Finite isospin groups and their experimental consequences

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Under the assumption that the observed isospin symmetry is a manifestation of the group structures of hadrons and their interactions, it is attempted to determine the order of the symmetry group, if finite, and to clarify the physical meaning of each group element. Our scheme is based on the observations that (1) classifications of particles according to irreducible representations of both the finite and the continuous groups are possible under certain restrictions, and (2) the transformation law of particles under continuous rotations in isospin space cannot be established directly by experiments. In particular, we consider the polyhedral kaleidoscope groups. The consistent formulation by finite groups needs a selection rule to exclude unobserved exotic states, which turns out to be a requirement of charge conservation. Several experiments are suggested to test our assumptions in strong-interaction processes.

I. INTRODUCTION

Approximate isospin symmetry is one of the most important concepts of particle physics. This symmetry or charge independence, which originated in the study of nuclear forces, is usually formulated as a continuous, rotational symmetry in a hypothetical isospin space. The particles are classified into irreducible representations of the group SU(2) in analogy to the ordinary angular momentum. Then one asks about consequences of the assumption that scattering processes due to strong interactions conserve the total isospin and that scattering amplitudes depend, as for the isospin quantum numbers, only on the total isospin. This procedure is applied for most of the actual analyses. This symmetry has been successfully extended to SU(3) symmetry by including strangeness. The prediction and the subsequent discovery of Ω^-, in particular, seem to indicate that the group principle is really working in nature. [We remember that the J/ψ was not predicted although the SU(4) symmetry had been known.] Experiments have revealed, however, a remarkable property of hadrons that their isospins seem to have upper limits, $I=1$ for mesons, and $I=\frac{3}{2}$ for baryons.[1]

Now it is well known that the finite groups have only finite-dimensional irreducible representations. So the question naturally arises: Is the isospin group finite or infinite?

The continuous group is a special case of the latter. This question should be answered before forming the Clebsch-Gordan series for the products of two irreducible representations. A similar problem has also occurred for the SU(3) symmetry, in which mesons are classified into $\underline{1}$ and $\underline{8}$, while baryons are classified into $\underline{1}$, $\underline{8}$, and $\underline{10}$. The reason higher-dimensional multiplets do not appear in nature has not been clarified in a convincing way. An interesting fact in this case is that these representations are constructed in the following way[2]: $\underline{3}\times\underline{3}^*=\underline{1}+\underline{8}$ and $\underline{3}\times\underline{3}\times\underline{3}=\underline{1}+\underline{8}+\underline{8}+\underline{10}$. However, the group SU(3) itself does not contain any inherent rule to exclude the higher-dimensional representations. One of the motivations of our work presented here may be considered as an attempt to find such a framework.[3] In order to answer the question raised above in connection with the isospin symmetry, it will be necessary to examine the way in which the group SU(2) has been used. For the ordinary spin, relative angles between polarization vectors are measurable in principle to any degree of accuracy. This is the key point in establishing that electrons behave as spinors in the ordinary space. Similarly, in order to establish the transformation law under a continuous group, it is necessary to find some ways to observe the pions, the nucleons, and other particles at every angle θ_i ($i=1$, 2, and 3) with respect to some fixed coordinate system in isospin space, if such ever exists.

The customary reason to believe in SU(2) symmetry comes from an entirely different, indirect observation. The charge independence of systems with relatively small isospins can be conveniently described by adopting this symmetry.[4] Thus a far weaker symmetry than SU(2) symmetry may be sufficient to classify the particles. In this work we will try to formulate the isospin symmetry by using the finite subgroups of SU(2). If this method can explain all the evidence of charge independence, then we will lose the argument for the isospin symmetry under a continuous group. On the other hand, if such an attempt turns out to be impossible, we must perhaps go to a stronger symmetry. Thus we find the formal similarity with questions asked many years ago by Case

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et al.[5] and Fairbairn *et al.*[6]

In the formulation we shall encounter the basic problem: What is the physical meaning of each group element? This problem is not peculiar to the formulation by finite groups and arises because we may think the observed isospin symmetry is just like the bilateral—and the rotational—symmetries of various objects and dynamical laws in the real world.

One suggestion to this problem comes from the original formulation by Heisenberg.[7] Three Pauli matrices were introduced there to a hypothetical space to describe the different states of nucleons and transitions between them. According to *our* interpretation, three matrices and the group generated by them are related to a dicyclic group of order 8. The group elements, or more rigorously three Pauli matrices, have physical interpretations in this example. The finite group which appears here is also generated by three quaternions i, j, and k ($i^2=j^2=k^2=ijk=-1$) and is denoted by $\langle 2,2,2\rangle$.

The finite groups considered in our work are slight generalizations of it. They are binary tetrahedral, binary octahedral, and binary icosahedral groups. These groups are often denoted by $\langle 3,3,2\rangle$, $\langle 4,3,2\rangle$, and $\langle 5,3,2\rangle$, respectively.

Now it is not difficult to make the unitary representations of these groups. We can identify one generator with the rotation around the z axis and diagonalize it. Then what are the possible correspondences between the basis of irreducible representations and electric charges?

In Sec. II we will consider one natural choice of the correspondence. The other choices will be mentioned. We then assign mesons and baryons to irreducible representations of the binary tetrahedral group and the binary octahedral group, respectively. This is based on the possible dimensions of irreducible representations of these groups.

The observed isospins and several arguments suggest that hadrons actually belong to the representations of these groups. However, it should be stressed that we have no conclusive evidence for it at present. In the course of analyses, we shall find that the decomposition of the product of two irreducible representations into the Clebsch-Gordan series contains, in general, components that are not eigenstates of the electric charge. If such components are realized as particles, then they will lead to a violation of the charge-conservation law through scattering processes. A possible interpretation is suggested. In particular, we shall assume that only the states with definite electric charges can be realized as particles. This assumption still allows the appearance of incomplete isospin multiplets such as doublets with charges +1 and −1. These multiplets, if realized as particles, will lead to the violation of charge independence, yet will conserve the electric charge through scattering processes. We will suggest experiments to test such a possibility. If the incomplete multiplets are suppressed or forbidden as a whole, then our result is essentially the same, with respect to the classification of particles, as the conventional result with all the exotic contributions omitted. In this case, the test of our hypotheses will need a much more advanced framework and will be postponed. Our work presented here should be considered to be of preliminary nature in this sense.

In Sec. III, the scattering processes are considered and several experiments are suggested as possible tests of our hypotheses. Section IV contains concluding remarks.

II. CLASSIFICATION OF PARTICLES BY FINITE GROUPS

The conventional isospin group SU(2) or O(3) can classify particles with any values of the isospin ($I=0,\frac{1}{2},1,\ldots$). The observed mesons and baryons, however, seem to have only limited values of isospin. The satisfactory explanation of this remarkable fact is hitherto unknown. We are therefore tempted to classify particles by taking this restriction into account. In order to formulate it mathematically, the use of finite groups seems to be the most attractive method for this purpose. Many-body systems such as heavy nuclei and neutron stars will be assumed to belong to the reducible representations of these symmetry groups.

Now let us begin our discussions with finite subgroups of O(3). The possible finite subgroups are cyclical, dihedral, tetrahedral, octahedral, and icosahedral groups.

We know that quantum-mechanical states are represented by rays, rather than by vectors. So let us consider the ray representations of these groups. They are the same as the ordinary repre-

TABLE I. Character table of the binary tetrahedral group $\langle 3,3,2\rangle$, of order 24, $\omega=\exp(2\pi i/3)$. Γ_0, $\Gamma_{1/2}$, and Γ_1 are representations D_0, $D_{1/2}$, and D_1 of the group SU(2).

Rep. \ Class	E	R	$6C_2$	$4C_3$	$4C_3^1$	$4C_3^a$	$4C_3^{a'}$
Γ_0	1	1	1	1	1	1	1
Γ_0'	1	1	1	ω	ω	ω^2	ω^2
Γ_0''	1	1	1	ω^2	ω^2	ω	ω
Γ_1	3	3	−1	0	0	0	0
$\Gamma_{1/2}$	2	−2	0	1	−1	1	−1
$\Gamma_{1/2}'$	2	−2	0	ω	$-\omega$	ω^2	$-\omega^2$
$\Gamma_{1/2}''$	2	−2	0	ω^2	$-\omega^2$	ω	$-\omega$

TABLE II. Character table of the binary octahedral group $\langle 4,3,2\rangle$ of order 48. Γ_0, $\Gamma_{1/2}$, Γ_1, and $\Gamma_{3/2}$ are representations D_0, $D_{1/2}$, D_1, and $D_{3/2}$ of the group SU(2).

Rep. \ Class	E	R	$6C_2$	$6C_4'$	$6C_4$	$12C_2^a$	$8C_3$	$8C_3'$
Γ_0	1	1	1	1	1	1	1	1
Γ_0^*	1	1	1	−1	−1	−1	1	1
Γ_x	2	2	2	0	0	0	−1	−1
Γ_1	3	3	−1	1	1	−1	0	0
Γ_1^*	3	3	−1	−1	−1	1	0	0
$\Gamma_{1/2}$	2	−2	0	$\sqrt{2}$	$-\sqrt{2}$	0	1	−1
$\Gamma_{1/2}^*$	2	−2	0	$-\sqrt{2}$	$\sqrt{2}$	0	1	−1
$\Gamma_{3/2}$	4	−4	0	0	0	0	−1	1

sentations of corresponding finite subgroups of SU(2).[8] The character tables for them are given in Tables I to III.[9]

The familiar form for the generator of a cyclic group is given by

$$A=\exp(2\pi i/n)=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\,, \tag{1}$$

where n is a positive integer. The group elements are $1, A, \ldots, A^{n-1}$. In a similar way, finite subgroups of SU(2) are generated by three quaternions:

$$A=\exp(P\pi/p),\quad B=\exp(Q\pi/q),\quad C=\exp(R\pi/r)\,. \tag{2}$$

They are cyclic, dicyclic, binary tetrahedral, binary octahedral, and binary icosahedral groups. These groups are exactly related to the subgroups of O(3) in the same way SU(2) is related to O(3). In (2), P, Q, and R are pure unit quaternions and p, q, and r are positive integers. Geometrically P expresses a point on the unit sphere in the three-dimensional space which is spanned by three unit quaternions i, j, and k. Thus P, Q, and R can express a spherical triangle with angles π/p at P, π/q at Q, and π/r at R. All possible reflections on the sphere of this triangle, which is often called a "fundamental region," generate the desired finite group.

This notion is known to be quite general.[10] In another way, these groups are completely specified by defining relations

$$A^p=B^q=C^r=ABC=Z,\quad Z^2=1\,. \tag{3}$$

The resultant group is denoted by $\langle p,q,r\rangle$.

Let us turn to the representation of finite groups. It is easy to read off the character tables the possible dimensions of irreducible representations. They are 1, 2, 3 for $\langle 3,3,2\rangle$, 1, 2, 3, 4 for $\langle 4,3,2\rangle$, and 1, 2, 3, 4, 5, 6 for $\langle 5,3,2\rangle$. The group mentioned in Sec. I, $\langle 2,2,2\rangle$, has only one- and two-dimensional irreducible representations.

We may identify one generator of the finite group with the rotation around the z axis of the three-dimensional Euclidean space and diagonalize it. Such a generator is conveniently expressed by using a discrete angle $\theta=2\pi/n$ ($n=2$, 3, 4, and 5) and the usual infinitesimal generator I_z. Then it is $\exp(i\theta I_z)$.

Next, in order to apply to physical problems,

TABLE III. Character table of the binary icosahedral group $\langle 5,3,2\rangle$ of order 120, $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$. Γ_0, $\Gamma_{1/2}$, Γ_1, $\Gamma_{3/2}$, Γ_2, and $\Gamma_{5/2}$ are representations D_0, $D_{1/2}$, D_1, $D_{3/2}$, D_2, and $D_{5/2}$ of the group SU(2).

Rep. \ Class	E	R	$12C_5$	$12C_5'$	$12C_5^a$	$12C_5^{a'}$	$20C_3$	$20C_3'$	$30C_2$
Γ_0	1	1	1	1	1	1	1	1	1
Γ_1	3	3	α	α	β	β	0	0	−1
Γ_1^*	3	3	β	β	α	α	0	0	−1
Γ_y	4	4	−1	−1	−1	−1	1	1	0
Γ_2	5	5	0	0	0	0	−1	−1	1
$\Gamma_{1/2}$	2	−2	α	$-\alpha$	$-\beta$	β	1	−1	0
$\Gamma_{1/2}^*$	2	−2	β	$-\beta$	$-\alpha$	α	1	−1	0
$\Gamma_{3/2}$	4	−4	1	−1	−1	1	−1	1	0
$\Gamma_{5/2}$	6	−6	−1	1	1	−1	0	0	0

we need to assume some correspondence between the basis of irreducible representations and the electric charge. The most natural way is clearly to retain the Gell-Mann–Nishijima relation in the integrated form. We may require the equation

$$\exp(i\theta I_z) = \exp[i\theta(Q - Y/2)] \tag{4}$$

to hold for all possible discrete values of θ corresponding to a given finite group. In this equation, Q is the electric charge and Y is the hypercharge of a particle. Another possibility is realized if the charge states are permuted among themselves in an arbitrary way. We note that the quantum numbers Q, Y, and others, if needed, specify the eigenvalues of matrices $\exp(i\theta I_z)$, but that they are not group elements.

It is clear that finite groups considered as subgroups of SU(2) contain a finite number of discrete rotation angles.[11] However, it may be too early to conclude that such angles have direct physical meanings unless the metric is introduced into the underlying space in a physically meaningful way. The generators A, B, and C in (2) are the fundamental ingredients of finite groups. Therefore in any physical applications, their meanings should be clarified.

We simply note that the generating relations (3) can be realized by isodoublet fermion fields in the following way:

$$\begin{aligned} A\psi &= A_{\text{op}}^{-1}\psi A_{\text{op}}, \\ A_{\text{op}} &= \exp\left[\int d^3x\, \psi^\dagger(x) P(\pi/p)\psi(x)\right], \end{aligned} \tag{5}$$

and similar relations for B and C. In (5), P should be identified with $i\sigma_1$, which is a two-dimensional realization of the pure unit quaternion P in terms of the Pauli matrix, and $p = 2$.

III. SCATTERING AMPLITUDES

The isospin coordinate was introduced to describe the proton and the neutron as different states of the same particle. The group SU(2) is usually employed for the classification of particles. But the hypothesis of charge independence is a more complicated matter.

The experimental analyses have shown that the isospin symmetry of scattering amplitudes can be understood if we assume that the same symmetry holds for vertices of a diagram corresponding to the scattering and that the propagators (which are either resonances or Reggeons) constitute a complete isospin multiplet.[12]

The situation seems to be very general. In this scheme, a scattering amplitude can be constructed diagrammatically by combining vertices and propagators with no loop. One can even imagine that

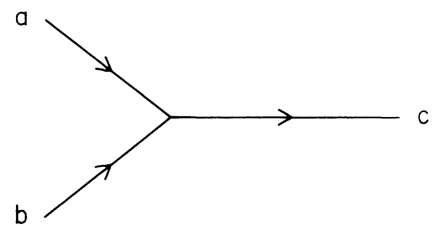

FIG. 1. The basic diagram for the scattering process. Three lines a, b, and c represent nonexotic particles.

the vertices are actually 3-vertices (Fig. 1).

We know that the apparent absence of exotic states of mesons and baryons has been confirmed by analyses based on such diagrams. The basic observation is that, to a good approximation, such 3-vertices are actually allowed only when three lines correspond to nonexotic particles.

So let us try to formulate the above rule as a basic law of scattering processes. We may require that the allowed vertices in the above sense should also occur in the Clebsch-Gordan decomposition of the product of two irreducible representations corresponding to a and b in Fig. 1. In $\pi\pi$ scattering, the usual decomposition contains $I = 0$, 1, and 2 states. Then is it possible to suppress the unobserved direct-channel resonances with $I = 2$ by a composition rule for the initial two-pion state?

In order to answer this question, we now propose the following set of assumptions[11]:

(a) There is no exotic stable or resonance state.

(b) The mesons (baryons) belong to the irreducible representations of the binary tetrahedral (binary octahedral) group.

(c) The assignment of electric charges to each member of the multiplet is done in the conventional way.

(d) Only the eigenstates of the electric charge are realized as particles.

Before assigning particles to the basis of each representation, Q and Y must be known beforehand for each particle. This is clear for any experimental situation. The converse problem, i.e., to define Q or Y from (4), does not arise.

Let us turn to the Clebsch-Gordan series for the two-pion system. It will contain the doubly charged components, e.g., $\pi^+\pi^+$ and $\pi^-\pi^-$. If these components are realized as particles, they will contradict our assumption (a). Evidently this process can lead to states with any values of the electric charge for sufficiently many pions if they are allowed. So some way to exclude such exotic states is essential for the success of our procedure. It may be accomplished by a selection rule. We will find such a rule in the following.

Next we assign mesons to Γ_0, $\Gamma_{1/2}$, and Γ_1

of $\langle 3,3,2\rangle$, and baryons to Γ_0, $\Gamma_{1/2}$, Γ_1, and $\Gamma_{3/2}$ of $\langle 4,3,2\rangle$. These representations are the same as the corresponding representations D_0, $D_{1/2}$, D_1, and $D_{3/2}$ of group SU(2). Of these representations, Γ_0, $\Gamma_{1/2}$, and Γ_1 belong to both $\langle 3,3,2\rangle$ and $\langle 4,3,2\rangle$.

Our assumption (b) implies that the system with baryon number 0 is to be decomposed into irreducible representations of $\langle 3,3,2\rangle$, while the system with baryon number ± 1 is always an irreducible representation of $\langle 4,3,2\rangle$. In particular, the baryon-antibaryon systems should be first decomposed into irreducible representations of $\langle 4,3,2\rangle$, and the latter representations should be further decomposed, if reducible under $\langle 3,3,2\rangle$, into irreducible representations of $\langle 3,3,2\rangle$. For the πN state, the decomposition is the same as the conventional case. This is seen from

$$\Gamma_{1/2}\times\Gamma_1=\Gamma_{1/2}+\Gamma_{3/2}\,. \tag{6}$$

In (6), N belongs to $\Gamma_{1/2}$ of $\langle 4,3,2\rangle$, while π belongs to Γ_1 of $\langle 3,3,2\rangle$. Thus $\Gamma_{1/2}$ and Γ_1 are apparently two irreducible representations of different groups. Now we know that

$$\mathrm{SU}(2)\supset\langle 4,3,2\rangle\supset\langle 3,3,2\rangle\,. \tag{7}$$

So we may use the decomposition rule of SU(2), followed by subduction (i.e., restriction to the subgroup), to reach (6). The total baryon number determines whether the group should be subducted to $\langle 4,3,2\rangle$ or $\langle 3,3,2\rangle$. The baryon states $\pi\Sigma$, $\bar{K}\Delta$, and $\pi\Delta$ have unconventional components in the decompositions.

For the $\pi\Sigma$ state we obtain

$$\Gamma_1\times\Gamma_1=\Gamma_0+\Gamma_1+\Gamma_1^*+\Gamma_x\,,$$

$$|\pi\rangle\times|\Sigma\rangle=\frac{1}{\sqrt3}(\pi^-\Sigma^+-\pi^0\Sigma^0+\pi^+\Sigma^-)
+\begin{cases}\frac{1}{\sqrt2}(\pi^+\Sigma^0-\pi^0\Sigma^+)\\ \frac{1}{\sqrt2}(\pi^+\Sigma^--\pi^-\Sigma^+)\\ -\frac{1}{\sqrt2}(\pi^-\Sigma^0-\pi^0\Sigma^-)\end{cases}
+\begin{cases}\frac{1}{\sqrt2}(\pi^-\Sigma^0+\pi^0\Sigma^-)\\ \frac{1}{\sqrt2}(\pi^+\Sigma^+-\pi^-\Sigma^-)\\ \frac{1}{\sqrt2}(\pi^+\Sigma^0+\pi^0\Sigma^+)\end{cases}
+\begin{cases}\frac{1}{\sqrt2 i}(\pi^+\Sigma^++\pi^-\Sigma^-)\\ \frac{1}{\sqrt6}(\pi^-\Sigma^++2\pi^0\Sigma^0+\pi^+\Sigma^-)\,.\end{cases} \tag{8}$$

The assignments of the $\pi\Sigma$ state to $\Gamma_1\times\Gamma_1^*$ and $\Gamma_1^*\times\Gamma_1^*$ lead to the same decomposition.

The $\bar{K}\Delta$ state is decomposed as

$$\Gamma_{1/2}\times\Gamma_{3/2}=\Gamma_1+\Gamma_1^*+\Gamma_x\,,$$

$$|\bar K\rangle\times|\Delta\rangle=\begin{cases}\frac12(\Delta^+\bar K^0-\sqrt3\Delta^{++}K^-)\\ \frac{1}{\sqrt2}(\Delta^0\bar K^0-\Delta^+K^-)\\ \frac12(\sqrt3\Delta^-\bar K^0-\Delta^0K^-)\end{cases}
+\begin{cases}\frac12(\Delta^-\bar K^0+\sqrt3\Delta^0K^-)\\ \frac{1}{\sqrt2}(\Delta^{++}K^0-\Delta^-K^-)\\ -\frac12(\sqrt3\Delta^+\bar K^0+\Delta^{++}K^-)\end{cases}
+\begin{cases}\frac{1}{\sqrt2 i}(\Delta^{++}\bar K^0+\Delta^-K^-)\\ \frac{1}{\sqrt2}(\Delta^0\bar K^0+\Delta^+K^-)\,.\end{cases} \tag{9}$$

The assignment of the $\bar K\Delta$ state to $\Gamma_{1/2}^*\times\Gamma_{3/2}$ leads to the same decomposition. The $\pi\Delta$ state is decomposed to

$$\Gamma_{3/2}\times\Gamma_1=\Gamma_{3/2}\times\Gamma_1^*=\Gamma_{3/2}+\Gamma_{1/2}+\Gamma_{3/2}+\Gamma_{1/2}^*\,,$$

$$|\pi\rangle\times|\Delta\rangle=\begin{cases}(\frac35)^{1/2}\Delta^{++}\pi^0-(\frac25)^{1/2}\Delta^+\pi^+\\ (\frac25)^{1/2}\Delta^{++}\pi^--\frac{1}{\sqrt{15}}\Delta^+\pi^0-(\frac{8}{15})^{1/2}\Delta^0\pi^+\\ (\frac{8}{15})^{1/2}\Delta^+\pi^--\frac{1}{\sqrt{15}}\Delta^0\pi^0-(\frac25)^{1/2}\Delta^-\pi^+\\ (\frac25)^{1/2}\Delta^0\pi^--(\frac35)^{1/2}\Delta^-\pi^0\end{cases}
+\begin{cases}\frac{1}{\sqrt2}\Delta^{++}\pi^--\frac{1}{\sqrt3}\Delta^+\pi^0+\frac{1}{\sqrt6}\Delta^0\pi^+\\ \frac{1}{\sqrt6}\Delta^+\pi^--\frac{1}{\sqrt3}\Delta^0\pi^0+\frac{1}{\sqrt2}\Delta^-\pi^+\end{cases}$$

$$+\begin{cases}\frac{1}{\sqrt{10}}\Delta^+\pi^++\frac{1}{\sqrt{15}}\Delta^{++}\pi^0+(\frac56)^{1/2}\Delta^-\pi^-\\ -(\frac{3}{10})^{1/2}\Delta^0\pi^++(\frac35)^{1/2}\Delta^+\pi^0+\frac{1}{\sqrt{10}}\Delta^{++}\pi^-\\ \frac{1}{\sqrt{10}}\Delta^-\pi^++(\frac35)^{1/2}\Delta^0\pi^0+(\frac{3}{10})^{1/2}\Delta^+\pi^-\\ -(\frac56)^{1/2}\Delta^{++}\pi^+-\frac{1}{\sqrt{15}}\Delta^-\pi^0-\frac{1}{\sqrt{10}}\Delta^0\pi^-\end{cases}
+\begin{cases}\frac{1}{\sqrt6}\Delta^{++}\pi^+-\frac{1}{\sqrt3}\Delta^-\pi^0-\frac{1}{\sqrt2}\Delta^0\pi^-\\ -\frac{1}{\sqrt2}\Delta^+\pi^+-\frac{1}{\sqrt3}\Delta^{++}\pi^0+\frac{1}{\sqrt6}\Delta^-\pi^-\,.\end{cases} \tag{10}$$

In (8), Γ_0 and Γ_1 are the same as the conventional $I=0$ and $I=1$ states, respectively. Γ_1^* contains a component which does not correspond to a definite charge.

If such a component is realized as a particle, it has no definite charge and the conservation law of the charge will be violated through the process $\pi^+ + \Sigma^+ \to (1/\sqrt{2})(\pi^+\Sigma^+ - \pi^-\Sigma^-) \to \pi^- + \Sigma^-$. This is one motivation for our assumption (d). It is interesting to note that all the exotic components appear in combination with ones of different charges. (This is also the case for mesons.)

Stated differently, all noneigenstates of electric charge contain at least one component, which is exotic. Therefore if such states are realized in nature, they will partially decay via exotic resonance states in contradiction with observations. From this fact we are led to the assumption (d) in a more convincing way. If this interpretation is correct, then all the particle states which can be realized are eigenstates of electric charge. When the electromagnetic interaction is introduced by a minimal coupling in this case, the Lagrangian will be automatically invariant under continuous gauge transformations. The classical Noether theorem is then clearly applicable and leads to the charge conservation law. There are still two possibilities: (a) Incomplete multiplets are realized in nature; (b) they are not realized. The basis of an incomplete multiplet no longer constitutes an irreducible representation of the group. Perhaps this is the most serious objection against (a). On the other hand, if the incomplete multiplet is forbidden as a whole as in (b), then it is equivalent to omit all the exotic multiplets in the conventional way ($I=2$ for the $\pi\pi$ state). One should ask whether such an incomplete multiplet violates well-established principles of physics. As noted before, the charge independence will be violated in scattering processes if such particles are exchanged. We found no further difficulty. For the $\pi\Sigma$ and the $\bar{K}\Delta$ scatterings, the gapped-charge states will be observed if Γ_1^* is dominant. These states simulate the $I=1$ state but are different from the latter in that the neutral components are absent in the particle spectrum.[13]

This is not so strange a possibility. We know one similar example in the case of the ordinary spin, i.e., polarization states of the real photon. One difficulty in identifying such resonances in the existing data lies in the fact that the usual analyses are always done by assuming conventional isospin symmetry.[14] A more definite conclusion may be obtained from the ratio

$$\sigma(\Delta^0K^-\to\Delta^0K^-):\sigma(\Delta^0K^-\to\Delta^-\bar{K}^0):\sigma(\Delta^{++}K^-\to\Delta^+\bar{K}^0):\sigma(\Delta^{++}K^-\to\Delta^{++}K^-)=1:3:3:9 \text{ for } \Gamma_1 = 9:3:3:1 \text{ for } \Gamma_1^* . \tag{11}$$

If the Δ^0-exchange contribution is sufficiently well separated from the n-exchange contribution in K^-p backward scattering, then this ratio will give us an interesting test of our assumptions.[15] We have as yet no conclusive evidence on these points.

It is interesting to see what should happen if the assumption (c) is relaxed. Clearly a quite general way of the assignment of the charge is obtained by a replacement, $\vec{\pi}\to M\vec{\pi}$, for pions for example, by using a unitary 3×3 matrix M. The permutation between definitely charged components will be a special case of it. However, we have no experimental evidence for such a generalization at present.

Finally we study the decomposition for the two-meson state. The $\pi\pi$ state is decomposed as

$$\Gamma_1\times\Gamma_1=\Gamma_0+\Gamma_1+\Gamma_1+\Gamma_0'+\Gamma_0'' ,$$

$$|\pi\rangle\times|\pi\rangle=\frac{1}{\sqrt{3}}(\pi^-\pi^+-\pi^0\pi^0+\pi^+\pi^-)$$

$$+\begin{cases}\frac{1}{\sqrt{2}}(\pi^+\pi^0-\pi^0\pi^+)\\ \frac{1}{\sqrt{2}}(\pi^+\pi^--\pi^-\pi^+)\\ -\frac{1}{\sqrt{2}}(\pi^-\pi^0-\pi^0\pi^-)\end{cases} + \begin{cases}\frac{1}{\sqrt{2}}(\pi^-\pi^0+\pi^0\pi^-)\\ \frac{1}{\sqrt{2}}(\pi^+\pi^+-\pi^-\pi^-)\\ -\frac{1}{\sqrt{2}}(\pi^+\pi^0+\pi^0\pi^+)\end{cases}$$

$$+\left[\frac{1}{\sqrt{6}}(\pi^-\pi^+-2\pi^0\pi^0+\pi^+\pi^-)-\frac{1}{\sqrt{2}}(\pi^+\pi^++\pi^-\pi^-)\right]C_1$$

$$+\left[\frac{1}{\sqrt{6}}(\pi^-\pi^+-2\pi^0\pi^0+\pi^+\pi^-)+\frac{1}{\sqrt{2}}(\pi^+\pi^++\pi^-\pi^-)\right]C_2 , \tag{12}$$

where C_1 and C_2 are normalization constants.

Another interesting example is the πK state. It is given by[16]

$$\Gamma_1 \times \Gamma_{1/2} = \Gamma_{1/2} + \Gamma'_{1/2} + \Gamma''_{1/2} ,$$

$$|\pi\rangle \times |K\rangle = \begin{cases} \frac{1}{\sqrt{3}} \pi^0 K^+ - (\frac{2}{3})^{1/2} \pi^+ K^0 \\ (\frac{2}{3})^{1/2} K^+ \pi^- - \frac{1}{\sqrt{3}} \pi^0 K^0 \end{cases} + \begin{cases} -\frac{1}{\sqrt{3}} \pi^- K^+ - (\frac{2}{3})^{1/2} \pi^0 K^0 \\ (\frac{2}{3})^{1/2} \pi^+ K^+ + \frac{1}{\sqrt{3}} \pi^- K^0 \end{cases} + \begin{cases} \frac{1}{\sqrt{3}} \pi^+ K^+ - (\frac{2}{3})^{1/2} \pi^- K^0 \\ -(\frac{2}{3})^{1/2} \pi^0 K^+ - \frac{1}{\sqrt{3}} \pi^+ K^0 . \end{cases} \quad (13)$$

In (13), $\Gamma_{1/2}$, $\Gamma'_{1/2}$, and $\Gamma''_{1/2}$ are equivalent ray representations. If incomplete multiplets can be neglected, we get only a conventional $I = \frac{1}{2}$ state ($\Gamma_{1/2}$). This decomposition is particularly interesting when we note that in the customary theory the effective Hamiltonian for the nonleptonic weak interaction is assumed to have the same isospin structure as the $K\pi$ system. More specifically,

$$H_{wk} = \text{const} \times \int d^3x J_\mu(x)^{\Delta S=0}_{\Delta I=1} \times J_\mu(x)^{\Delta S=1/2}_{\Delta I=0} .$$

Thus, under the assumption that incomplete multiplets ($\Gamma'_{1/2}$ and $\Gamma''_{1/2}$) do not occur, we are naturally led to the $\Delta I = \frac{1}{2}$ rule. Experimentally the $\Delta I = \frac{3}{2}$ part certainly exists and this fact may indicate either that the incomplete multiplet has contributions or that the assumed form of the Hamiltonian is not appropriate.[17] The most attractive way will be to relate the current operators with the generators of the finite group in the way of (5) and to write the Hamiltonian in terms of them in the usual way. However, we must wait for a much more detailed, quantitative analysis on this subject before reaching a definite conclusion.

IV. CONCLUDING REMARKS

If we accept the group concept seriously in any physical applications, we should find some physical method to determine the precise structure of the group. The symmetry under a continuous group can be established and meaningful only when some experimental procedure is actually given to prove the symmetry at every value of the continuous parameters. The precise group structure responsible for the isospin symmetry is not yet known in this sense, even if the group concept is actually relevant in this symmetry. We note, however, that this is essential in forming the Clebsch-Gordan series.

In this work we made a preliminary attempt to clarify the isospin symmetry. We classified particles into the irreducible ray representations of finite subgroups of O(3). If exotic mesons and baryons are established by experiment, then we may still classify them by an icosahedral group as long as their isospins are sufficiently small.

In atomic physics, discrete energy levels of hydrogen atoms were explained by the standing-wave condition for the de Broglie wave. Our assignment of mesons and baryons into the ray representations of finite subgroups of O(3) is therefore in close similarity to it. It is quite a remarkable fact that these finite groups are generated by reflections. They are analogous to familiar C, P, T transformations.

Finally we should stress again that we have not as yet any conclusive evidence for our scheme. If incomplete multiplets are not observed, it simply means that we cannot distinguish between our formulation and the conventional one. Then a far more elaborate framework is certainly necessary to test our assumptions. The detailed knowledge from electromagnetic and weak-interaction processes will be indispensable in that case.

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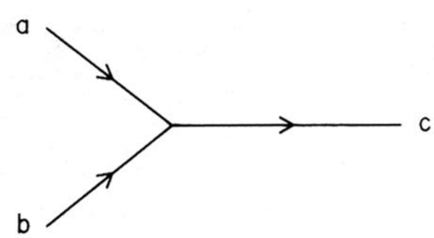

FIG. 1. The basic diagram for the scattering process. Three lines a, b, and c represent nonexotic particles.