PHYSICAL REVIEW D VOLUME 18, NUMBER 2 15 JULY 1978

Exactly solvable wave equation with a linear confining potential. I. Asymmetric model

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(Received 14 February 1978)

A wave equation is presented which describes the confinement of a Dirac particle by means of a static linear potential. The model is exactly solvable and all solutions are tabulated. Also a complete set of commuting observables is given which can be used to characterize these solutions uniquely. The eigenvalues for the total angular momentum, j, and the energy squared, E^2, arrange themselves into linear and parallel Regge trajectories. Finally, a discussion is given of the high-mass limit.

I. INTRODUCTION

During the last few years linear-potential models[1] have become popular in high-energy physics for several reasons. They seem to be able to explain the charmonium spectrum reasonably well and they are a natural consequence of certain non-Abelian field theories involving quarks, gluons, and strings.

The models in which a linear potential is introduced in a nonrelativistic Schrödinger equation are exactly solvable only when the orbital angular momentum is equal to zero. The solutions then involve Airy functions.[2] The asymptotic behavior of the leading Regge trajectory which describes the bound states of highest angular momentum l as a function of energy E is given by $l \sim E^{3/2}$. Not much else is known about the behavior of the trajectories, but there certainly is no simple regularity in the energy spectrum as there is in the isotropic harmonic-oscillator model or in dual resonance models. Nevertheless, if the system is coupled to a scattering channel[3] there probably exists a high-energy limit in the sense of Dolen, Horn, and Schmid.[4]

It is easy to show that if a conventional linear potential is introduced into the Dirac equation, which for high energies is more appropriate to do, then this will give rise to at least asymptotically linear leading Regge trajectories in the sense that $l \sim s = E^2$. The exact behavior of the trajectories is not known. It is the purpose of this article to present a potential model, based on the Dirac equation, which contains a modified linear confining potential and which gives rise to an infinite set of linear and parallel Regge trajectories. The model is exactly solvable and has a perfectly regular energy spectrum. The complete solution is presented in this article and a set of commuting observables including the Hamiltonian, the total angular momentum squared, and the z component of the total angular momentum is found of which it will be proved that it is a complete set.

The main ingredient is a linear-potential term which, as distinct to ordinary linear potentials, has a smooth behavior at the origin, but it does not find as yet its justification in field theory.

There are also problems of interpretation which are the result of the asymmetric appearance of two spin operators. However, the model already contains all necessary ingredients for a satisfactory adaptation as will be shown in the following article.

II. DEFINITION OF THE MODEL

The model which will be studied throughout this article is defined by a Hamiltonian of the form ($\hbar = c = 1$)

$$\mathcal{H} = \vec{\pi} \cdot \vec{p} + \alpha \vec{\rho} \cdot \vec{r} + \zeta m \,, \tag{2.1}$$

where $\vec{p}$ and $\vec{r}$ are the momentum and position operators satisfying the usual commutation relations

$$[p_i, r_j] = -i\delta_{ij} \quad (i,j = 1,2,3)\,; \tag{2.2}$$

$\alpha > 0$ defines the strength of the linear potential represented by the second term on the right-hand side of Eq. (2.1); m is a real number representing some sort of rest mass. The remaining quantities $\vec{\pi}$, $\vec{\rho}$, and ζ are Hermitian matrices, independent of $\vec{p}$ and $\vec{r}$, which satisfy the anticommutation relations

$$\begin{aligned} &\{\pi_i, \pi_j\} = 2\delta_{ij}, \quad \{\rho_i, \rho_j\} = 2\delta_{ij}, \quad \{\pi_i, \rho_j\} = 0\,, \\ &\{\pi_i, \zeta\} = 0\,, \quad \{\rho_i, \zeta\} = 0\,, \quad \zeta^2 = 1\,. \end{aligned} \tag{2.3}$$

A representation of these matrices can easily be found:

$$\begin{aligned} &\vec{\pi} = \vec{\sigma} \otimes I_2 \otimes \sigma_x\,, \\ &\vec{\rho} = I_2 \otimes \vec{\sigma} \otimes \sigma_y\,, \\ &\zeta = I_2 \otimes I_2 \otimes \sigma_z = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \end{aligned} \tag{2.4}$$

Here $\vec{\sigma}$ is a set of Pauli spin matrices and I_n represents the $n \times n$ unit matrix. Apparently the matrices defined by (2.4) have eight rows and columns.

There is a conserved axial vector $\vec{J}$ defined by

$$\vec{J}=\vec{r}\times\vec{p}+\frac{\vec{\pi}\times\vec{\pi}+\vec{\rho}\times\vec{\rho}}{4i} \tag{2.5}$$

satisfying

$$[J_i, J_j]=i\epsilon_{ijk}J_k\,, \tag{2.6}$$

which can be interpreted as the total angular momentum of the system with respect to the origin $\vec{r}=0$. The term $\vec{L}=\vec{r}\times\vec{p}$ is to be interpreted as the orbital angular momentum with respect to $\vec{r}=0$, and

$$\vec{S}=\tfrac{1}{2}(\vec{\Sigma}_p+\vec{\Sigma}_r)\,, \tag{2.7}$$

where

$$\vec{\Sigma}_p=\frac{\vec{\pi}\times\vec{\pi}}{2i}=\vec{\sigma}\otimes I_2\otimes I_2 \tag{2.8}$$

and

$$\vec{\Sigma}_r=\frac{\vec{\rho}\times\vec{\rho}}{2i}=I_2\otimes\vec{\sigma}\otimes I_2\,, \tag{2.9}$$

is to be interpreted as the total spin.

If $\alpha=0$, Eq. (2.1) describes a free Dirac particle and it is therefore suggested that $\vec{\Sigma}_p$ describes the spin $(\frac{1}{2})$ of the particle itself, while $\vec{\Sigma}_r$ is a spin which is somehow carried by the potential. Both spins commute with each other and with ζ.

By squaring one obtains from (2.1)

$$\mathcal{H}^2=p^2+\alpha^2r^2+m^2+\alpha\zeta\vec{\Sigma}_p\cdot\vec{\Sigma}_r\,. \tag{2.10}$$

The operator $\zeta\vec{\Sigma}_p\cdot\vec{\Sigma}_r$ has one eigenvalue $+3$, one eigenvalue -3, three eigenvalues $+1$, and three eigenvalues -1. The eigenvalues of $\mathcal{H}^2$ are therefore, remembering that $m^2+3\alpha$ is the lowest eigenvalue of $p^2+\alpha^2r^2+m^2$ and the spectrum is equally spaced with spacing 2α,

$$\begin{aligned}
&\lambda_1(n)=m^2+2n\alpha\,, \quad \text{nondegenerate for } n=0\\
&\lambda_2(n)=m^2+2(n+1)\alpha\,, \quad \text{triply degenerate for } n=0\\
&\lambda_3(n)=m^2+2(n+2)\alpha\,, \quad \text{triply degenerate for } n=0\\
&\lambda_4(n)=m^2+2(n+3)\alpha\,, \quad \text{nondegenerate for } n=0\,.
\end{aligned} \tag{2.11}$$

Here $n=0,1,2,\ldots$ and the degeneracy for $n\neq0$ is either once or three times the degeneracy which is normal for the ordinary isotropic harmonic oscillator. Apparently the spectrum of $\mathcal{H}^2$ is equidistant with spacing 2α. The operator $\mathcal{H}^2$ and therefore also $\mathcal{H}$ lacks a continuum, which means that the potential appearing in $\mathcal{H}$ is confining. We now discuss parity and time reversal.

Let $\overline{P}$ be the ordinary parity operation,

$$\begin{aligned}
&\overline{P}\vec{r}=-\vec{r}\overline{P}\,,\\
&\overline{P}\vec{p}=-\vec{p}\overline{P}\,,
\end{aligned} \tag{2.12}$$

and define the parity operation P as follows:

$$P=\overline{P}\zeta\,. \tag{2.13}$$

Then $\mathcal{H}$ commutes with P and the solutions can be labeled according to parity. However, this labeling turns out to be trivial. The form of P is in agreement with the interpretation of the model as describing a confined Dirac particle.

The time-reversal operation T can be defined by the antilinear operator

$$T=K\sigma_y\otimes\sigma_y\otimes I_2\,, \tag{2.14}$$

where K represents complex conjugation. We have

$$\begin{aligned}
&T\vec{r}=\vec{r}T\,,\\
&T\vec{p}=-\vec{p}T\,,\\
&T\vec{\rho}=\vec{\rho}T\,,\\
&T\vec{\pi}=-\vec{\pi}T\,,\\
&T\zeta=\zeta T\,,
\end{aligned} \tag{2.15}$$

and we find that $\mathcal{H}$ commutes with T. Apparently the model is parity conserving and time reversal invariant.

III. SOLUTIONS OF THE EIGENVALUE EQUATION

Simple solutions of the equation

$$\mathcal{H}\psi=\left(\frac{1}{i}\,\vec{\pi}\cdot\nabla+\alpha\vec{\rho}\cdot\vec{r}+\zeta m\right)\psi=E\psi \tag{3.1}$$

can easily be found with the ansatz

$$\psi(\vec{r})=\phi(\vec{r})e^{-\alpha r^2/2}. \tag{3.2}$$

The equation for $\phi(\vec{r})$ then becomes

$$\left(\frac{1}{i}\,\vec{\pi}\cdot\nabla+\sqrt{2}\alpha\vec{\chi}\cdot\vec{r}+\zeta m\right)\phi(\vec{r})=E\phi(\vec{r}), \tag{3.3}$$

where

$$\vec{\chi}=\frac{\vec{\rho}+i\vec{\pi}}{\sqrt{2}} \tag{3.4}$$

has the property

$$\{\chi_i,\chi_j\}=0 \quad (i,j=1,2,3). \tag{3.5}$$

The scalar

$$R=\chi_i\chi_j\chi_k\epsilon_{ijk} \tag{3.6}$$

satisfies

$$\vec{\chi}R=R\vec{\chi}=0, \tag{3.7}$$

and we see that

$$Re^{-\alpha r^2/2}$$

represents a set of solutions of (3.1) with $E=\pm m$. However, if we define

$$\eta = I_2 \otimes I_2 \otimes \sigma_x = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \quad (3.8)$$

and

$$\theta = I_2 \otimes I_2 \otimes \sigma_y = \begin{pmatrix} 0 & -iI_4 \\ iI_4 & 0 \end{pmatrix} \quad (3.9)$$

we find that

$$R = \tfrac{3}{2}\sqrt{2}\,(\eta + i\theta)(I_8 - \vec{\Sigma}_p \cdot \vec{\Sigma}_r), \quad (3.10)$$

and since $\frac{1}{4}(I_8 - \vec{\Sigma}_p \cdot \vec{\Sigma}_r)$ projects out states with $S^2 = 0$ and

$$\eta + i\theta = \zeta(\eta + i\theta) = \begin{pmatrix} 0 & 2I_4 \\ 0 & 0 \end{pmatrix}, \quad (3.11)$$

we find that $Re^{-\alpha r^2/2}$ contains only one independent column, representing the one solution ψ_0 for which $S^2 = 0$ and $E = m$. There does not exist a corresponding solution with $E = -m$, which means that the symmetry between positive- and negative-energy solutions is broken. It makes a difference whether m is chosen positive or negative. It will be convenient to indicate the explicit dependence of the Hamiltonian $\mathcal{H}$ on m and we write (2.1) in the form

$$\mathcal{H}(m) = \vec{\pi} \cdot \vec{p} + \alpha\vec{\rho} \cdot \vec{r} + \zeta m. \quad (3.12)$$

If the set of operators $\vec{a}^\dagger$ is defined by

$$\vec{a}^\dagger = \frac{\alpha\vec{r} - i\vec{p}}{\sqrt{2\alpha}}, \quad (3.13)$$

then the following relations can easily be derived:

$$[\mathcal{H}(m), \vec{a}^\dagger] = \sqrt{\alpha}\,\vec{\chi}^\dagger, \quad (3.14)$$

$$[\mathcal{H}(m), \zeta\vec{\chi}^\dagger] = -2\sqrt{\alpha}\,\zeta\vec{a}^\dagger + 2m\,\vec{\chi}^\dagger, \quad (3.15)$$

which leads to

$$[\mathcal{H}(m), \vec{a}^\dagger + \bar{\beta}_\pm \zeta\vec{\chi}^\dagger] = -2\bar{\beta}_\pm \zeta\sqrt{\alpha}(\vec{a}^\dagger + \bar{\beta}_\pm \zeta\vec{\chi}^\dagger) \quad (3.16)$$

with

$$\bar{\beta}_\pm = \frac{1}{\sqrt{\alpha}}\,[-\tfrac{1}{2}m \pm \tfrac{1}{2}(m^2 - 2\alpha)^{1/2}]. \quad (3.17)$$

From (3.16) we obtain, by replacing m by $(m^2 + 2\alpha)^{1/2}$,

$$\mathcal{H}(\pm m)(\vec{a}^\dagger + \beta_\pm \zeta\vec{\chi}^\dagger) = (\vec{a}^\dagger + \beta_\pm \zeta\vec{\chi}^\dagger)\mathcal{H}((m^2 + 2\alpha)^{1/2})$$

with (3.18)

$$\beta_\pm = \frac{1}{\sqrt{\alpha}}\,[-\tfrac{1}{2}(m^2 + 2\alpha)^{1/2} \pm \tfrac{1}{2}m]. \quad (3.19)$$

We find that if ψ satisfies

$$\mathcal{H}((m^2 + 2\alpha)^{1/2})\psi = E\psi \quad (3.20)$$

and if $\vec{U}^\dagger_\pm$ is defined by

$$\vec{U}^\dagger_\pm = \vec{a}^\dagger + \beta_\pm \zeta\vec{\chi}^\dagger \quad (3.21)$$

then

$$\mathcal{H}(\pm m)\vec{U}^\dagger_\pm \psi = E\vec{U}^\dagger_\pm \psi. \quad (3.22)$$

Thus we find that $\vec{U}^\dagger$ does not cause the energy to change, but the Hamiltonian. When $\vec{U}^\dagger_-\psi$ is multiplied by ζ, the resulting expression is an eigenstate of $\mathcal{H}(m)$ belonging to the energy $-E$. It is clear that in this way by repeated application of operators such as $\vec{U}^\dagger_\pm$ on the solution ψ_0 which we have already found, an infinite set of positive- and negative-energy solutions of the equation

$$\mathcal{H}(m)\psi = E\psi \quad (3.23)$$

can be obtained. Note that if n operators of the type (3.21) are multiplied together the result is an operator which is homogeneous of nth degree in $\vec{a}^\dagger$ and $\zeta\vec{\chi}^\dagger$ combined. Observe also that

$$[a_i^\dagger, a_j^\dagger] = 0 \quad (3.24)$$

and

$$\{\zeta\chi_i^\dagger, \zeta\chi_j^\dagger\} = 0. \quad (3.25)$$

If we partially reduce such an nth-degree operator by symmetrization and antisymmetrization we thus find that only a limited number of Young diagrams can contribute (see Fig. 1).

There are three categories:

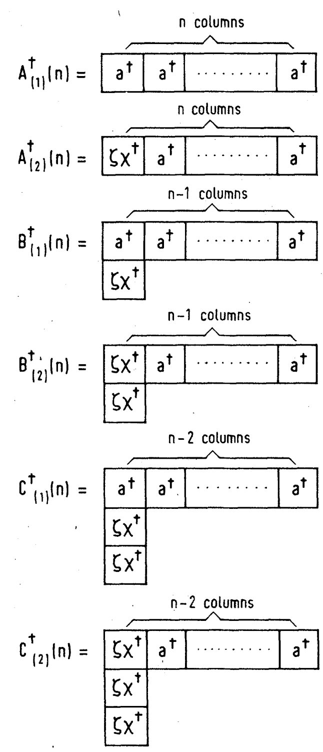

FIG. 1. Representation of the operators $A^\dagger_{(1,2)}$, $B^\dagger_{(1,2)}$, and $C^\dagger_{(1,2)}$ by means of Young diagrams. For the definition of the operators see Eqs. (3.26), (3.27), (3.35), (3.36), (3.43), and (3.44).

(a) Two completely symmetric operators of nth degree can be constructed. We have

$$A^{\dagger}_{(1)}(n)=a^{\dagger}_{i_1}a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_n} \tag{3.26}$$

and

$$A^{\dagger}_{(2)}(n)=\frac{1}{n}\left(\zeta\chi^{\dagger}_{i_1}a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_n}+a^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_2}a^{\dagger}_{i_3}\cdots a^{\dagger}_{i_n}+\cdots\right). \tag{3.27}$$

From (3.14) we have

$$[\mathcal{H}(m),A^{\dagger}_{(1)}(n)]=n\sqrt{\alpha}\,\zeta\,A^{\dagger}_{(2)}(n)\,, \tag{3.28}$$

and from (3.14) and (3.15),

$$[\mathcal{H}(m),A^{\dagger}_{(2)}(n)]=-2\zeta\sqrt{\alpha}\,A^{\dagger}_{(1)}(n)+2m\zeta A^{\dagger}_{(2)}(n). \tag{3.29}$$

This leads to

$$\mathcal{H}(\pm m)A^{\dagger}_{\pm}(n)=A^{\dagger}_{\pm}(n)\mathcal{H}((m^2+2n\alpha)^{1/2})\,, \tag{3.30}$$

where $A^{\dagger}_{\pm}(n)$ is defined as follows:

$$A^{\dagger}_{\pm}(n)=A^{\dagger}_{(1)}(n)+\beta_{\pm}(n)A^{\dagger}_{(2)}(n) \tag{3.31}$$

with

$$\beta_{\pm}(n)=\frac{1}{\sqrt{\alpha}}\left[-\tfrac{1}{2}(m^2+2n\alpha)^{1/2}\pm\tfrac{1}{2}m\right]. \tag{3.32}$$

Apparently

$$A^{\dagger}_{+}(n)\psi_0 \tag{3.33}$$

and

$$\zeta A^{\dagger}_{-}(n)\psi_0 \tag{3.34}$$

are eigenstates of $\mathcal{H}(m)$ with eigenvalues $+(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$, respectively.

(b) There are two nth-degree operators of mixed symmetry corresponding to two-row Young diagrams. They are

$$B^{\dagger}_{(1)}(n)=\tfrac{1}{2}(a^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_n}-a^{\dagger}_{i_n}\zeta\chi^{\dagger}_{i_1})a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_{n-1}} \tag{3.35}$$

and

$$B^{\dagger}_{(2)}(n)=\frac{1}{n-1}\left[\zeta\chi^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_n}a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_{n-1}}+\tfrac{1}{2}(a^{\dagger}_{i_1}\chi^{\dagger}_{i_n}-a^{\dagger}_{i_n}\chi^{\dagger}_{i_1})(\chi^{\dagger}_{i_2}a^{\dagger}_{i_3}\cdots a^{\dagger}_{i_{n-1}}+a^{\dagger}_{i_2}\chi^{\dagger}_{i_3}a^{\dagger}_{i_4}\cdots a^{\dagger}_{i_{n-1}}+\cdots)\right]. \tag{3.36}$$

We find after some algebra

$$[\mathcal{H}(m),B^{\dagger}_{(1)}(n)]=\sqrt{\alpha}\,(n-1)\zeta\,B^{\dagger}_{(2)}(n)+2\,m\zeta\,B^{\dagger}_{(1)}(n) \tag{3.37}$$

and

$$[\mathcal{H}(m),B^{\dagger}_{(2)}(n)]=\frac{-2n}{n-1}\,\zeta\,\sqrt{\alpha}\;B^{\dagger}_{(1)}(n). \tag{3.38}$$

This leads to

$$\mathcal{H}(\pm m)B^{\dagger}_{\pm}(n)=B^{\dagger}_{\pm}(n)\,\mathcal{H}((m^2+2n\alpha)^{1/2})\,, \tag{3.39}$$

where $B^{\dagger}_{\pm}(n)$ is defined as

$$B^{\dagger}_{\pm}(n)=B^{\dagger}_{(1)}(n)-\frac{n-1}{n}\,\beta_{\mp}(n)B^{\dagger}_{(2)}(n) \tag{3.40}$$

with $\beta_{\pm}(n)$ given by (3.32). (Note that $B^{\dagger}_{+}$ contains β_{-}.) Apparently

$$B^{\dagger}_{+}(n)\psi_0 \tag{3.41}$$

and

$$\zeta B^{\dagger}_{-}(n)\psi_0 \tag{3.42}$$

are again eigenstates of $\mathcal{H}(m)$ with eigenvalues $+(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$, respectively.

(c) There are two nth-degree operators of mixed symmetry corresponding to three-row Young diagrams. They are

$$C^{\dagger}_{(1)}(n)=\tfrac{1}{3}(a^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_{n-1}}\zeta\chi^{\dagger}_{i_n}+\text{cycl})a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_{n-2}} \tag{3.43}$$

and

$$C^{\dagger}_{(2)}(n)=\frac{1}{n-2}\,\zeta\chi^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_{n-1}}\zeta\chi^{\dagger}_{i_n}a^{\dagger}_{i_2}\cdots a^{\dagger}_{i_{n-2}}+\frac{1}{3(n-2)}\,(a^{\dagger}_{i_1}\zeta\chi^{\dagger}_{i_{n-1}}\zeta\chi^{\dagger}_{i_n}+\text{cycl})(\zeta\chi^{\dagger}_{i_2}a^{\dagger}_{i_3}\cdots a^{\dagger}_{i_{n-2}}+a^{\dagger}_{i_2}\zeta\chi^{\dagger}_{i_3}a^{\dagger}_{i_4}\cdots a^{\dagger}_{i_{n-2}}+\cdots)\,. \tag{3.44}$$

This gives

$$[\mathcal{H}(m),C^{\dagger}_{(1)}(n)]=\sqrt{\alpha}\,(n-2)\zeta C^{\dagger}_{(2)}(n) \tag{3.45}$$

and

$$[\mathcal{H}(m),C^{\dagger}_{(2)}(n)]=-\frac{2n\zeta}{n-2}\,\sqrt{\alpha}\;C^{\dagger}_{(1)}(n)+2\,m\zeta\,C^{\dagger}_{(2)}(n)\,. \tag{3.46}$$

This gives

$$\mathcal{H}(\pm m)C^{\dagger}_{\pm}(n)=C^{\dagger}_{\pm}(n)\,\mathcal{H}((m^2+2n\alpha)^{1/2})\,, \tag{3.47}$$

where $C^{\dagger}_{\pm}(n)$ is defined as

$$C^{\dagger}_{\pm}(n)=C^{\dagger}_{(1)}(n)+\frac{n-2}{n}\beta_{\pm}(n)C^{\dagger}_{(2)}(n) \tag{3.48}$$

with $\beta_{\pm}(n)$ again given by (3.32). We thus find that

$$C_+^\dagger(n)\psi_0 \tag{3.49}$$

and

$$\zeta C_-^\dagger(n)\psi_0 \tag{3.50}$$

are eigenstates of $\mathcal{H}(m)$ with eigenvalues $+(m^2+2n\alpha)^{1/2}$ and $-(m^2+2n\alpha)^{1/2}$, respectively. This exhausts the number of possibilities. We see also that except for ψ_0 all positive-energy states have their negative-energy counterparts.

The operators $A_\pm^\dagger(n)$, $B_\pm^\dagger(n)$, and $C_\pm^\dagger(n)$ can be reduced further. Let $\{A_\pm, n, j\}$, $\{B_\pm, n, j\}$, and $\{C_\pm, n, j\}$ be the irreducible operators with total angular momentum j obtained by reduction of $A_\pm^\dagger(n)$, $B_\pm^\dagger(n)$, and $C_\pm^\dagger(n)$. These are nondegenerate. We have

$$A_\pm^\dagger(n) = \{A_\pm, n, n\} + \{A_\pm, n, n-2\} + \cdots + \{A_\pm, n, 0 \text{ or } 1\}, \tag{3.51}$$

$$B_\pm^\dagger(n) = \{B_\pm, n, n-1\} + \{B_\pm, n, n-2\} + \cdots + \{B_\pm, n, 1\}, \tag{3.52}$$

$$C_\pm^\dagger(n) = \{C_\pm, n, n-3\} + \{C_\pm, n, n-5\} + \cdots + \{C_\pm, n, 0 \text{ or } 1\}. \tag{3.53}$$

From a simple counting argument it follows that in this way all positive- and negative-energy states can be constructed. The E^2 spectrum of the positive-energy eigenstates is displayed in Fig. 2.

States with energy squared $m^2+2n\alpha$ have parity $(-1)^n$, so we see that the parity operation is unable to lift the degeneracy of the spectrum. The negative-energy spectrum has the same structure, with only the state with $E=-m$ $(m>0)$ missing.

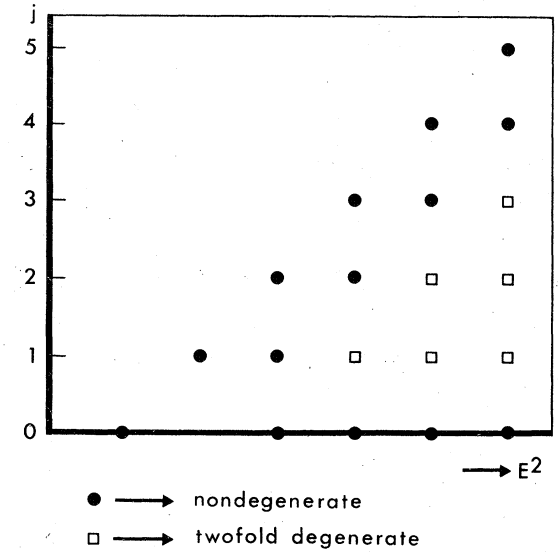

FIG. 2. The total angular momentum j plotted against the square of the total energy for the positive-energy solutions of $\mathcal{H}(m)\psi = E\psi$ $(m>0)$.

IV. CONSTRUCTION OF A COMPLETE SET OF COMMUTING OBSERVABLES

As we have seen in Sec. III, the operators $\mathcal{H}$, J^2, and J_z do not form a complete set of commuting observables. We shall now prove that all remaining degeneracies can be lifted by adding the following Hermitian operator:

$$Q = i\zeta(\vec{\rho}\cdot\vec{p} - \alpha\vec{\pi}\cdot\vec{r}) + i\,m\vec{\rho}\cdot\vec{\pi}. \tag{4.1}$$

This operator commutes with $\mathcal{H}$ and manifestly commutes with J^2 and J_z. We have

$$[Q, \vec{a}^\dagger] = \sqrt{\alpha}\,\zeta\vec{\chi}^\dagger, \tag{4.2}$$

$$[Q, \vec{\chi}^\dagger] = -2\sqrt{\alpha}\,\zeta\vec{a}^\dagger + 2m\vec{\chi}^\dagger. \tag{4.3}$$

From this the following relations can be derived:

$$[Q, A_{(1)}^\dagger(n)] = n\sqrt{\alpha}\,\zeta A_{(2)}^\dagger(n), \tag{4.4}$$

$$[Q, A_{(2)}^\dagger(n)] = -2\sqrt{\alpha}\,\zeta A_{(1)}^\dagger(n) + 2mA_{(2)}^\dagger(n), \tag{4.5}$$

$$[Q, B_{(1)}^\dagger(n)] = (n-1)\sqrt{\alpha}\,\zeta B_{(2)}^\dagger(n) + 2mB_{(1)}^\dagger(n), \tag{4.6}$$

$$[Q, B_{(2)}^\dagger(n)] = \frac{-2n}{n-1}\sqrt{\alpha}\,\zeta B_{(1)}^\dagger(n) + 4mB_{(2)}^\dagger(n), \tag{4.7}$$

$$[Q, C_{(1)}^\dagger(n)] = (n-2)\sqrt{\alpha}\,\zeta C_{(2)}^\dagger(n) + 4mC_{(1)}^\dagger(n), \tag{4.8}$$

$$[Q, C_{(2)}^\dagger(n)] = \frac{-2n}{n-2}\sqrt{\alpha}\,\zeta C_{(1)}^\dagger(n) + 6mC_{(2)}^\dagger(n). \tag{4.9}$$

Remembering that $\zeta\psi_0 = \psi_0$ and using the fact that ψ_0 is an eigenstate of Q,

$$Q\psi_0 = -3m\psi_0, \tag{4.10}$$

we obtain from the relations (4.4)—(4.9) the following results:

$$QA_+^\dagger(n)\psi_0 = [-2m - (m^2+2n\alpha)^{1/2}]A_+^\dagger(n)\psi_0, \tag{4.11}$$

$$QB_+^\dagger(n)\psi_0 = (m^2+2n\alpha)^{1/2}B_+^\dagger(n)\psi_0, \tag{4.12}$$

$$QC_+^\dagger(n)\psi_0 = [2m - (m^2+2n\alpha)^{1/2}]C_+^\dagger(n)\psi_0, \tag{4.13}$$

$$Q\zeta A_-^\dagger(n)\psi_0 = [-2m + (m^2+2n\alpha)^{1/2}]\zeta A_-^\dagger(n)\psi_0, \tag{4.14}$$

$$Q\zeta B_-^\dagger(n)\psi_0 = -(m^2+2n\alpha)^{1/2}\zeta B_-^\dagger(n)\psi_0, \tag{4.15}$$

$$Q\zeta C_-^\dagger(n)\psi_0 = [2m + (m^2+2n\alpha)^{1/2}]\zeta C_-^\dagger(n)\psi_0. \tag{4.16}$$

From (3.51), (3.52), and (3.53) it now follows that Q is able to distinguish between states with the same energy and total angular momentum under all circumstances. Thus we find that

$\mathcal{H}$, J^2, J_z, and Q

form a complete set of commuting observables.

V. THE HIGH-m LIMIT

When m is large and n small the dominating term in (2.10) is m^2. We write for the positive-energy states

$$\mathcal{H} = \mathcal{H}' + m \tag{5.1}$$

and substitute this into (2.10), after which we neglect $\mathcal{H}'^2$:

$$\mathcal{H}' = \frac{p^2}{2m} + \frac{\alpha^2 r^2}{2m} + \frac{\alpha}{2m}\zeta\vec{\Sigma}_p\cdot\vec{\Sigma}_r \ . \tag{5.2}$$

Except for the last term on the right-hand side, this is just the nonrelativistic isotropic harmonic oscillator with characteristic frequency $\omega = \alpha/m$.

The energy spectrum becomes

$$E' = E - m = (m^2 + 2n\alpha)^{1/2} - m \simeq \frac{\alpha}{m}n$$
$$= \omega n \quad (n = 0, 1, \ldots) . \tag{5.3}$$

Let us now look at the behavior of the solutions of the wave equation. Since

$$\alpha = O(|p|^2) ,$$
$$\zeta\vec{\chi}^\dagger = O(1) , \tag{5.4}$$
$$\vec{a}^\dagger = O(1) ,$$

and

$$|p| \ll m , \tag{5.5}$$

we find, with

$$\beta_+(n) \to -\frac{n\sqrt{\alpha}}{2m} \tag{5.6}$$

and

$$\beta_-(n) \to -\frac{m}{\sqrt{\alpha}} , \tag{5.7}$$

that in $A_+^\dagger(n)$ as given by (3.31) only the first term survives:

$$A_+^\dagger(n) \to A_{(1)}^\dagger(n) . \tag{5.8}$$

In $B_+^\dagger(n)$ as given in (3.40) it is the second term which dominates the first term:

$$B_+^\dagger(n) \to \frac{m}{\sqrt{\alpha}}\,\frac{n-1}{n}\,B_{(2)}^\dagger(n) . \tag{5.9}$$

Finally, in $C_+^\dagger(n)$ as given by (3.48), the first term dominates again:

$$C_+^\dagger(n) \to C_{(1)}^\dagger(n) . \tag{5.10}$$

Applied to ψ_0 we find that in all cases the lowest four components of the 8-spinors vanish in the high-m limit, since they are eigenstates of ζ with eigenvalue +1. This is exactly what one should expect.

The operator Q becomes very simple in the large-m limit:

$$Q \to im\vec{p}\cdot\vec{\pi} = m\zeta\vec{\Sigma}_r\cdot\vec{\Sigma}_p \ , \tag{5.11}$$

which for positive-energy states is equivalent to $m\vec{\Sigma}_r\cdot\vec{\Sigma}_p$. Apparently Q measures the total spin of the system when m is large. We have from (4.11), (4.12), and (4.13)

$$QA_+^\dagger(n)\psi_0 = -3mA_+^\dagger(n)\psi_0 , \tag{5.12}$$

$$QB_+^\dagger(n)\psi_0 = mB_+^\dagger(n)\psi_0 , \tag{5.13}$$

$$QC_+^\dagger(n)\psi_0 = mC_+^\dagger(n)\psi_0 . \tag{5.14}$$

The interpretation is obviously the following:

In $A_+^\dagger(n)\psi_0$ the particle spin is opposite to the spin carried by the potential, while in $B_+^\dagger(n)\psi_0$ and $C_+^\dagger(n)\psi_0$ these spins are parallel.

VI. CONCLUSION

In the foregoing sections it has been proved that the linear-potential model as presented in the first section can be solved exactly and the explicit expressions for the solutions have been given. The spectrum is discrete and resembles that of the isotropic harmonic oscillator. There is an infinity of linear Regge trajectories in terms of the square of the total energy. These are all parallel.

One could imagine the potential as being associated with an infinitely heavy Dirac particle located at the origin. A light Dirac particle as well as its antiparticle can then be bound by the potential. The lack of symmetry between positive- and negative-energy states as is evident from (3.31), (3.40), and (3.48) is then the result of the apparent differences between potentials carried by particles and antiparticles.

The above interpretation is not without difficulties, since it would mean that the interpretation of E as the total energy of the system is incorrect and a connection with quark dynamics is hard to make.

However, a modification is possible which restores the symmetry between the source of the linear potential and the particle which is affected by it. In that way the description of neutral mesons becomes possible. This is the subject of the following article.

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