Oscillations among three neutrino types and CP violation

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If ν_e, ν_μ, and ν_τ are treated symmetrically in a CP-invariant theory, it is impossible to have vacuum neutrino oscillations in which on the average each of the three neutrino types becomes an equal mixture of all three. It is shown how this becomes possible for a CP-noninvariant theory.

A number of experiments have been proposed[1,2] to search for neutrino oscillations from one type of neutrino (as ν_e) to another type (as ν_μ). In the theory of vacuum oscillations[3] these arise as a consequence of a nondiagonal mass matrix. In the case of two neutrino types, the oscillations are a maximum if the eigenvectors of the mass matrix are

$$|\nu_1\rangle = (|\nu_a\rangle + |\nu_b\rangle)/\sqrt{2},$$

$$|\nu_2\rangle = (|\nu_a\rangle - |\nu_b\rangle)/\sqrt{2},$$

where ν_a and ν_b are the neutrino types as defined by the usual charged currents.

With the discovery of a third charged lepton τ there is now considerable reason[4] to believe that there may be a third neutrino type ν_τ in addition to ν_μ and ν_e. The possibility of neutrino oscillations among three or more neutrino types has been discussed by a number of authors[1,5,6]; in particular, it has been suggested[6] that the solar neutrino flux could naturally be reduced by a factor greater than 2 in this case. For three neutrinos if we can express ν_e in terms of the mass eigenvectors ν_1, ν_2, ν_3 as

$$|\nu_e\rangle = \frac{1}{\sqrt{3}}\{|\nu_1\rangle + |\nu_2\rangle + |\nu_3\rangle\} \tag{1}$$

then the probability that a ν_e is still a ν_e after a time t is

$$|\langle \nu_e|\nu_e(t)\rangle|^2 = \tfrac{1}{3} + \tfrac{2}{9}(\cos\omega_{12}t + \cos\omega_{13}t + \cos\omega_{23}t), \tag{2}$$

where $\omega_{ij} = (m_i^2 - m_j^2)/2k$. For an average over a broad band of neutrino momenta k as in the solar neutrino problem[7] one expects the cosine terms to average to zero and obtain the result $\frac{1}{3}$, provided all ω_{ij} are sufficiently different from zero.

However, it has been pointed out by Nussinov[5] that with the assumption of CP invariance it is impossible to have a solution of the type (1) with three different mass eigenvalues and at the same time treat all three neutrino types symmetrically. If all three neutrino types enter a CP-invariant mass matrix symmetrically, it is easy to see that one eigenvector ν_1 will be a symmetric combination of the three neutrino types, but that the other two eigenvectors must be degenerate corresponding to the two-dimensional representation of the permutation group (mixed symmetry). In terms of the eigenvectors we have

$$|\nu_e\rangle = \frac{1}{\sqrt{3}}|\nu_1\rangle + (\tfrac{2}{3})^{1/2}|\nu_2\rangle, \tag{3a}$$

$$|\nu_\mu\rangle = \frac{1}{\sqrt{3}}|\nu_1\rangle + (\tfrac{2}{3})^{1/2}|\nu_2'\rangle, \tag{3b}$$

$$|\nu_\tau\rangle = \frac{1}{\sqrt{3}}|\nu_1\rangle + (\tfrac{2}{3})^{1/2}|\nu_2''\rangle \tag{3c}$$

where ν_2, ν_2', ν_2'' are three different combinations of the degenerate eigenvectors. If we choose as a basis ν_2 and the orthogonal vector ν_3, then

$$|\nu_2'\rangle = -\tfrac{1}{2}|\nu_2\rangle + \frac{\sqrt{3}}{2}|\nu_3\rangle, \tag{3d}$$

$$|\nu_2''\rangle = -\tfrac{1}{2}|\nu_2\rangle - \frac{\sqrt{3}}{2}|\nu_3\rangle. \tag{3e}$$

Rather than Eq. (2) we now find

$$\langle \nu_e|\nu_e(t)\rangle^2 = \tfrac{5}{9} + \tfrac{4}{9}\cos\omega_{12}t. \tag{4}$$

The average over neutrino momenta yields $\frac{5}{9}$ instead of $\frac{1}{3}$. The symmetry among neutrino types is manifest in that Eq. (4) holds when ν_e is replaced by ν_μ or ν_τ. In contrast, if a solution of the form (1) is demanded with nondegenerate eigenvectors and CP invariance, the equations for ν_μ and ν_τ have the general form[8]

$$|\nu_\mu\rangle = \cos\theta|\nu_4\rangle + \sin\theta|\nu_5\rangle,$$

$$|\nu_\tau\rangle = -\sin\theta|\nu_4\rangle + \cos\theta|\nu_5\rangle,$$

$$|\nu_4\rangle \equiv \frac{1}{\sqrt{2}}(|\nu_2\rangle - |\nu_3\rangle),$$

$$|\nu_5\rangle = -\frac{2}{\sqrt{6}}|\nu_1\rangle + \frac{1}{\sqrt{6}}(|\nu_2\rangle + |\nu_3\rangle).$$

Instead of Eq. (2) ν_μ and ν_τ then have time dependence of the form

$$|\langle\nu_\mu|\nu_\mu(t)\rangle|^2 = \tfrac{1}{2} + \text{oscillating term}, \tag{5}$$

so that they average to $\frac{1}{2}$ when averaged over the oscillations.

It is possible to break the degeneracy between ν_2 and ν_3 and still maintain the symmetry between the three types of neutrinos by introducing a small amount of CP violation. This can be done by a mass term of the form

$$i\sum_P (-1)^P \bar{\nu}_\alpha \nu_\beta\,, \tag{6a}$$

where the sum is over the permutations of the ordered set of indices e, μ, τ, and P is the permutation index.[9] In the ν_1, ν_2, ν_3 representation of Eq. (3) the mass matrix becomes

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & i\delta \\ 0 & -i\delta & m_2 \end{pmatrix}, \tag{6b}$$

where δ is the CP-violating parameter ($\delta < m_2$). The new eigenvectors are

$$|\tilde{\nu}_2\rangle = (|\nu_2\rangle + i|\nu_3\rangle)/\sqrt{2},$$
$$|\tilde{\nu}_3\rangle = (|\nu_2\rangle - i|\nu_3\rangle)/\sqrt{2},$$

with $\tilde{m}_2 = m_2 - \delta$, $\tilde{m}_3 = m_2 + \delta$. In place of Eqs. (3) we now have

$$\begin{aligned} |\nu_e\rangle &= \frac{1}{\sqrt{3}}(|\nu_1\rangle + |\tilde{\nu}_2\rangle + |\tilde{\nu}_3\rangle),\\ |\nu_\mu\rangle &= \frac{1}{\sqrt{3}}(|\nu_1\rangle + e^{-i\alpha}|\tilde{\nu}_2\rangle + e^{-2i\alpha}|\tilde{\nu}_3\rangle),\\ |\nu_\tau\rangle &= \frac{1}{\sqrt{3}}(|\nu_1\rangle + e^{+i\alpha}|\tilde{\nu}_2\rangle + e^{+2i\alpha}|\tilde{\nu}_3\rangle), \end{aligned} \tag{7}$$

where $\alpha = 2\pi/3$. Because of the symmetry, each of the eigenvectors should be an irreducible representation of the cyclic permutation group; this is achieved here by having ν_1, $\tilde{\nu}_2$, and $\tilde{\nu}_3$ all one-dimensional representations with the values 1 or $e^{\pm i\alpha}$ under cyclic permutations. Clearly CP violation is necessary in order to introduce these complex factors.[10]

Each of the three neutrino types now satisfies Eq. (2) so that the average of the diagonal oscillation probability $|\langle \nu_i | \nu_i(t)\rangle|^2$ is $\frac{1}{3}$ for each if $\delta \neq 0$. It is interesting to note that this is impossible without CP violation. Thus, for this particular example it is possible in principle to find definitive, although indirect, evidence for CP violation from a quantitative study of diagonal oscillations for three types of neutrinos. More direct tests of CP violation for this case [Eq. (7)] involving the comparison of off-diagonal oscillations for ν and $\bar{\nu}$ have been discussed by Cabibbo.[11]

For this case the neutrino oscillation equation satisfied by each type of neutrino is given by Eq. (2), which can now be written

$$|\langle \nu_\alpha | \nu_\alpha(t)\rangle|^2 = \tfrac{1}{9} + \tfrac{4}{9}\cos\Delta t\,(\cos\omega t + \cos\Delta t), \tag{8}$$
$$\omega = e, \mu, \tau,$$
$$\alpha = \frac{m_1 - m_2{}^2 - \delta^2}{2k}\,,$$
$$\Delta = -m_2\delta/k.$$

An intersting possibility is that the CP-violating term $\Delta \ll \omega$, in which case it is conceivable that for terrestrial experiments Δ can be approximated as zero and Eq. (4) will hold, whereas for the case of the solar neutrinos the full form Eq. (8) must be used. The off-diagonal oscillations are given by

$$|\langle \nu_\alpha | \nu_e(t)\rangle|^2 = \tfrac{4}{9} - \tfrac{2}{9}\cos\Delta t(\cos\Delta t + \cos\omega t) \pm (2\sqrt{3}/9)\sin\Delta t(\cos\Delta t - \cos\omega t),$$

where the + sign holds for $\nu_\alpha = \nu_\mu$ and the − sign holds for $\nu_\alpha = \nu_\tau$. The ± sign indicates a kind of asymmetry between the neutrino types, which has its origin in the ordering in Eq. (6a). Since this sign changes[11] in going from ν to $\bar{\nu}$, this asymmetry vanishes if the equations are summed over ν and $\bar{\nu}$.

The discussion in this Comment has been limited to vacuum oscillations; as discussed elsewhere,[12] the equations need to be modified if they are to be applied to neutrinos passing through matter. It should also be emphasized that there is no basic reason to demand symmetry among the three neutrino types. The major point of this note is really that CP noninvariance enlarges the domain of the set of diagonal oscillation probabilities when three neutrino types are considered. The domain of the three diagonal oscillation probabilities is also enlarged if there are more than three neutrino types available; thus, strictly, the observation of the enlarged domain is a sign of CP violation or of extra neutrino types. Observations of the off-diagonal oscillation probabilities are necessary to distinguish between these.

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[8]Equation (12) of Ref. 1 has this general form with $\theta = 0$ and the interchanges $\nu_\mu \rightleftarrows \nu_e$, $\nu_1 \rightleftarrows \nu_2$.

[9]As emphasized by M. Kobayashi and K. Maskawa, Prog. Theor. Phys. 49, 652 (1973), there is in general one phase factor that cannot be defined away when three types of massive particles are involved. Se also Ref. 11.

[10]Equation (7) is also a possible form for the solution when *CP* invariance holds provided two of the eigenvectors are degenerate. This can be seen directly from Eqs. (3) using the unitary transformation on the degenerate eigenvectors, $|\nu_2\rangle = (|\tilde{\nu}_2\rangle + |\tilde{\nu}_3\rangle)/\sqrt{2}$ and $|\nu_3\rangle = -i(|\tilde{\nu}_2\rangle - |\tilde{\nu}_3\rangle)/\sqrt{2}$, which transforms Eqs. (3) into Eq. (7).

[11]N. Cabibbo, Phys. Lett. 72B, 333 (1978). Cabibbo shows that direct tests of *CP* violation require studying the probability as a function of time for one type of neutrino starting with another type, that is $|\langle\nu_j|\nu_i(t)\rangle|^2$ with $i \neq j$.

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