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Why are there only two meson nonet structures? SU(4) provides an answer*

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In the theoretical framework of asymptotic SU(4), chiral SU(4) ⊗ (4) charge algebras, SU(4) breaking characterized by exotic commutators, and the hypothesis of asymptotic 16-plet realization of SU(4) in the algebra $[A_i, A_j] = if_{ijk}V_k$, three general pure mass relations are derived without using any approximation. One of the remarkable features of the theory is that its SU(3) limit predicts the existence of only two distinct nonet mass splittings which accommodate the existing patterns well. The other is that it enables us to predict reasonable masses of η_c, D, and F *solely* from an input of pseudoscalar-nonet masses.

I. INTRODUCTION

The unusual pattern of mass splittings within the 0^{-+} nonet, as compared to those within other SU(3) nonets (1^{--}, 2^{++}, etc.), has been one of the puzzles of hadron physics. Schwinger's nonet mass formula (SNF), which was discovered[1] in various ways in SU(3) and is well satisfied by the 1^{--} and 2^{++} nonets, fails for the 0^{-+} nonet since it predicts the mass of the ninth 0^{-+} meson η' to be around 1.6 GeV, while the best candidate is $\eta'(958)$. In this paper, we demonstrate that the observed patterns of SU(3)-nonet splittings (*including* that of the 0^{-+} nonet) can be accommodated in the framework of SU(4) [but *not* SU(3)] by taking an SU(3) limit of the general 15 ⊕ 1-plet mass relations in SU(4) which we derive. Furthermore, the small *deviation* of the observed SU(3) 0^{-+} mass splitting from that predicted in the SU(3) limit sets a reasonable scale, for example, for the mass of the charmed meson D *without adding* any further considerations. Our result is also consistent with the mass of $\eta_c(0^{-+})$ in the range 2.7–2.8 GeV.[2]

II. DERIVATION OF SU(4) CONSTRAINTS

We denote the 15 ⊕ 1-plet as (π_r, K_r, η_r, η_{cr}, D_r, F_r and η_r') where r denotes the J^{PC} and other quantum numbers. A nonperturbative approach[3] to broken SU(4)—the method of asymptotic SU(4) and asymptotic algebraic realization of SU(4) in the chiral SU(4)⊗ SU(4) charge algebras of V_α and A_α—is used. The same approach in broken SU(3) produced[4] the SNF

$$(3\eta_r'^2 + \pi_r^2 - 4K_r^2)(3\eta_r^2 + \pi_r^2 - 4K_r^2) = -8(K_r^2 - \pi_r^2)^2 \tag{1}$$

as the *general* nonet mass formula. In SU(4) we obtain three mass formulas, which can be used to predict D_r, F_r, and η_{cr} using *only* nonet masses as input. According to asymptotic SU(4), we define the SU(4) η_r-η_r'-η_{cr} mixing parameters, among the annihilation operators of physical particles and the SU(4)-representation operators a_8^r, a_0^r, and a_{15}^r, in the infinite-momentum limit as follows:

$$\begin{bmatrix} a_\eta^r \\ a_{\eta'}^r \\ a_{\eta c}^r \end{bmatrix} = \begin{bmatrix} \alpha_8^r & \alpha_0^r & \alpha_{15}^r \\ \beta_8^r & \beta_0^r & \beta_{15}^r \\ \gamma_8^r & \gamma_0^r & \gamma_{15}^r \end{bmatrix} \begin{bmatrix} a_8^r \\ a_0^r \\ a_{15}^r \end{bmatrix} . \tag{2}$$

The mixing parameter α_8, β_8, etc. can be expressed[5] in terms of three mixing angles (θ, ϕ, ψ). By realizing (in our asymptotic limit) the exotic commutators[3] $[\dot{V}_\alpha, V_\beta] = 0$, where $\dot{V}_\alpha = (d/dt)V_\alpha$ and (α, β) stands for all the exotic combinations of the physical SU(4) indices, four independent SU(4) constraints (in mass squared) are obtained,[3]

$$(\alpha_8^r)^2\eta_r^2 + (\beta_8^r)^2\eta_r'^2 + (\gamma_8^r)^2\eta_{cr}^2 = \tfrac{1}{3}(4K_r^2 - \pi_r^2) , \tag{3}$$

$$\alpha_8^r\alpha_{15}^r\eta_r^2 + \beta_8^r\beta_{15}^r\eta_r'^2 + \gamma_8^r\gamma_{15}^r\eta_{cr}^2 = -\frac{\sqrt{2}}{3}(K_r^2 - \pi_r^2) , \tag{4}$$

$$(\alpha_{15}^r)^2\eta_r^2 + (\beta_{15}^r)^2\eta_r'^2 + (\gamma_{15}^r)^2\eta_{cr}^2 = \tfrac{1}{6}(9D_r^2 + K_r^2 - 4\pi_r^2) , \tag{5}$$

$$(F_r^2 - D_r^2) = (K_r^2 - \pi_r^2) . \tag{6}$$

For the case $C_rC_t = +1$, we obtain[3] two more independent constraints from the exotic commutators[6] $[\dot{V}_\alpha, A_\beta] = 0$, namely,

$$\alpha_8^r(\eta_r^2 - \pi_r^2)A^{rt} + \beta_8^r(\eta_r'^2 - \pi_r^2)B^{rt} + \gamma_8^r(\eta_{cr}^2 - \pi_r^2)C^{rt} = 0 , \tag{7}$$

$$\alpha_{15}^r(\eta_r^2 - \pi_r^2)A^{rt} + \beta_{15}^r(\eta_r'^2 - \pi_r^2)B^{rt} + \gamma_{15}^r(\eta_{cr}^2 - \pi_r^2)C^{rt} = 0, \tag{8}$$

where we define $A^{rt} \equiv \langle \eta_r | A_{\pi^-} | \pi_t^+ \rangle$, $B^{rt} \equiv \langle \eta_r' | A_{\pi^-} | \pi_t^+ \rangle$ and $C^{rt} \equiv \langle \eta_{cr} | A_{\pi^-} | \pi_t^+ \rangle$. The commutators $[V_i, A_j] = if_{ijk}A_k$ yield the additional constraint among the asymptotic matrix elements A^{rt}, B^{rt}, and C^{rt},

$$(\alpha_8^r - \sqrt{2}\,\alpha_{15}^r)A^{rt} + (\beta_8^r - \sqrt{2}\,\beta_{15}^r)B^{rt} + (\gamma_8^r - \sqrt{2}\,\gamma_{15}^r)C^{rt} = 0 . \tag{9}$$

These constraints, Eqs. (3)–(9), have been studied[3] with approximation. Making a straightforward extension from SU(3) to SU(4), we now add the hypothesis of asymptotic algebraic (or level) realization[4,7] of SU(4) in the algebra $[A_i, A_j] = i f_{ijk} V_k$ which yields one last constraint,

$$(A^{rt})^2 + (B^{rt})^2 + (C^{rt})^2 = 3(\alpha_8^r A^{rt} + \beta_8^r B^{rt} + \gamma_8^r C^{rt})^2 . \tag{10}$$

The result of this paper as well as the successful result[7] in SU(3) seem to lend strong support for the hypothesis introduced.

III. EXACT SOLUTION OF SU(4) CONSTRAINTS

We now solve Eqs. (3)–(10) exactly. Using Eq. (9) we eliminate C^{rt} from Eqs. (7), (8), and (10). The ratio (A^{rt}/B^{rt}) can then be eliminated from Eq. (10) using either Eq. (7) or Eq. (8) yielding two independent mass-mixing angle equations which, after simplification using Eqs. (3), (4), and (5), become

$$3(\eta_{cr}^2 - \eta_r^2)(\eta_r'^2 - \eta_r^2)(\alpha_8^r)^2 = N_r + (3\eta_r'^2 + \pi_r^2 - 4K_r^2)(\eta_{cr}^2 - \eta_r^2) , \tag{11}$$

$$6(\eta_{cr}^2 - \eta_r^2)(\eta_r'^2 - \eta_r^2)(\alpha_{15}^r)^2 = (2D_r^2 - \eta_{cr}^2 - \pi_r^2)(9D_r^2 + K_r^2 - 6\eta_r'^2 - 4\pi_r^2) - (D_r^2 - K_r^2)(2K_r^2 + \pi_r^2 - 3\eta_r'^2) + 2(K_r^2 - \pi_r^2)(\eta_r'^2 - \pi_r^2) . \tag{12}$$

Here

$$N_r = \tfrac{1}{3}[(4K_r^2 - \pi_r^2 - 3\eta_r^2)(4K_r^2 - \pi_r^2 - 3\eta_r'^2) + 8(K_r^2 - \pi_r^2)^2]$$

and represents deviations from exact satisfaction of SNF. Substituting our mixing-angle expressions for α_8, β_8, etc. in Eqs. (3), (4), (5), (11), and (12) we obtain three equations defining the mixing angles and two (long awaited) pure mass formulas [in addition to Eq. (6)]:

$$3(\eta_{cr}^2 - \eta_r^2)(\eta_r'^2 - \eta_r^2)\sin^2\theta^s = (4K_r^2 - \pi_r^2 - 3\eta_r^2)(\eta_{cr}^2 - \eta_r^2) - N_r , \tag{13}$$

$$3(\eta_{cr}^2 - \eta_r'^2)\sin^2\theta^r \sin^2\psi^r = (4K_r^2 - \pi_r^2 - 3\eta_r^2) - 3(\eta_r'^2 - \eta_r^2)\sin^2\theta^r , \tag{14}$$

$$18(\eta_{cr}^2 - \eta_r^2)(\eta_r'^2 - \eta_r^2)\sin^2\theta^r \cos^2\phi^r = (9D_r^2 + K_r^2 - 6\eta_r^2 - 4\pi_r^2)(4K_r^2 - \pi_r^2 - 3\eta_r^2) - 4(K_r^2 - \pi_r^2)^2 , \tag{15}$$

$$(3\eta_{cr}^2 + \pi_r^2 - 4K_r^2)[2(2K_r^2 - \eta_r^2 - \eta_r'^2) - L_r] = [L_r^2 + L_r(4K_r^2 - 2\pi_r^2 - \eta_r^2 - \eta_r'^2) - 2(2K_r^2 - \eta_r^2 - \eta_r'^2)(4K_r^2 - 2\pi_r^2 - \eta_r^2 - \eta_r'^2) + 4N_r] , \tag{16}$$

$$N_r[N_r - 2(K_r^2 - \pi_r^2)^2](3\eta_{cr}^2 + \pi_r^2 - 4K_r^2)^2 + 2N_r(K_r^2 - \pi_r^2)^2[L_r - (8K_r^2 - 2\pi_r^2 - 3\eta_r'^2 - 3\eta_r^2)](3\eta_{cr}^2 + \pi_r^2 - 4K_r^2) + 4(K_r^2 - \pi_r^2)^4(L_r^2 + 4N_r) = 0 , \tag{17}$$

where $L_r \equiv (6D_r^2 - 3\pi_r^2 - 3\eta_{cr}^2 - \eta_r^2 - \eta_r'^2 + 2K_r^2)$. If we impose that η_{cr} is a pure $c\bar{c}$ state (i.e., $\phi^r = 30°$ and $\psi^r = 0°$), these five equations *produce* the ideal nonet mass relations, $\eta_r'^2 = \pi_r^2$ and $\eta_r^2 - K_s^2 = K_s^2 - \pi_s^2$, the equal spacing $\eta_{cs}^2 - D_s^2 = D_s^2 - \pi_s^2$, and the remarkable selection rule $A^{st} = C^{st} = 0$ (t is arbitrary).

IV. PREDICTION ON THE NONET MESON MASS SPLITTINGS

It is natural to study first the SU(3) limit by taking a limit of infinite charmed-quark mass. If the limit, $D_r^2 \to \infty$ and $\eta_{cr}^2 \to \infty$, is applied to Eq. (16) one obtains $2D_r^2 = \eta_{cr}^2$, a statement of equal spacing. Then the same limit applied to Eq. (17) yields $N_r[N_r - 2(K_r^2 - \pi_r^2)^2] = 0$. Therefore, in the SU(3) limit we now obtain two possible solutions, hence we *predict two* possible nonet structures. The first solution, $N_r = 0$, is the SNF [i.e., Eq. (1)] and is well satisfied by the almost ideal[8] nonets, such as 1^{--} and 2^{++}. The second solution

$$[N_r - 2(K_r^2 - \pi_r^2)^2] = 0$$

or

$$\left(\eta_r^2 - \frac{4K_r^2 - \pi_r^2}{3}\right)\left(\eta_r'^2 - \frac{4K_r^2 - \pi_r^2}{3}\right) + \tfrac{2}{9}(K_r^2 - \pi_r^2)^2 = 0 \tag{18}$$

coincides with *one* of the three *15-plet* mass formulas[9] derived in the very early work on SU(4) and is *well* satisfied by the 0^{-+} nonet (η' is predicted to be around 0.943 GeV). Therefore, we find that the SU(3) limit of our mass formulas accommodates nicely the observed *two distinct patterns* of SU(3) nonets. We show below that the finite masses of η_c and D do not modify the result significantly as long as they are reasonably heavy. Since Eq. (18) is found only in the framework of SU(4), the somewhat eccentric structure of the 0^{-+} nonet may be considered an early manifestation of an SU(4) effect.[10]

The width of the η_{cr} is greatly affected by the size of the small SU(4) angle ψ^r. Using Eq. (13), we rewrite Eq. (14) as $3(\eta_{cr}^2-\eta_r^2)(\eta_{cr}^2-\eta_r'^2)\sin^2\theta^r \sin^2\psi^r = N_r$. Thus there is a clear difference in the size of ψ^r for the two types of 16-plets. If N_r is very small, as is the case for the 1^{--} and 2^{++}, ψ^r will also be small and the width of η_{cr} will be narrow. For the 0^{-+}, N_r is much larger, leading to a value of ψ^r and consequently a much broader η_{cr} width. From Eq. (13) the SU(4) *correction* to the SU(3) mass formula $3(\eta_r'^2-\eta_r^2)\sin^2\theta^r=(4K_r^2-\pi_r^2-3\eta_r^2)$ is $N_r/(\eta_{cr}^2-\eta_r^2)$. Thus the η_r-η_r' mixing angle will remain largely unchanged for the ideal nonet while the angle for the 0^{-+} nonet could be significantly changed.

V. CALCULATION OF THE MASSES OF D AND η_c

Using Eqs. (16) and (17), the masses of D_r and η_{cr} can be calculated *solely*[11] in terms of the SU(3)-nonet particles π_r, K_r, η_r, and η_r'. In general, there will be four solutions i.e., four pairs of masses (D_r, η_{cr}). We have done the calculation for the 0^{-+} $15\oplus 1$-plet, inputting the masses of π_r, K_r, η_r, and η_r', using the average of charged and neutral mass squareds for π_r and K_r. Our predicted values are $D=1.65$ GeV and $\eta_c=2.39$ GeV. The other three possible solutions are unphysical and can be discarded. To test the sensitivity of our physical solution, we varied η' between 0.938 and 1.000 GeV. Our results are presented graphically in Fig. 1. The masses of η_c and D, of course, go asymptotically to infinity[12] for $\eta'=0.9425$ GeV. For $\eta' < 0.942$ GeV, the mass of η_c becomes imaginary (i.e., η_c^2 is negative). For $\eta' \gtrsim 0.980$ GeV, η_c again becomes complex. Between $\eta'=0.960$ and 0.951 GeV, η_c gradually increases from 2.26 to 3.12 GeV and at $\eta'=0.954$, a shift of only 4 MeV from its measured value, we obtain $D=1.87$, $F=1.94$, and $\eta_c=2.69$ GeV, reasonably consistent with present experimental values.[2] (The $c\bar{c}$ contents are 0.03% for η, 0.53% for η', and 99.44% for η_c.) Therefore, our mass

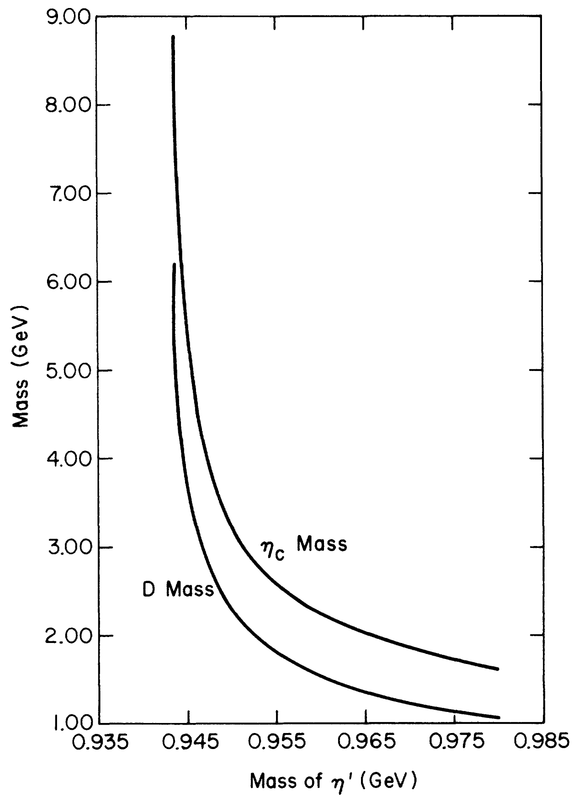

FIG. 1. Dependence of the η_c and D masses on the input mass of η'.

formulas give a realistic scale for charmed masses and, with a small shift in the measured mass of the η', predict reasonable masses for D, F, and η_c.

Although the qualitative features of the 0^{-+} 16-plet now seem to be much clarified, detailed agreement with experiment will require the inclusion of the effects of SU(2) breaking, SU(4) mixing with the radially excited states,[13] and the possible existence of more quarks. These effects will also certainly play a role for the decays involving the η_c such as $J/\psi \to \eta_c + \gamma$. One immediate consequence (compare with Ref. 3) of the now favored choice of $\eta' \equiv X(958)$ over $\eta' \equiv E(1420)$ is that the hadronic η_c width becomes smaller (< 50 MeV) and the main modes will be $\eta_c \to K^*\bar{K}$ and $A_2\pi$.

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*This paper contains part of the thesis research of H. L. Hallock to be submitted to the University of Maryland in partial fulfillment of the requirements for a Ph. D. degree in physics.

[1]J. Schwinger, Phys. Rev. Lett. 12, 237 (1964) and the references cited in Refs. 3 and 4.

[2]S. Yamada, in *Proceedings of INS International Symposium on New Particles and the Structure of Hadrons, University of Tokyo, 1977*, edited by K. Fujikawa, Y. Hara, and H. Terazawa (Institute for Nuclear Study, Tokyo, 1977), p. 33; R. Schwitters, *ibid.* p. 2.

[3]E. Takasugi and S. Oneda, Phys. Rev. D 12, 198 (1975); 13, 70 (1976). Earlier works on 16-plets based on conventional SU(4) are cited there.

[4]Seisaku Matsuda and S. Oneda, Phys. Lett. 37B, 105 (1971). See also Milton D. Slaughter and S. Oneda, Phys. Rev. D 15, 879 (1977). This SU(3) calculation, which included SU(2) breaking, still predicts mass values in the vicinity of those predicted by the SNF.

[5]We use $\alpha_8 = \cos\theta$, $\beta_8 = \sin\theta\cos\psi$, $\gamma_8 = \sin\theta\sin\psi$, $\alpha_0 = -\sin\theta\cos\phi$, $\beta_0 = -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi$, $\gamma_0 = \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi$, $\alpha_{15} = -\sin\theta\sin\phi$, $\beta_{15} = \cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi$, $\gamma_{15} = -\cos\phi\cos\psi + \cos\theta\sin\phi\sin\psi$.

[6]$[\dot{V}_\alpha, V_\beta] = 0$ and $[\dot{V}_\alpha, A_\beta] = 0$ are weaker assumptions than the usual pure $(4, 4^*) \oplus (4^*, 4)$ breaking.

[7]For a recent review, see S. Oneda, in *Proceedings of INS International Symposium on New Particles and the Structure of Hadrons, University of Tokyo, 1977,* edited by K. Fujikawa, Y. Hara, and H. Terazawa (Institute for Nuclear Study, Tokyo, 1977), p. 94.

[8]However, SNF is a much better mass formula for the 1^{--} and 2^{++} than the ideal nonet formula.

[9]J. D. Bjorken and S. L. Glashow, Phys. Lett. 11, 255 (1964); D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Nuovo Cimento 34, 1732 (1964). Two other mass formulas derived predict too low masses for the D and F. Assumption of the 15-plet is certainly not realistic. Z. Maki, T. Maskawa, and I. Umemura [Prog. Theor. Phys. 47, 1682 (1972)] used Eq. (18) as an empirical relationship in their early work on the 16-plet. For review on the 15- and 16-plet, see M. Nakagawa, Meijo University Report No. DP-MJU-601, 1976 (unpublished).

[10]We note that the papers cited in Ref. 3, in which constraint (10) was not imposed, produce results consistent with ours, *provided* the input of the physical masses of both η' and D are made. In fact, on the basis of hadronic decay rate calculations, these earlier papers seem to favor the choice of $\eta' = X$. It is amusing to note that applying the limiting procedure employed in Refs. 3 (i.e., $\eta_c{}^2 \to \infty$, $\gamma_8^s \to 0$ but $\gamma_8^s \eta_c{}^2 \to 0$) to our Eqs. (13) and (14), we obtain the SNF, as was the case in Refs. 3. However, this simple limit is not sufficient to derive our other nonet formulas, Eq. (18).

[11]However, for multiplets other than the 0^{-+} this method is not practical, since the masses involve large uncertainties.

[12]Corresponding to SNF, another singular behavior takes place around $\eta' \simeq 1.6$ GeV.

[13]For example, D. H. Boal, Phys. Rev. Lett. 37, 1333 (1976).

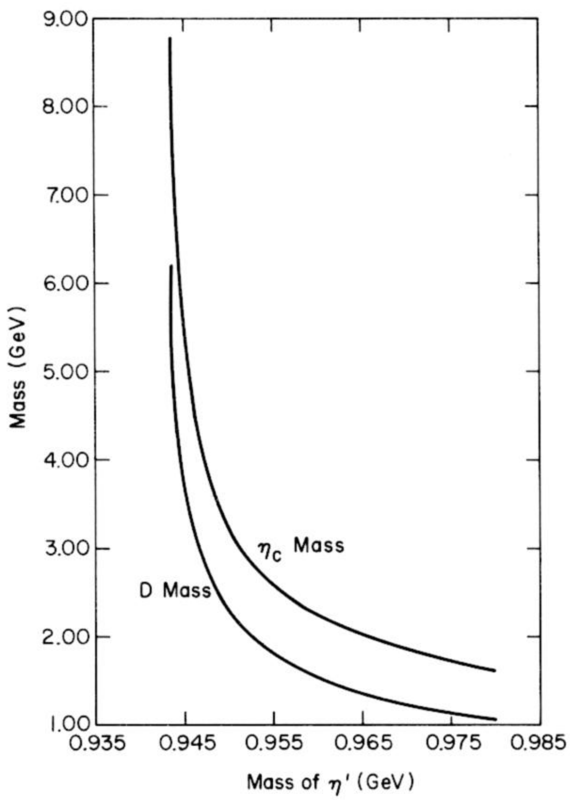

FIG. 1. Dependence of the η_c and D masses on the input mass of η'.