either purely positive or purely negative. This corresponds to considering states which describe an expanding (or static) or a contracting (or static) model, but excludes states which mix expanding and contracting modes. On these subspaces the usual Fock inner product is always time independent (i.e., independent of α). Note that a state of nonzero graviton number is necessarily expanding or contracting and cannot remain static. This accords with the classical result that any gravitational wave (i.e., TT mode) excitation forces the universe to expand or contract. Ignoring the second-order constraint, one could construct coherent quantum states with the nonclassical behavior of admitting large gravitational wave excitations without expansion or contraction.

If the metric g admits a Killing field Y then the spacetime metric (3.1) will admit a Killing field ${}^{(4)}Y$ which is tangent to the t = const. hypersurfaces. The associated second-order constraint is

$$: \int_M d^3x \, \hat{\omega}^{TTij} (L_Y \hat{h}^{TT})_{ij} : |\Psi(\alpha)\rangle = 0 \ , \tag{3.12}$$

where L_Y is the Lie derivative with respect to Y. This condition demands the invariance of physical states under translations in the symmetry direction given by Y. The description of physical phenomena which are localized in the symmetry directions requires again an intrinsic point of view.

To show how translationally invariant states can be understood to represent localized phenomena we consider the simple example of a hydrogen atom in nonrelativistic quantum mechanics. The quantum states can be written $\Psi(\vec{x}, \vec{X}, t)$, where $\vec{X}$ is the position vector of the center of mass and $\vec{x}$ is the relative position of electron to proton. Let us (arbitrarily) restrict attention to the translationally invariant states defined by $\vec{P}\Psi(\vec{x}, \vec{X}, t) = 0$, where $\vec{P}$ is the total momentum operator. These states are independent of $\vec{X}$ and thus fail completely to localize the atom with respect to the background space. However, the dependence upon the relative coordinates is unaffected by this extra constraint. In particular, the bound-state wave functions still describe the electron localized within about an angstrom of the proton. One could of course extend this argument to complicated many-body systems as well. It is clear that the restriction to translationally invariant states does not prohibit the localization of one part of the system relative to another but only excludes localization of the system as a whole (e.g., its center of mass) with respect to the background space.

The exclusive use of invariant states prevents one from sensibly regarding the observer or his measuring apparatus as independent of the quantum system observed and localized in the background space. Clearly, to obtain a reasonable interpretation of invariant physical states one needs to regard the measuring apparatus as an intrinsic part of the quantum system rather than as an external entity. This viewpoint has long been espoused by Wheeler[1] and colleagues[2,14,15] in the context of geometrodynamical quantization of gravity. We see here that it may play an essential role in even the more conventional approach to quantization.

The arguments of this paper can be extended to the treatment of various matter fields coupled to gravity and to the inclusion of a cosmological constant. Typically one expects to need second-order constraints in linearized theory whenever the background matter and gravitational fields possess a simultaneous symmetry. If Yang-Mills fields are considered, second-order constraints can also arise from gauge symmetries or even from mixtures of gauge and spacetime symmetries.[7]

An interesting application of quantized matter and gravitational perturbations in a symmetric background would be the study of the Hawking process in de Sitter space. Hawking and Gibbons[16] have recently shown that the Hawking process for scalar particles in de Sitter space leads to a thermal radiation bath of nonzero temperature and seemingly paradoxical transformation properties. The radiation bath appears the same to every timelike geodesic "observer" in the spacetime. Such observers are equivalent under de Sitter group transformations and are in general boosted with respect to one another. In effect, the quantum state of the radiation is invariant under the transformations of the de Sitter group. To study the gravitational reaction effects of this Hawking radiation would seem to require a quantum treatment of the gravitational perturbations rather than a semiclassical treatment based on classical metrics and expectation values of quantum stress tensors. This conclusion follows from the observation of Gibbons and Hawking that such a group-invariant radiation bath (of nonzero temperature) cannot be described by a classical stress tensor field. The tensor property required of any source in the classical Einstein equations conflicts with the group invariance of their computed thermal radiation bath. In this connection, and perhaps in other problems involving quantum processes in a closed universe, the considerations of this paper may prove to be relevant.

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