Exactly solvable wave equation with a linear confining potential. II. Symmetric model

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A model based on an exactly solvable wave equation with a linear confining potential is constructed which is time-reversal invariant and which allows the definition of a conserved parity and C parity. Sets of linear and parallel Regge trajectories are obtained. A discussion is given of the states which have the properties of the ρ^0 and π^0 mesons.

I. INTRODUCTION

In the preceding article[1] a model has been presented which possibly describes the motion of a light quark in the field of a heavy (static) antiquark. The main characteristic of this model which is based on the Dirac equation in first quantization (hole theory) is the appearance of a linear potential term in the wave equation. The way this potential has been constructed is such that the ensuing equations are rendered exactly solvable and lead to Regge trajectories which are linear when the total angular momentum is plotted against the square of the total energy of the light quark.

Due to the impossibility of defining a workable charge-conjugation operation this model is not fitted for the description of mesons which possess a well-defined C or G parity, but may lead to an approximate description of strange or charmed mesons in which a heavy and a light quark or antiquark can be distinguished.

We shall now make use of the results of the preceding article in order to construct a model for which such a parity can indeed be defined. We shall call this the symmetric model, which possibly describes the interaction between a quark and its own antiquark. The equations can again be solved exactly and lead to Regge trajectories which turn out to be linear when the total angular momentum is plotted against the mass squared of the composite particle.

Since this model is based on the Dirac equation in first quantization, difficulties of interpretation appear which are the typical result of using hole theory for the description of two interacting Dirac particles. At this moment we have not found a satisfactory model based on second quantization and the unpleasant features must be dealt with *ad hoc*.

There appear to be two ways of constructing a symmetric model. They have the special feature that they seem to be able to produce states which have the properties of the ρ^0 and π^0 mesons. It is for this reason that special attention is given to these states.

In Sec. II the two symmetric models are constructed, starting from the asymmetric Hamiltonian of Ref. 1. Both allow the definition of a suitable conserved C parity. In order for the equations to describe particles such as the π^0 or ρ^0 meson which are their own antiparticles another requirement must be met. This is discussed in Sec. III. In Sec. IV and V the eigenstates and eigenvalues of the new Hamiltonians are written down. It appears that spurious states come about. These are separately studied in Sec. VI. Sections VII and VIII are devoted to the states which describe the physical ρ^0 and π^0 mesons.

II. DEFINITION OF THE MODEL

As has been shown in the preceding article,[1] which shall henceforth be called I, the "asymmetric" Hamiltonian ($\hbar = c = 1$)

$$\mathcal{H}_{as}(m) = \vec{\pi} \cdot \vec{p} + \alpha\vec{\rho} \cdot \vec{r} + m\zeta \quad (\alpha > 0)\,, \tag{2.1}$$

where $\vec{p}$ and $\vec{r}$ satisfy canonical commutation relations

$$[p_i, r_j] = \frac{1}{i}\delta_{ij} \quad (i,j = 1,2,3) \tag{2.2}$$

and where $\vec{\pi}$, $\vec{\rho}$, and ζ are 8×8 Hermitian matrices of the form

$$\begin{aligned} \vec{\pi} &= \vec{\sigma}\otimes I_2 \otimes \sigma_x\,, \\ \vec{\rho} &= I_2 \otimes \vec{\sigma} \otimes \sigma_y\,, \\ \zeta &= I_2 \otimes I_2 \otimes \sigma_z\,, \end{aligned} \tag{2.3}$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\vec{\sigma}$ is a set of Pauli spin matrices, leads to wave equations which are exactly solvable. The solutions have been tabulated in I.

It has also been shown in I that parity and time-reversal operators can be defined and that these operators commute with $\mathcal{H}_{as}$. Also a conserved total angular momentum $\vec{J}$ can be defined:

$$\vec{J}=\vec{r}\times\vec{p}+\tfrac{1}{2}(\vec{\Sigma}_p+\vec{\Sigma}_r)\,, \tag{2.4}$$

with

$$\vec{\Sigma}_p=\vec{\sigma}\otimes I_2\otimes I_2=\eta\vec{\pi}$$

and

$$\vec{\Sigma}_r=I_2\otimes\vec{\sigma}\otimes I_2=\theta\vec{\rho}\,. \tag{2.5}$$

The "asymmetry" of the model comes from the fact that $\vec{\Sigma}_p$ must be associated with the spin of the light quark while $\vec{\Sigma}_r$ must be associated with the spin of the heavy quark. They do not occur in a symmetric way in the Hamiltonian. Associated with this is the impossibility of defining a conserved quantity which has the characteristics of a C parity. Such an operator should at least contain the spin exchange operator

$$\mathfrak{U}=\tfrac{1}{2}(I+\vec{\Sigma}_p\circ\vec{\Sigma}_r) \tag{2.6}$$

which has the property

$$\mathfrak{U}\vec{\Sigma}_p=\vec{\Sigma}_r\mathfrak{U}\,,\quad \mathfrak{U}\vec{\Sigma}_r=\vec{\Sigma}_p\mathfrak{U}. \tag{2.7}$$

However, in I a Hermitian quantity Q has been introduced which commutes with $\mathcal{H}_{as}$ and which itself shows characteristics of a Hamiltonian,

$$Q(m)=i\zeta(\vec{\rho}\circ\vec{p}-\alpha\vec{\pi}\circ\vec{r})+im\vec{\rho}\circ\vec{\pi}\,. \tag{2.8}$$

It is with the help of Q that Hamiltonians can be constructed which allow a satisfactory definition of C parity. We have

$$[\mathcal{H}_{as}(m),\mathfrak{U}]=-[Q(m),\mathfrak{U}]$$
$$=-i(\vec{\Sigma}_p\times\vec{\Sigma}_r)\cdot(\eta\vec{p}-\alpha\theta\vec{r})\,, \tag{2.9}$$

from which it follows that

$$[\mathcal{H}_{as}(m)+Q(m),\mathfrak{U}]=0 \tag{2.10}$$

We have also

$$[\mathcal{H}_{as}(m),\zeta\mathfrak{U}]=[Q(m),\zeta\mathfrak{U}]$$
$$=\zeta(\vec{\Sigma}_p+\vec{\Sigma}_r)\circ(\eta\vec{p}+\alpha\theta\vec{r})\,, \tag{2.11}$$

from which we obtain

$$[\mathcal{H}_{as}(m)-Q(m),\zeta\mathfrak{U}]=0\,. \tag{2.12}$$

If P is the parity operation defined by

$$P=\epsilon\bar{P}\zeta\,, \tag{2.13}$$

where $\bar{P}$ is the ordinary parity operation and ϵ is the intrinsic parity, we can now define charge-conjugation parity together with a new Hamiltonian in two ways:

(a) Let the new Hamiltonian be

$$\mathcal{H}^{I}(m)=\mathcal{H}_{as}(m)+Q(m)\,, \tag{2.14}$$

and the charge-conjugation operator

$$C^{I}=P\mathfrak{U}\,. \tag{2.15}$$

Then

$$\begin{aligned}
&C^I\vec{\Sigma}_p=\vec{\Sigma}_rC^I\,,\\
&C^I\vec{\Sigma}_r=\vec{\Sigma}_pC^I\,,\\
&C^I\mathcal{H}^I(m)=\mathcal{H}^I(m)C^I\,,\\
&C^I\vec{p}=-\vec{p}C^I\,,\\
&C^I\vec{r}=-\vec{r}C^I\,,\\
&C^I\eta=-\eta C^I\,,\\
&C^I\theta=-\theta C^I\,.
\end{aligned} \tag{2.16}$$

We obtain for $\mathcal{H}^I(m)$ the following explicit form:

$$\mathcal{H}^I(m)=2\vec{S}\circ(\eta\vec{p}+\alpha\theta\vec{r})+2\zeta(S^2-1)m\,, \tag{2.17}$$

where

$$\vec{S}=\tfrac{1}{2}(\vec{\Sigma}_p+\vec{\Sigma}_r)\,. \tag{2.18}$$

Here the symmetric appearance of $\vec{\Sigma}_p$ and $\vec{\Sigma}_r$ is evident.

(b) If the new Hamiltonian is defined by

$$\mathcal{H}^{II}(m)=\mathcal{H}_{as}(m)-Q(m)\,, \tag{2.19}$$

and the charge-conjugation operator by

$$C^{II}=\zeta P\mathfrak{U}, \tag{2.20}$$

then we have

$$\begin{aligned}
&C^{II}\vec{\Sigma}_p=\vec{\Sigma}_rC^{II}\,,\\
&C^{II}\vec{\Sigma}_r=\vec{\Sigma}_pC^{II}\,,\\
&C^{II}\mathcal{H}^{II}(m)=\mathcal{H}^{II}(m)C^{II}\,,\\
&C^{II}\vec{p}=-\vec{p}C^{II}\,,\\
&C^{II}\vec{r}=-\vec{r}C^{II}\,,\\
&C^{II}\eta=\eta C^{II}\,,\\
&C^{II}\theta=\theta C^{II}\,.
\end{aligned} \tag{2.21}$$

The explicit form for $\mathcal{H}^{II}$ is the following:

$$\mathcal{H}^{II}(m)=2\vec{S}'\cdot(\eta\vec{p}-\alpha\theta\vec{r})+2\zeta(2-S^2)m\,, \tag{2.22}$$

where

$$\vec{S}'=\tfrac{1}{2}(\vec{\Sigma}_p-\vec{\Sigma}_r)\,. \tag{2.23}$$

Here $\vec{\Sigma}_p$ and $\vec{\Sigma}_r$ appear antisymmetrically.

III. RESTRICTIONS FOR $\mathcal{H}^{I}$ AND $\mathcal{H}^{II}$

For a satisfactory description of neutral mesons it is not enough to be able to define a conserved C parity. There should be the added requirement that the absence of negative-energy particles is interpreted as the presence of positive-energy antiparticles of opposite total angular momentum. However, particles and antiparticles are identical. We must therefore be able to find an operator which anticommutes with the Hamiltonian and with the total angular momentum. Such an operator is by necessity antilinear.

With the time-reversal operator T defined as in I,

$$T = K\sigma_y \otimes \sigma_y \otimes I_2\,, \tag{3.1}$$

which commutes with the Hamiltonians $\mathcal{H}^{\mathrm{I}}$ and $\mathcal{H}^{\mathrm{II}}$, we can construct an antilinear operator Θ defined by

$$\Theta = T\zeta \tag{3.2}$$

which has the following properties:

$$\Theta\mathcal{H}^{\mathrm{I}}(m) = -\mathcal{H}^{\mathrm{I}}(-m)\Theta\,, \tag{3.3}$$

$$\Theta\mathcal{H}^{\mathrm{II}}(m) = -\mathcal{H}^{\mathrm{II}}(-m)\Theta\,, \tag{3.4}$$

$$\Theta\vec{J} = -\vec{J}\,\Theta\,. \tag{3.5}$$

The operator Θ would qualify only if $m=0$.

This turns out to be acceptable for the Hamiltonian $\mathcal{H}^{\mathrm{II}}$ and then leads to a massless pion. For $\mathcal{H}^{\mathrm{I}}(m)$ the choice $m=0$ is too severe a restriction. If m is chosen purely imaginary we find

$$\Theta\mathcal{H}^{\mathrm{I}}(m) = -\mathcal{H}^{\mathrm{I}}(-m^*)\Theta = -\mathcal{H}^{\mathrm{I}}(m)\Theta\,, \tag{3.6}$$

so that the requirement is formally met. However, we now have to deal with a non-Hermitian Hamiltonian and with a seeming breakdown of time-reversal invariance. Part of the ill effects can be eliminated by carrying out a similarity transformation. The remaining bad features turn out to be peculiarities of a model based on hole theory and have no observable effects. We come back to this in Secs. IV, VI, and VII.

IV. PROPERTIES OF THE MODEL HAMILTONIAN $\mathcal{H}^{\mathrm{I}}(m)$

The eigenstates and eigenvalues of the equation

$$\mathcal{H}^{\mathrm{I}}(m)\psi = E\psi \tag{4.1}$$

can be found immediately with the help of the formalism developed in I. Let

$$\vec{a} = \frac{\alpha\vec{r} + i\vec{p}}{(2\alpha)^{1/2}} \tag{4.2}$$

and

$$\vec{\chi} = \frac{\vec{\mathfrak{p}} + i\vec{\pi}}{\sqrt{2}}\,. \tag{4.3}$$

Let R be defined by

$$R = \chi_i\chi_j\chi_k\epsilon_{ijk}\,, \tag{4.4}$$

and let ψ_0 be proportional to the one independent column of

$$Re^{-\alpha r^2/2}\,. \tag{4.5}$$

Moreover, let

$$A^\dagger_\pm(n) = A^\dagger_{(1)}(n) + \beta_\pm(n)A^\dagger_{(2)}(n)\,, \tag{4.6}$$

where

$$A^\dagger_{(1)}(n) = a^\dagger_{i_1}a^\dagger_{i_2}\cdots a^\dagger_{i_n}\,, \tag{4.7}$$

$$A^\dagger_{(2)}(n) = \frac{1}{n}(\zeta\chi^\dagger_{i_1}a^\dagger_{i_2}\cdots a^\dagger_{i_n} + a^\dagger_{i_1}\zeta\chi^\dagger_{i_2}a^\dagger_{i_3}\cdots a^\dagger_{i_n} + \cdots)\,, \tag{4.8}$$

and

$$\beta_\pm(n) = \frac{1}{\sqrt{\alpha}}\left[-\tfrac{1}{2}(m^2 + 2n\alpha)^{1/2} \pm \tfrac{1}{2}m\right]. \tag{4.9}$$

Then

$$A^\dagger_+(n)\psi_0 \quad\text{and}\quad \zeta A^\dagger_-(n)\psi_0 \tag{4.10}$$

are eigenstates of $\mathcal{H}_{\mathrm{as}}$ with eigenvalues $+(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$, respectively. They are also eigenstates of Q with eigenvalues $-2m-(m^2+2n\alpha)^{1/2}$ and $-2m+(m^2+2n\alpha)^{1/2}$, respectively, so we find

$$\mathcal{H}^{\mathrm{I}}(m)A^\dagger_+(n)\psi_0 = -2mA^\dagger_+(n)\psi_0 \tag{4.11}$$

and

$$\mathcal{H}^{\mathrm{I}}(m)\zeta A^\dagger_-(n)\psi_0 = -2m\zeta A^\dagger_-(n)\psi_0\,. \tag{4.12}$$

Since these states all have the same eigenvalue $-2m$, independent of n, they have no physical significance. This is one of two sets of spurious states which occur. These states are a necessary consequence of hole theory. We come back to this later. Next we define

$$B^\dagger_\pm(n) = B^\dagger_{(1)}(n) - \frac{n-1}{n}\beta_\mp(n)B^\dagger_{(2)}(n)\,, \tag{4.13}$$

where

$$B^\dagger_{(1)}(n) = \tfrac{1}{2}(a^\dagger_{i_1}\zeta\chi^\dagger_{i_n} - a^\dagger_{i_n}\zeta\chi^\dagger_{i_1})a^\dagger_{i_2}\cdots a^\dagger_{i_{n-1}} \tag{4.14}$$

and

$$B^\dagger_{(2)}(n) = \frac{1}{n-1}\left[\zeta\chi^\dagger_{i_1}\zeta\chi^\dagger_{i_n}a^\dagger_{i_2}\cdots a^\dagger_{i_{n-1}} + \tfrac{1}{2}(a^\dagger_{i_1}\chi^\dagger_{i_n} - a^\dagger_{i_n}\chi^\dagger_{i_1}) \times (\chi^\dagger_{i_2}a^\dagger_{i_3}\cdots a^\dagger_{i_{n-1}} + a^\dagger_{i_2}\chi^\dagger_{i_3}a^\dagger_{i_4}\cdots a^\dagger_{i_{n-1}} + \cdots)\right]. \tag{4.15}$$

Then

$$B^\dagger_+(n)\psi_0 \quad\text{and}\quad \zeta B^\dagger_-(n)\psi_0 \tag{4.16}$$

are again eigenstates of $\mathcal{H}_{\mathrm{as}}(m)$ with eigenvalues $+(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$ $(n\geq 2)$, respectively. They are also eigenstates of $Q(m)$ with eigenvalues $(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$, respectively, so we find

$$\mathcal{H}^{\mathrm{I}}(m)B^\dagger_+(n)\psi_0 = 2(m^2+2n\alpha)^{1/2}B^\dagger_+(n)\psi_0 \tag{4.17}$$

and

$$\mathcal{H}^{\mathrm{I}}(m)\zeta B^\dagger_-(n)\psi_0 = -2(m^2+2n\alpha)^{1/2}\zeta B^\dagger_-(n)\psi_0\,. \tag{4.18}$$

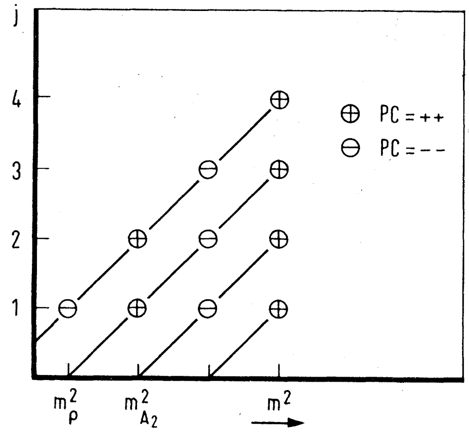


FIG. 1. Physical eigenvalue spectrum ($E > 0$) of the model Hamiltonian $\mathcal{H}^{\mathrm{I}}$.

This is a set of states with meaningful eigenvalues. When the total angular momentum is plotted against the square of the total energy, linear Regge trajectories appear. The ground state of the positive-energy spectrum has spin 1, negative parity (if ϵ is chosen to be -1), and negative C parity. All states belonging to this spectrum have the property

$$P = C^{\mathrm{I}}, \tag{4.19}$$

see Fig. 1. Finally we define

$$C^\dagger_\pm(n) = C^\dagger_{(1)}(n) + \frac{n-2}{n}\beta_\pm(n) C^\dagger_{(2)}(n)\,, \tag{4.20}$$

where

$$C^\dagger_{(1)}(n) = \tfrac{1}{3}(a^\dagger_{i_1}\zeta\chi^\dagger_{i_{n-1}}\zeta\chi^\dagger_{i_n} + \text{cycl})a^\dagger_{i_2}\cdots a^\dagger_{i_{n-2}} \tag{4.21}$$

and

$$C^\dagger_{(2)}(n) = \frac{1}{n-2}\zeta\chi^\dagger_{i_1}\zeta\chi^\dagger_{i_{n-1}}\zeta\chi^\dagger_{i_n}a^\dagger_{i_2}\cdots a^\dagger_{i_{n-2}} + \frac{1}{3(n-2)}(a^\dagger_{i_1}\zeta\chi^\dagger_{i_{n-1}}\zeta\chi^\dagger_{i_n} + \text{cycl}) \times (\zeta\chi^\dagger_{i_2}a^\dagger_{i_3}\cdots a^\dagger_{i_{n-2}} + \cdots)\,. \tag{4.22}$$

Then

$$C^\dagger_+(n)\psi_0 \quad\text{and}\quad \zeta C^\dagger_-(n)\psi_0 \tag{4.23}$$

are again eigenstates of $\mathcal{H}_{\mathrm{as}}$ with eigenvalues $(m^2 + 2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$ ($n \geq 3$), respectively. They are also eigenstates of Q with eigenvalues $2m - (m^2+2n\alpha)^{1/2}$ and $2m + (m^2+2n\alpha)^{1/2}$, respectively, so we find

$$\mathcal{H}^{\mathrm{I}}(m)C^\dagger_+(n)\psi_0 = 2mC^\dagger_+(n)\psi_0 \tag{4.24}$$

and

$$\mathcal{H}^{\mathrm{I}}(m)\zeta C^\dagger_-(n)\psi_0 = 2m\zeta C^\dagger_-(n)\psi_0\,. \tag{4.25}$$

Since these states all have the same eigenvalue $2m$, independent of n, they have no physical significance. This is the second set of spurious states as required by hole theory. The number of significant states is apparently equal to the number of spurious states.

When m is taken purely imaginary, as suggested in Sec. III, we find that the spurious states have purely imaginary energy eigenvalues. As long as $|m^2| < 4\alpha$, the physical states have real eigenvalues. Moreover, the positive-energy eigenstates can be chosen mutually orthogonal and so can the negative-energy eigenstates. The non-Hermiticity of the Hamiltonian manifests itself through the imaginary eigenvalues of the spurious states and through the fact that negative-energy states are not orthogonal to their *corresponding* positive-energy states.

It is an attractive possibility to identify the physical state of lowest positive energy with the ρ^0 meson. The physical mass of the ρ^0 meson indeed requires a purely imaginary value for m.

V. PROPERTIES OF THE MODEL HAMILTONIAN $\mathcal{H}^{\mathrm{II}}(m)$

With the help of the information provided in the preceding section we can immediately find the eigenstates and eigenvalues of the equation

$$\mathcal{H}^{\mathrm{II}}(m)\psi = E\psi\,. \tag{5.1}$$

The results are as follows:

$$\mathcal{H}^{\mathrm{II}}(m)\psi_0 = 4m\psi_0\,, \tag{5.2}$$

$$\mathcal{H}^{\mathrm{II}}(m)A^\dagger_+(n)\psi_0 = 2[m + (m^2+2n\alpha)^{1/2}]A^\dagger_+(n)\psi_0\,, \tag{5.3}$$

$$\mathcal{H}^{\mathrm{II}}(m)\zeta A^\dagger_-(n)\psi_0 = 2[m - (m^2+2n\alpha)^{1/2}]\zeta A^\dagger_-(n)\psi_0\,, \tag{5.4}$$

$$\mathcal{H}^{\mathrm{II}}(m)B^\dagger_+(n)\psi_0 = 0\,, \tag{5.5}$$

$$\mathcal{H}^{\mathrm{II}}(m)\zeta B^\dagger_-(n)\psi_0 = 0\,, \tag{5.6}$$

$$\mathcal{H}^{\mathrm{II}}(m)C^\dagger_+(n)\psi_0 = 2[-m + (m^2+2n\alpha)^{1/2}]C^\dagger_+(n)\psi_0\,, \tag{5.7}$$

$$\mathcal{H}^{\mathrm{II}}(m)\zeta C^\dagger_-(n)\psi_0 = 2[-m - (m^2+2n\alpha)^{1/2}]\zeta C^\dagger_-(n)\psi_0\,. \tag{5.8}$$

We now see that the roles of spurious and physical states have been interchanged. In order to meet the requirements of Sec. III the only realistic choice for m is zero.

For the states (5.3) and (5.4) we find

$$P = -C^{\mathrm{II}}, \tag{5.9}$$

and for the states (5.7) and (5.8) we have

$$P = C^{\mathrm{II}}. \tag{5.10}$$

Figure 2 shows the positive-energy spectrum (in-

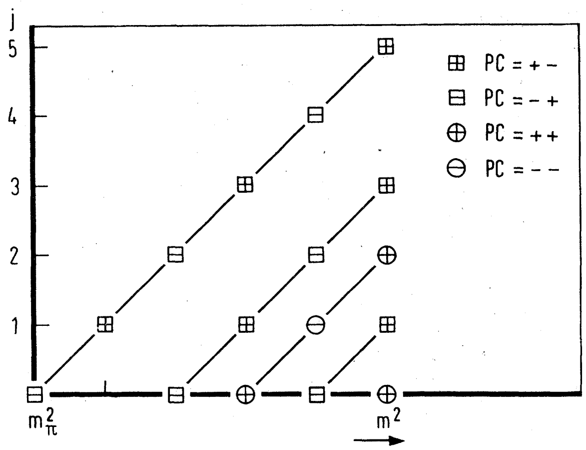


FIG. 2. Physical eigenvalue spectrum ($E \geq 0$) of the model Hamiltonian $\mathcal{H}^{\text{II}}$.

cluding ψ_0) for the case $m=0$.

It is an attractive possibility to identify the state ψ_0 with the π^0 meson, which would then be massless. The nonzero physical mass is apparently due to perturbations.

VI. INTERPRETATION OF THE SPURIOUS STATES

In order to obtain an insight into the meaning of the spurious states which appear in the symmetric models let us consider a system consisting of a *massless* quark and a *massless* antiquark in the center-of-mass system. There are two types of massless quarks. One is left-handed and is comparable to the neutrino, the other is right-handed and has no neutrino counterpart. Similarly there are two types of antiquarks. We have

$$\mathcal{H}_\pm^R = \vec{\sigma}\cdot\vec{p} \quad \text{and} \quad \mathcal{H}_\pm^L = -\vec{\sigma}\cdot\vec{p}\,, \tag{6.1}$$

where the + sign refers to the quark and the − sign to the antiquark. Both Hamiltonians have positive- as well as negative-energy eigenstates.

If a right-handed quark of momentum $\vec{p}_q$ and spin $\frac{1}{2}\vec{\Sigma}_q = \frac{1}{2}\vec{\sigma}\otimes I_2$ is combined with a left-handed antiquark with momentum $\vec{p}_{\bar{q}}$ and spin $\frac{1}{2}\vec{\Sigma}_{\bar{q}} = \frac{1}{2}I_2\otimes\vec{\sigma}$, then the total Hamiltonian for that system is equal to

$$\mathcal{H}_T^R = \vec{\Sigma}_q\cdot\vec{p}_q - \vec{\Sigma}_{\bar{q}}\cdot\vec{p}_{\bar{q}}\,. \tag{6.2}$$

If the center-of-mass motion is split off and disregarded, this effectively becomes, in the c.m. system,

$$\mathcal{H}_T^R = (\vec{\Sigma}_q + \vec{\Sigma}_{\bar{q}})\cdot\vec{p} = 2\vec{S}\cdot\vec{p}\,, \tag{6.3}$$

where

$$\vec{p} = \vec{p}_q = -\vec{p}_{\bar{q}}\,.$$

The Hamiltonian (6.3) does not conserve parity. There should be coupling with the mirror-image world for which the Hamiltonian $\mathcal{H}_T^L$ is equal to

$$\mathcal{H}_T^L = -2\vec{S}\cdot\vec{p}\,. \tag{6.4}$$

We now define

$$\mathcal{H}_T^0 = \begin{bmatrix} \mathcal{H}_T^R & 0 \\ 0 & \mathcal{H}_T^L \end{bmatrix} = \begin{bmatrix} 2\vec{S}\cdot\vec{p} & 0 \\ 0 & -2\vec{S}\cdot\vec{p} \end{bmatrix} \tag{6.5}$$

as the free Hamiltonian describing a system consisting either of a right-handed quark combined with a left-handed antiquark or of a left-handed quark combined with a right-handed antiquark. If we make the following redefinition in accordance with the definition of $\vec{S}$ used in the previous sections:

$$\vec{S}\otimes I_2 \to \vec{S}\,, \tag{6.6}$$

then we can write

$$\mathcal{H}_T^0 = 2\vec{S}\cdot\zeta\vec{p}\,. \tag{6.7}$$

The parity operator should then be defined by

$$P = \epsilon\bar{P}\eta \tag{6.8}$$

and is clearly conserved.

The spectrum of $\mathcal{H}_T^0$ already shows spurious states of energy zero together with positive-energy states of energy $2|\vec{p}|$ and negative-energy states of energy $-2|\vec{p}|$.

The zero-energy states are sums of products of positive-energy quark and negative-energy antiquark states and products of negative-energy quark and positive-energy antiquark states. If the negative- and positive-energy states are called "physical" then there are just as many physical as spurious states. The latter are therefore a necessary consequence of hole theory.

If the system consists of positive-energy quark and antiquark states then because their momenta are opposite and they have opposite "handedness" their spins line up in the direction of $\vec{p}$. When both energies are negative the spins line up in the opposite direction.

Interaction between the massless quarks by means of a linear potential can be built-in in a parity-conserving way:

$$\mathcal{H}_T = \begin{bmatrix} 2\vec{S}\cdot\vec{p} & 2\alpha i\vec{S}\cdot\vec{r} \\ -2\alpha i\vec{S}\cdot\vec{r} & -2\vec{S}\cdot\vec{p} \end{bmatrix}\,, \tag{6.9}$$

where $\vec{r}$ is the quark-antiquark distance. With the substitution (6.6) we find

$$\mathcal{H}_T = 2\vec{S}\cdot(\zeta\vec{p} - \alpha\theta\vec{r})\,. \tag{6.10}$$

By carrying out a unitary transformation one can make the following substitution:

$$(\zeta, -\theta, \eta) \to (\eta, \theta, \zeta), \tag{6.11}$$

so that $\mathcal{H}_T$ as given by (6.10) appears to be equivalent to $\mathcal{H}^{\text{I}}(0)$. The parity P as given by (6.8) transforms into P as defined by (2.13).

As follows from Sec. III, half of the eigenstates of $\mathcal{H}^{\text{I}}(0)$ have physical eigenvalues, while the remainder have the eigenvalue zero. They are the result of coupling a "negative-energy quark" with a "positive-energy antiquark" or a "positive-energy quark" with a "negative-energy antiquark".

Let us now discuss the C-parity operator in connection with $\mathcal{H}_T$. This operator should turn the right-handed quark into a right-handed antiquark of the same momentum, while the left-handed antiquark is turned into a left-handed quark with the same momentum. This transformation is effected by changing $\vec{r}$ into $-\vec{r}$ and $\vec{p}$ into $-\vec{p}$, by interchanging $\vec{\Sigma}_q$ and $\vec{\Sigma}_{\bar{q}}$, and by interchanging "world" and "mirror-image world". An extra minus sign is needed to correct for the wrong sequence in which the quark and antiquark states are treated (which is just the minus sign which results from interchanging two fermion creation operators in second-quantized theory). Since ϵ as occurring in the parity operation is equal to -1, because the intrinsic parities of quarks and antiquarks are opposite, we find immediately that

$$C = P\mathfrak{U}, \tag{6.12}$$

where P is given by (6.8) and $\mathfrak{U}$ by (2.7). For $\mathcal{H}^{\text{I}}(0)$ expression (6.8) should be replaced by (2.13). Note that expression (6.12) then is equal to the C^{I} parity defined in Sec. II.

If a right-handed quark of momentum $\vec{p}_q$ and spin $\frac{1}{2}\vec{\Sigma}_q$ is combined with a right-handed antiquark of momentum $\vec{p}_{\bar{q}}$ and spin $\frac{1}{2}\vec{\Sigma}_{\bar{q}}$ then the total Hamiltonian is

$$\tilde{\mathcal{H}}_T^R = \vec{\Sigma}_q \cdot \vec{p}_q + \vec{\Sigma}_{\bar{q}} \cdot \vec{p}_{\bar{q}}. \tag{6.13}$$

Disregarding the center-of-mass motion we find that this Hamiltonian effectively becomes, in the c.m. system,

$$\tilde{\mathcal{H}}_T^R = (\vec{\Sigma}_q - \vec{\Sigma}_{\bar{q}}) \cdot \vec{p} = 2\vec{S}' \cdot \vec{p}. \tag{6.14}$$

Also the Hamiltonian (6.14) does not conserve parity. There should again be coupling with the mirror-image world for which the Hamiltonian is

$$\tilde{\mathcal{H}}_T^L = -2\vec{S}' \cdot \vec{p}. \tag{6.15}$$

Therefore we define

$$\tilde{\mathcal{H}}_T^0 = \begin{pmatrix} \tilde{\mathcal{H}}_T^R & 0 \\ 0 & \tilde{\mathcal{H}}_T^L \end{pmatrix} = \begin{pmatrix} 2\vec{S}' \cdot \vec{p} & 0 \\ 0 & -2\vec{S}' \cdot \vec{p} \end{pmatrix} \tag{6.16}$$

as the free Hamiltonian describing a system consisting either of a right-handed quark combined with a right-handed antiquark or a left-handed quark combined with a left-handed antiquark. If we make the following redefinition:

$$\vec{S}' \otimes I_2 \to \vec{S}', \tag{6.17}$$

then we can write

$$\tilde{\mathcal{H}}_T^0 = 2\vec{S}' \cdot \zeta \vec{p}, \tag{6.18}$$

The parity operator is the same as that of (6.8) and is again conserved.

The spectrum of $\tilde{\mathcal{H}}_T^0$ shows again spurious states of energy zero, positive-energy states of energy $2|\vec{p}|$ and negative-energy states of energy $-2|\vec{p}|$. Now, however, the positive- and negative-energy states have their quark spins antiparallel if measured in the $\vec{p}$ direction. We thus find that the physical states are mixtures of spin-0 and transversal spin-1 states.

Interaction between the massless quarks can again be built-in in a parity-conserving way by means of a linear potential and we obtain

$$\tilde{\mathcal{H}}_T = \begin{pmatrix} 2\vec{S}' \cdot \vec{p} & -2\alpha i \vec{S}' \cdot \vec{r} \\ 2\alpha i \vec{S}' \cdot \vec{r} & -2\vec{S}' \cdot \vec{p} \end{pmatrix}, \tag{6.19}$$

which by means of the substitution (6.17) becomes

$$\tilde{\mathcal{H}}_T = 2\vec{S}' \cdot (\zeta \vec{p} + \alpha \theta \vec{r}). \tag{6.20}$$

By means of a unitary transformation this transforms into $\mathcal{H}^{\text{II}}(0)$ while P defined by (6.8) transforms into expression (2.13). As follows from Sec. V, one-half of the eigenstates of $\mathcal{H}^{\text{II}}(0)$ have physical eigenvalues, while the remainder have an eigenvalue of zero. The reason for calling the latter spurious is the same as before.

We now discuss the C parity in connection with $\tilde{\mathcal{H}}_T$. Again C should turn a quark into an antiquark and an antiquark into a quark with the conservation of "handedness" and momentum. This transformation is effected by changing $\vec{r}$ into $-\vec{r}$ and $\vec{p}$ into $-\vec{p}$ and by interchanging $\vec{\Sigma}_q$ and $\vec{\Sigma}_{\bar{q}}$. A transformation of "world" into "mirror-image world" is not necessary anymore. An extra minus sign is still needed and we find

$$C = \eta P \mathfrak{U}, \tag{6.21}$$

where P is given by (6.8) and $\mathfrak{U}$ by (2.7). For $\mathcal{H}^{\text{II}}(0)$ nothing is changed. If expression (6.8) is replaced by expression (2.13) we find C to be equal to C^{II}.

In this section we have shown that the spurious states are a necessary consequence of hole theory when applied to two-particle systems. When an imaginary mass term is introduced into the Hamiltonian $\mathcal{H}^{\text{I}}$, the spurious states automatically acquire an imaginary energy eigenvalue. We shall from now on consider only the Hilbert space of "physical" states of positive and negative energy.

As a consequence of this, operators like $\vec{p}$ and $\vec{r}$ are not observables anymore since their eigenstates are mixtures of physical and spurious states. Let Λ project out the physical states, then $\Lambda=\Lambda^\dagger$ and $\vec{p}_r=\Lambda\vec{p}\Lambda$ and $\vec{r}_r=\Lambda\vec{r}\Lambda$ are Hermitian operators, restricted to the physical Hilbert space. As such they can be called observables.

VII. THE PHYSICAL ρ^0 MESON

Since the ground state of the positive-energy $\mathcal{H}^{\text{I}}(m)$ spectrum has spin 1, negative parity, and negative C parity it is attractive to interpret this state as representing a ρ^0 meson if internal-symmetry considerations are disregarded. To obtain the correct ρ^0 mass we are forced to introduce an imaginary "mass" term into the Hamiltonian and we find

$$\mathcal{H}^{\text{I}}(m)=2\vec{S}\cdot(\eta\vec{p}+\alpha\theta\vec{r})\pm 2i|m|(S^2-1)\zeta\,, \tag{7.1}$$

which is not Hermitian. Let us recall the properties of (7.1).

Since the second term anticommutes with the first term on the right-hand side of (7.1) we find $[\mathcal{H}^{\text{I}}(m)]^2$ to be a Hermitian operator. We may conclude that the eigenvalues of $\mathcal{H}^{\text{I}}(m)$ are real or purely imaginary and that the eigenstates can be chosen to form an orthonormal set. Thus it follows that the spurious states with purely imaginary energy eigenvalues are orthogonal to the "physical" states which have real-energy eigenvalues as long as $|m|^2\leq 4\alpha$. It follows also that the positive-energy states can be chosen to form an orthogonal set among themselves. This is also the case for the negative-energy states.

It is not true anymore that the positive-energy states are orthogonal to their corresponding negative-energy states.

In order to correct for this we carry out a similarity transformation.

Let $|\psi_{\pm i}\rangle$ be a set of two corresponding states which satisfy the equation

$$\mathcal{H}^{\text{I}}(m)|\psi_{\pm i}\rangle=\pm E_i|\psi_{\pm i}\rangle\,, \tag{7.2}$$

where

$$|\psi_{-i}\rangle=T\zeta|\psi_i\rangle\,. \tag{7.3}$$

Since $[\mathcal{H}^{\text{I}}(m)]^2$ contains m only in a term of the form $4m^2$, the eigenstates of $\mathcal{H}^{\text{I}}(m)$ can be written in the form

$$|\psi_i\rangle=\lambda(m)|\psi_i^{(1)}\rangle+\mu(m)|\psi_i^{(2)}\rangle\,, \tag{7.4}$$

where $|\psi_i^{(1)}\rangle$ and $|\psi_i^{(2)}\rangle$ are a set of m-independent orthonormal eigenstates of $[\mathcal{H}^{\text{I}}(m)]^2$. We choose them to be eigenstates of ζ such that

$$\zeta|\psi_i^{(1)}\rangle=|\psi_i^{(1)}\rangle\,, \tag{7.5}$$

$$\zeta|\psi_i^{(2)}\rangle=-|\psi_i^{(2)}\rangle\,, \tag{7.6}$$

and where, moreover, the phase is determined by

$$T|\psi_i^{(1)}\rangle=|\psi_i^{(1)}\rangle\,, \tag{7.7}$$

$$T|\psi_i^{(2)}\rangle=|\psi_i^{(2)}\rangle\,. \tag{7.8}$$

Thus we find that

$$|\psi_{-i}\rangle=\lambda^*(m)|\psi_i^{(1)}\rangle-\mu^*(m)|\psi_i^{(2)}\rangle\,. \tag{7.9}$$

We have also that

$$\begin{aligned}\langle\psi_i|\zeta|\psi_j\rangle&=\frac{1}{4i|m|}\langle\psi_i|[\mathcal{H}^{\text{I}}(m)-\mathcal{H}^{\text{I}\dagger}(m)]|\psi_j\rangle\\&=\frac{E_j-E_i}{4i|m|}\langle\psi_i|\psi_j\rangle=0\,,\quad\text{unless } i=-j\,.\end{aligned} \tag{7.10}$$

Thus we have

$$\begin{aligned}\langle\psi_i|\zeta|\psi_i\rangle&=[\lambda^*(m)\langle\psi_i^{(1)}|+\mu^*(m)\langle\psi_i^{(2)}|]\\&\quad\times\zeta[\lambda(m)|\psi_i^{(1)}\rangle+\mu(m)|\psi_i^{(2)}\rangle]\\&=|\lambda(m)|^2-|\mu(m)|^2=0\,.\end{aligned} \tag{7.11}$$

We make the following choice:

$$\lambda(m)=\frac{1}{\sqrt{2}}\,,\quad \mu(m)=\frac{e^{i\phi(m)}}{\sqrt{2}}\,,\quad \phi(m)\text{ real}\,. \tag{7.12}$$

With this we obtain

$$\begin{bmatrix}\langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_{-i}\rangle\\ \langle\psi_{-i}|\psi_i\rangle & \langle\psi_{-i}|\psi_{-i}\rangle\end{bmatrix}=\begin{bmatrix}1 & \dfrac{1-e^{2i\phi}}{2}\\ \dfrac{1-e^{-2i\phi}}{2} & 1\end{bmatrix}. \tag{7.13}$$

Our purpose is to restore orthonormality by replacing $|\psi_i\rangle$ with $|\tilde{\psi}_i\rangle$ such that

$$\langle\tilde{\psi}_i|\tilde{\psi}_j\rangle=\delta_{ij}\,. \tag{7.14}$$

Let

$$|\psi_i\rangle=A|\tilde{\psi}_i\rangle\,. \tag{7.15}$$

Then (7.13) is a set of submatrix elements of $A^\dagger A$. From this one can solve for A which is determined up to a unitary transformation.

A possible choice for $|\tilde{\psi}_i\rangle$ is the following:

$$|\tilde{\psi}_i\rangle=|\psi_i\rangle|_{m=0}\,. \tag{7.16}$$

With this choice we explicitly find

$$\begin{bmatrix}\langle\tilde{\psi}_i|A|\tilde{\psi}_i\rangle & \langle\tilde{\psi}_i|A|\tilde{\psi}_{-i}\rangle\\ \langle\tilde{\psi}_{-i}|A|\tilde{\psi}_i\rangle & \langle\tilde{\psi}_{-i}|A|\tilde{\psi}_{-i}\rangle\end{bmatrix}=\begin{bmatrix}\dfrac{1+e^{i\phi}}{2} & \dfrac{1-e^{-i\phi}}{2}\\ \dfrac{1-e^{i\phi}}{2} & \dfrac{1+e^{-i\phi}}{2}\end{bmatrix}. \tag{7.17}$$

Besides introducing the least possible change of the system, we find that A commutes with parity, C parity, and total angular momentum. Our corrected Hamiltonian becomes

$$\hat{\mathcal{H}}^{\mathrm{I}}(m)=A^{-1}\mathcal{H}^{\mathrm{I}}(m)A\,. \tag{7.18}$$

The eigenstates of $\hat{\mathcal{H}}^{\mathrm{I}}(m)$ are those of $\mathcal{H}^{\mathrm{I}}(0)$, while if E is an eigenvalue of $\mathcal{H}^{\mathrm{I}}(0)$ the corresponding eigenvalue of $\hat{\mathcal{H}}^{\mathrm{I}}(m)$ becomes $\pm(E^2-|m|^2)^{1/2}$.

With $\vec{a}$ and $\vec{\chi}$ defined by (4.2) and (4.3) and with the operators defined by (4.14) and (4.15) we now find for the ρ^0-meson wave function

$$\psi_{\rho^0}=\mathfrak{N}_{\rho^0}[B^{\dagger}_{(1)}(2)+\tfrac{1}{2}B^{\dagger}_{(2)}(2)]\psi_0\,, \tag{7.19}$$

where ψ_0 is defined in accordance with (4.4) and (4.5):

$$\psi_0=\frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right\}\otimes\begin{bmatrix}1\\0\\0\end{bmatrix}e^{-\alpha r^2/2}\,. \tag{7.20}$$

Explicitly, (7.19) leads to

$$\vec{\psi}_{\rho^0}=-\mathfrak{N}_{\rho^0}[\sqrt{\alpha}\,\theta(\vec{r}\times\vec{\Sigma}_r)+i\vec{\Sigma}_r]\psi_0\,, \tag{7.21}$$

where

$$\mathfrak{N}_{\rho^0}=\frac{1}{\sqrt{2}}\left(\frac{\alpha}{\pi}\right)^{3/4}. \tag{7.22}$$

The parameter α can be determined from the slope of the ρ trajectory

$$\alpha\sim\tfrac{1}{4}m_\rho^{\,2}\,, \tag{7.23}$$

so that the wave function is completely known. It can be used for several purposes. The *size* of the ρ^0 could be defined as the square root of the expectation value of r^2. We find

$$(\langle r^2\rangle_{\mathrm{av}})^{1/2}=\left(\frac{2}{\alpha}\right)^{1/2}\sim\frac{2\sqrt{2}}{m_\rho}\sim 0.66\text{ fermi}\,. \tag{7.24}$$

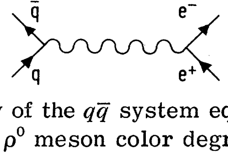
For the computation of the partial width of the decay mode

$$\rho^0\to e^+e^-\ \text{ or }\ \mu^+\mu^-\,, \tag{7.25}$$

the value of the absolute square of the wave function at $\vec{r}=0$ should be known. We find

$$|\psi_{\rho_z^0}(0)|^2=\mathfrak{N}_{\rho^0}^{\,2}=\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^{3/2}\sim\frac{1}{16\pi\sqrt{\pi}}\,m_\rho^{\,3}\,. \tag{7.26}$$

The relatively small dimensions of the ρ^0 meson in comparison with other models must result in a larger width for electromagnetic decays. A crude estimate is obtained by using the approximate expression

$\Gamma\sim|\psi(0)|^2\cdot\Big|$  $\Big|^2$ (7.27)

where the energy of the $q\bar{q}$ system equals the ρ^0 mass. If for the ρ^0 meson color degrees of freedom are taken into account we find with (7.27) and with the data from Ref. 2 for the partial width of the reactions (7.25) that

$$\Gamma\sim 3.5\text{ keV}\,,$$

which must be compared with the experimental value 6.5–10 keV. This result is already of the same magnitude as the result in Ref. 3. More accurate calculations with the present model will be presented in a subsequent paper.

VIII. THE PHYSICAL π^0 MESON

The Hamiltonian which should describe the physical π^0 meson is given by

$$\mathcal{H}^{\mathrm{II}}(0)=2\vec{S}'\cdot(\eta\vec{p}-\alpha\theta\vec{r})\,. \tag{8.1}$$

Indeed we find that the parity of the π^0 is -1. Taking into account that ϵ in (2.22) is equal to -1 we find the C parity to be $+1$ as it should be. The pion mass is zero. The size of the π^0 can be computed using the pion wave function

$$\psi_{\pi^0}=\mathfrak{N}_{\pi^0}\psi_0\,, \tag{8.2}$$

where ψ_0 is given by (7.20) and where

$$\mathfrak{N}_{\pi^0}=\left(\frac{\alpha}{\pi}\right)^{3/4}. \tag{8.3}$$

We find

$$(\langle r^2\rangle_{\mathrm{av}})^{1/2}=\left(\frac{3}{2\alpha}\right)^{1/2}\sim\frac{\sqrt{6}}{m_\rho}\sim 0.57\text{ fermi}\,. \tag{8.4}$$

IX. DISCUSSION

The model developed in this article for the description of neutral mesons, although based on Dirac hole theory, shows a number of favorable characteristics:

(a) It leads to exactly solvable equations.

(b) It contains linear confining potentials.

(c) It gives rise to linear and parallel Regge trajectories.

(d) It reproduces the main characteristics of the ρ^0 and π^0 mesons, including zero mass of the π^0. (The real mass of the π^0 is considered to be due to perturbations or refinements.)

The sizes of the mesons turn out to be of acceptable order of magnitude. The estimated probability for decay of the ρ^0 into charged leptons is somewhat on the low side.[2] In order to improve on this a better understanding of electromagnetic interactions in connection with this model is necessary.

When second quantization is introduced and the model is made truly Lorentz covariant (if at all possible), exact solvability will surely be lost.

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[1]C. Dullemond, preceding paper, Phys. Rev. D 18, 574 (1978).
[2]Particle Data Group, Rev. Mod. Phys. 48, S1 (1976).
[3]P. Hays and M. V. K. Ûlehla, Phys. Rev. D 13, 1339 (1976).