PHYSICAL REVIEW D VOLUME 18, NUMBER 2 15 JULY 1978

Interactions between 't Hooft-Polyakov monopoles

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(Received 27 March 1978)

We use the stress-energy tensor and the associated energy-momentum conservation to study the interactions between two widely separated monopoles (or monopole and antimonopole). By defining a set of minimal conditions to represent the above systems, we show how the problem reduced mathematically to a known electrostatic problem. The force between the monopoles (or monopole and antimonopole) is then, to the leading order, the expected repulsive (attractive) Coulomb force. We also discuss how the Prasad-Sommerfield limit alters the problem, leading to twice the Coulomb force between a monopole and antimonopole and zero force between two monopoles.

I. INTRODUCTION

The 't Hooft-Polyakov[1] model of a monopole is described by classical field configurations of finite energy. These configurations are static solutions to a set of nonlinear field equations derived from a local SO(3) gauge-invariant Lagrangian consisting of a triplet of isovector Yang-Mills gauge fields and a triplet of Higgs fields. They exhibit particlelike character and are stable against decaying into zero-energy solutions on account of their topological charge. As they are static solutions, their time evolution is trivial, that is, the configurations are unchanged under the motion generated by a constant timelike vector field. However, one could contemplate a more complex situation which represents a monopole-antimonopole (or a monopole-monopole) pair and ask how they interact with each other.

Even at the classical level this is a nontrivial problem since the configurations are governed by a set of coupled nonlinear differential equations. An obvious approach to the problem is the following: Choose any initial data set for the Yang-Mills-Higgs system which represents the pair and let the field equations determine the time evolution of the system. One could, by studying the time evolution, determine the effect of one member of the pair on the motion of the other. However, the complete time evolution of a given initial data set is very difficult to compute in the case of nonlinear systems. Furthermore, even if one has such a complete time evolution it may not be possible, in general, to abstract from the time dependence of the fields the nature of the interaction between the two objects. It is necessary, therefore, to specify or choose a proper set of initial data guided by physical considerations.

Our main purpose in this paper is to formulate a solution to this problem using the stress-energy tensor and the associated energy-momentum conservation. For this purpose we note that in the 't Hooft-Polyakov solutions, there is a region in which the Yang-Mills field equations have a nonnegligible source current term due to Higgs fields. We shall call such a region the "core" of the monopole or antimonopole. Outside this core, the Yang-Mills fields obey the free field equations. We choose the initial data to represent fields which are "generated by two widely separated cores of monopoles". Then we compute the four-momentum of the core at the initial instant of time and its first time derivative using the field equations. In calculating the first time derivative, we do not need any explicit solutions of the field equations but only the fact that the energy-momentum is conserved. The method is analogous to the one used by Dirac[2] in studying the interaction of the electron and radiation within the framework of classical Maxwell theory.

This paper is organized as follows: In the next section, we describe the model and give the expression for the stress-energy tensor. We show how, given a set of initial data, we can calculate the rate of change of momentum of the core. In Sec. III, we discuss the choice of the initial data set for the (a) monopole-antimonopole pair and (b) monopole-monopole pair. By a judicious choice of the gauge we can treat both problems (a) and (b) in the same manner. The problem then reduces to finding the solutions to Laplace's equation with specified boundary conditions on the boundaries of the regions surrounding the cores and in the asymptotic region. We show that the force is the expected Coulomb force. We consider the Prasad-Sommerfield[3] limit in Sec. IV. In this limit, there are two contributions to the stress-energy tensor, one coming from the long-range gauge field and the other from the Higgs field. They are such that they lead to twice the magnitude of the expected Coulomb force between a monopole and antimonopole and zero force between two monopoles. Identical results were obtained by Manton[4] previously by using a different method. We discuss his method and show that even though his conclusions are correct, his method does not imply a specific interaction. In other words, one can show that his approximation scheme is consistent with an arbitrary force law. The final section is devoted to a summary and discussion.

II. ENERGY-MOMENTUM CONSERVATION

Let A^a_μ and Φ^a be a set of Yang-Mills gauge and Higgs fields, respectively, belonging to the adjoint representation of an SO(3) symmetry group. The Lagrangian $\mathcal{L}(x)$ invariant under the local SO(3) gauge transformations is given by

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu a} F^{\mu\nu a} - \tfrac{1}{2} D_\mu \Phi_a D^\mu \Phi^a - \tfrac{1}{4}\lambda(\Phi^a\Phi_a - c^2)^2 , \tag{2.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc} A_{\mu b} A_{\nu c} \tag{2.2}$$

and

$$D_\mu \Phi^a = \partial_\mu \Phi^a + e\epsilon^{abc} A_{\mu b} \Phi_c . \tag{2.3}$$

The field equations which follow from the Lagrangian are

$$D^\mu D_\mu \Phi^a = \lambda(\Phi^b\Phi_b - c^2)\Phi^a , \tag{2.4}$$

$$D^\mu F^a_{\mu\nu} = e\epsilon^{abc}\Phi_b D_\nu \Phi_c \equiv J^a_\nu . \tag{2.5}$$

The stress-energy tensor of this system is given by

$$\begin{aligned} T^{\mu\nu} = {} & F^{\mu\rho a} F^\nu{}_{\rho a} - \tfrac{1}{4} g^{\mu\nu}(F^{\rho\lambda a}F_{\rho\lambda a}) + D^\mu\Phi^a D^\nu \Phi_a \\ & - \tfrac{1}{2} g^{\mu\nu}(D^\rho\Phi^a D_\rho\Phi_a) - \tfrac{1}{4}\lambda g^{\mu\nu}(\Phi^a\Phi_a - c^2)^2 . \end{aligned} \tag{2.6}$$

Using Eqs. (2.4) and (2.5), we can easily verify that

$$\partial_\mu T^{\mu\nu} = 0 . \tag{2.7}$$

Given an initial data set, consisting of the values of the fields and their first time derivatives at a given time t, the time evolution of this data set is determined by the field equations. The time evolution will be such that Eq. (2.7) will be valid. Suppose we have an initial data set on the $t=0$ surface such that the "matter field" current J^a_μ is confined inside two balls B_+ and B_- whose radii are R and whose centers are, respectively, at $z=+d/2$ and $z=-d/2$ of the z axis, where $d \gg R$. Then, using the conservation of the stress-energy tensor, we can compute the time rate of the four-momentum change of the cores at this instant of time.

To do this, consider a small cylindrical spacetime region associated with B_+, i.e., $B_+ \otimes [t=0, t=\epsilon]$. Since the divergence of the stress-energy tensor vanishes, the following three-dimensional surface integral over the boundary of the cylindrical region vanishes on account of Stokes's theorem:

$$\oint_{\partial\{B_+\otimes[0,\epsilon]\}} T^{\mu\nu}\xi_\mu dS_\nu = \int_{B_+\otimes[0,\epsilon]} \partial_\nu (T^{\mu\nu}\xi_\mu) dV = 0 , \tag{2.8}$$

where ξ_μ is any constant unit vector field. The surface integral of (2.8) can be decomposed into three pieces two of which are over spacelike surfaces at $t=0$ and $t=\epsilon$ and the remaining one is over a timelike surface. It follows then that

$$\begin{aligned} 0 &= \oint_{\partial\{B_+\otimes[0,\epsilon]\}} T^{\mu\nu}\xi_\mu dS_\mu \\ &= -\int_{B_+\otimes[0]} (T^{\mu\nu}\xi_\nu \hat{t}_\nu) d^3x + \int_{B_+\otimes[\epsilon]} (T^{\mu\nu}\xi_\mu \hat{t}_\nu) d^3x \\ &\quad + \int_{\partial B_+\otimes[0,\epsilon]} (T^{\mu\nu}\xi_\mu \hat{r}_{+\nu}) R^2 \sin\theta \, dt\, d\theta\, d\phi , \end{aligned} \tag{2.9}$$

where $\hat{t}_\nu = \partial_\nu t$, $\hat{r}_{+\nu} = \partial_\nu r_+$, and r_+ is the radial distance from the center of B_+. The first and second terms of the above equation give the four-momentum P^μ of B_+ in the direction $(-\xi_\mu)$ evaluated at $t=0$ and in the direction ξ_μ evaluated at $t=\epsilon$, respectively. Hence, taking the limit as $\epsilon \to 0$, we obtain from (2.9)

$$\begin{aligned} \frac{d}{dt} P^\mu \xi_\mu \Big|_{t=0} &\equiv \lim_{\epsilon\to 0} \frac{\xi_\mu P^\mu|_{t=\epsilon} - \xi_\mu P^\mu|_{t=0}}{\epsilon} \\ &= -\int_{B_+\otimes[0]} (T^{\mu\nu}\xi_\mu \hat{r}_{+\nu}) R^2 \sin\theta \, d\theta\, d\phi . \end{aligned} \tag{2.10}$$

Therefore, once we know $T^{\mu\nu}$ on the initial surface, we can compute the force, i.e., dP^μ/dt, exerted on the core at the initial moment. Of course, $T^{\mu\nu}$ on the initial surface can be obtained from the initial data set of the field Eqs. (2.4) and (2.5).

III. INTERACTION BETWEEN MONOPOLES

We shall work in the gauge in which $\hat{t}^\mu A^a_\mu = 0$. Then the initial data set on the $t=0$ surface consists of the spatial components of the gauge field A^a_i, the Higgs field Φ^a, and their first time derivatives evaluated on the $t=0$ surface. We shall consider a system initially at rest, i.e., the first time derivatives of the fields are zero. To specify an initial data set, all we have to do is to choose A^a_i and Φ^a such that they represent fields which are generated by monopole or antimonopole cores. To do this we choose the values of A^a_i and Φ^a on the surfaces of B_+ and B_-, so that they are consistent with the assumption that there is a monopole core (or an antimonopole core) of the 't Hooft-Polyakov type inside each ball. Then, to specify the A_i^a and Φ^a outside of B_+ and B_-, we solve the *static* field equations outside the balls with the chosen boundary conditions on the surfaces of B_+ and B_- and with the usual asymptotic conditions for $r\to\infty$. We impose the condition that the "matter field" current J_μ^a vanishes outside the B_+ and B_-. The fields outside are then only those which are generated by the cores in B_+ and B_-.

The situation can be compared with the one in the following electrostatic problem. To find the interaction between two conducting spheres which are at rest at $t=0$, we solve the Laplace equation for the scalar potential, i.e., the *static* Maxwell equation, with the appropriate boundary conditions on the sphere and an asymptotic condition for $r\to\infty$. Then, using this solution, we can compute the four-momentum transfer across the surfaces of the conductors using the method described in Sec. II, and thereby obtain a force exerted on each conductor.

Before we specify the choice of initial data set, let us consider some of the general requirements to be imposed upon A_i^a and Φ^a on the boundaries of B_+ and B_-. First, the choice of A_i^a and Φ^a on these spheres should be smooth functions. Second, the magnitude of Φ^a should be c on these spheres. This implies that we require R to be much larger than the characteristic length of the theory, i.e.,

$$\eta\equiv 1/(2e^2\lambda)^{1/2}. \tag{3.1}$$

Third, the normalized Higgs field, $\hat{\Phi}^a$, should be such that the integral[5]

$$m\equiv -\frac{1}{4\pi e}\int \epsilon_{abc}\hat{\Phi}^a\partial_i\hat{\Phi}^b\partial_j\hat{\Phi}^c dS^{ij} \tag{3.2}$$

over the boundary of B_+ is $+1$ and over the boundary of B_- is -1 for a monopole-antimonopole pair. For a monopole-monopole pair both integrals are 1. These are the minimal conditions to ensure that the cores of the 't Hooft-Polyakov type. These minimal conditions turn out to be enough to determine the interaction up to the zeroth order of R/d, when the separation distance d is much larger than the radii R of B_+ and B_-.

In what follows we shall first consider a monopole-antimonopole pair and then discuss what modifications are necessary to treat the monopole-monopole pair.

A. Interaction between a monopole and an antimonopole

In order to specify the values of Φ^a and A_i^a on the boundaries of B_+ and B_-, we consider the most general form of $\hat{\Phi}^a=\Phi^a/(\Phi^b\Phi_b)^{1/2}$ and A_i^a outside B_+ and B_-. We note that locally a smooth $\hat{\Phi}^a$ field can be transformed into any other smooth $\hat{\Phi}^a$ by an appropriate gauge transformation. The only restriction on $\hat{\Phi}^a$ is of a global nature. Namely, $\hat{\Phi}^a$ on the surface must satisfy the second condition states before and $\hat{\Phi}^a$ is such that the integral (3.2) over a sphere in the asymptotic region is zero. The following $\hat{\Phi}^a$ satisfies all these requirements:

$$\hat{\Phi}^a=\sin\chi\cos\phi\,\hat{x}^a+\sin\chi\cos\phi\,\hat{y}^a+\cos\chi\,\hat{z}^a, \tag{3.3}$$

where ϕ is the aximuthal angle, and χ is a smooth function of ρ and z which satisfies the conditions

$$\begin{aligned}&\chi=0 \text{ for } 0\le\theta_+\le\delta,\\ &\chi=\pi \text{ for } 0\le\rho\le\delta \text{ and } -\tfrac{1}{2}d+R\le z\le\tfrac{1}{2}d-R,\\ &\text{and}\\ &\chi=2\pi \text{ for } \pi-\delta\le\theta_-\le\pi,\end{aligned} \tag{3.4}$$

where θ_+ (θ_-) is the angle between the z axis and $\hat{r}_+^i$ ($\hat{r}_-^i$), and δ is an infinitesimal number. Any other $\hat{\Phi}^a$, which satisfies the global requirements, can be gauge transformed into the above one. With (3.3), the general Higgs field takes the form

$$\Phi^a=H(z,\rho)\hat{\Phi}^a. \tag{3.5}$$

We now use the condition that J_μ^a vanishes outside of B_+ and B_- to specify the general form of smooth A_i^a fields. This condition can be rewritten as

$$0=J_i^a=H^2\epsilon^{abc}\hat{\Phi}_b(\partial_i\hat{\Phi}_c+e\epsilon_{cde}A_i^d\hat{\Phi}^e). \tag{3.6}$$

The most general A_i^a which satisfies (3.6) is given by

$$eA_i^a=-\epsilon^{abc}\hat{\phi}_b\partial_i\hat{\Phi}_c+\hat{\Phi}^aG_i, \tag{3.7}$$

where G_i is any smooth vector field.[6] Since $\hat{\Phi}^a$ is smooth, (3.7) gives a smooth eA_i^a. The fields F_{ij}^a derived from (3.7) are given by

$$eF_{ij}^a=\hat{\Phi}^a(-\epsilon^{bcd}\hat{\Phi}_b\partial_i\hat{\Phi}_c\partial_j\hat{\Phi}_d+\partial_iG_j-\partial_jG_i) \tag{3.8}$$

and the static field equations reduce to

$$\partial^i\partial_iH=0, \tag{3.9}$$

$$\partial^i(\partial_iG_j-\partial_jG_i)=\partial^i(\epsilon^{bcd}\hat{\Phi}_b\partial_i\hat{\Phi}_c\partial_j\hat{\Phi}_d). \tag{3.10}$$

We can make use of the gauge freedom to choose a specific χ subject to (3.4), say $\bar{\chi}$. The right-hand side of (3.10) is then a known function. It is also evident from (3.10) that if G_i is a solution, so is $G_i+\partial_iG$ where G is an arbitrary function. We can use this remaining gauge freedom to set $\partial^iG_i=0$ so that our problem reduces to finding the solutions to a pair of linear equations, the linear homogeneous Eq. (3.9) for H, and the following linear inhomogeneous equation for G_i:

$$\partial^i\partial_iG_j=\partial^i(\epsilon^{bcd}\hat{\Phi}_b\partial_i\hat{\Phi}_c\partial_j\hat{\Phi}_c), \tag{3.11}$$

with prescribed boundary values on the surfaces ∂B_+ and ∂B_- and at infinity.

From the second of the minimal conditions discussed earlier, $H=c$ on ∂B_+ and ∂B_-. H approaches c as $r\to\infty$. Consequently,

$$H(z,\rho)=c \tag{3.12}$$

is the unique solution of (3.9). To choose the boundary conditions for G_i, we note that because of the symmetry of the problem, we can choose it to be axially symmetric. Further, let the boundary values of G_i on B_+ and B_- be chosen in such a way that they do not differ very much from those for an isolated monopole or an antimonopole. Since for an isolated monopole or an antimonopole, G_i is proportional to $\hat{\phi}_i$ (unit vector in the azimuthal direction), we shall assume that this property holds even when we have a system of a monopole and an antimonopole. This assumption is only to make our treatment simple. One can easily generalize our treatment to take into consideration components of G_i proportional to $\hat{\rho}$ and $\hat{z}_i$ on the boundaries of B_+ and B_-. Hence, let

$$G_i=\begin{cases} b_+(R,\theta_+)\hat{\phi}_i \text{ on } \partial B_+, & (3.13)\\ b_-(R,\theta_-)\hat{\phi}_i \text{ on } \partial B_-, & (3.14)\end{cases}$$

and

$$|G_i|=O(1/r) \text{ for } r\to\infty\,. \tag{3.15}$$

The solution of Eq. (3.11) with conditions (3.13)–(3.15) is given by

$$G_i(z,\rho)=\left[-\frac{1}{e}\,\frac{\cos\overline{\chi}(z,\rho)}{\rho}+f(z,\rho)\right]\hat{\phi}_i\,, \tag{3.16}$$

where $f(z,\rho)$ is a function defined by the following path integral:

$$f(\beta,\alpha)-f(d/2-R,0)=\frac{1}{\alpha}\int_{\substack{\rho=0\\ z=d/2-R}}^{\substack{\rho=\alpha\\ z=\beta}}\rho\left(\frac{\partial U}{\partial z}\,\hat{\rho}_i-\frac{\partial U}{\partial\rho}\,\hat{z}_i\right)dl^i\,. \tag{3.17}$$

The path joining the two end points is an arbitrary one which lies outside or on the spheres. The $U(z,\rho)$, in turn, is a smooth scalar potential which is a solution of Laplace's equation

$$\partial^i\partial_i U=0 \tag{3.18}$$

with the boundary conditions

$$\hat{r}^i_+\partial_i U=\epsilon^{ijk}\hat{r}_{+1}\partial_j\left\{\left[b(R,\theta_+)+\frac{1}{e}\,\frac{\cos\overline{\chi}}{\rho}\right]\hat{\phi}_k\right\}\text{on }\partial B_+, \tag{3.19}$$

$$\hat{r}^i_-\partial_i U=\epsilon^{ijk}\hat{r}_{-1}\partial_j\left\{\left[b(R,\theta_-)+\frac{1}{e}\,\frac{\cos\overline{\chi}}{\rho}\right]\hat{\phi}_k\right\}\text{ on }\partial B_-, \tag{3.20}$$

and $|\partial_i U|=O(1/r^2)$ for $r\to\infty$. Near the z axis, $(\cos\overline{\chi}/\rho)\hat{\phi}_i$ behaves like $\pm\hat{\phi}_i/\rho$, but $\epsilon^{ijk}\partial_j[(\cos\overline{\chi}/\rho)\hat{\phi}_k]$ is zero. In Eqs. (3.19) and (3.20) we take the values of $\epsilon^{ijk}\partial_j[(\cos\overline{\chi}/\rho)\hat{\phi}_k]$ on the z axis as zero. Notice that the path integral of (3.17) is independent of the choice of the path since U satisfies Laplace's equation.

That (3.16) is indeed the required solution can be seen as follows. First, notice that the first term on the right-hand side of (3.16) is a particular solution of the linear homogeneous equation (3.11). Second, notice that $f\hat{\phi}_i$ is a solution of the linear homogeneous equation (3.11). Third, using (3.19) and (3.20) check that the boundary values of G_i given by (3.16) agree with those of (3.13) and (3.14). Finally, observe that the singularities of $f\hat{\phi}_i$ on the z axis exactly cancel the singularities of $(-\cos\overline{\chi}/\rho)\hat{\phi}_i$. Hence (3.16) yields a smooth $G_i(z,\rho)$ which is the solution of (3.11) with the prescribed boundary conditions (3.13)–(3.15).

Even though the details of the Neumann boundary conditions, (3.19) and (3.20), over ∂B_+ and ∂B_- depend on our choice of G_i over the surfaces, the flux integrals of $\partial_i U$ over the surfaces are already specified, i.e.,

$$\begin{aligned}\int_{\partial B_+}\partial_k U\,da^k&=\int_{\partial B_+}\left[\partial_{[j}G_{k]}+\frac{1}{e}\,\partial_{[j}\left(\frac{\cos\overline{\chi}}{\rho}\,\psi_{k]}\right)\right]dS^{jk}\\&=\frac{2\pi}{e}\,[\cos\overline{\chi}(\theta=0)-\cos\overline{\chi}(\theta=\pi)]\\&=4\pi/e\,.\end{aligned} \tag{3.21}$$

(The square brackets indicate antisymmetry in the enclosed indices.) Similarly, the total flux of $\partial_k U$ over the boundary of B_- is $(-4\pi/e)$. This reduced problem, i.e., the boundary-value problem on U is mathematically the same as an electrostatic problem whose boundary condition is given in such a way that the total charge inside B_+ and B_- is $1/e$ and $(-1/e)$, respectively. This analogy can be extended further. In terms of U and H, the stress-energy tensor of the Yang-Mills-Higgs system, given by (2/6), takes the form

$$\begin{aligned}T^{\mu\nu}=&\tfrac{1}{2}(\partial^k U\partial_k U)g^{\mu\nu}-\partial^\mu U\partial^\nu U+(\partial^k U\partial_k U)\hat{t}^\mu\hat{t}^\nu\\&-\tfrac{1}{2}(\partial^k H\partial_k H)g^{\mu\nu}+\partial^\mu H\partial^\nu H.\end{aligned} \tag{3.22}$$

The dependence of $T^{\mu\nu}$ on U is exactly the same as the dependence of electromagnetic stress-energy tensor on the electrostatic potential. Since H is constant everywhere outside spheres, the terms involving H do not contribute to $T^{\mu\nu}$.

Once we observe this mathematical similarity, it is immediate that the force on the monopole core in B_+, computed using the method of Sec. II, will be

$$\left.\frac{d}{dt}P^\mu\right|_{t=0}=-\frac{1}{e^2d^2}\,\hat{z}^\mu \tag{3.23}$$

in the zeroth order of R/d. Since $d \gg R$, the change in the force due to the multipole moments of the charge distribution in B^+ and B^- is negligible.

B. Interaction between a monopole-monopole pair

With just a few modifications we can obtain the interaction between a monopole-monopole pair. The $\hat{\Phi}^a$ field now should be such that the topological charge integral of (3.2) is (+1) for both B_+ and B_-. The following $\hat{\Phi}^a$ satisfies the new requirements:

$$\hat{\Phi}^a = \sin\bar{\chi}\cos\phi\hat{x}^a + \sin\bar{\chi}\sin\phi\hat{y}^a + \cos\bar{\chi}\hat{z}^a \quad \text{for } z \geq 0, \tag{3.24}$$

and

$$\hat{\Phi}^a = \sin\bar{\psi}\cos(-\phi)\hat{x}^a + \sin\bar{\psi}\sin(-\phi)\hat{y}^a + \cos\bar{\psi}\hat{z}^a \quad \text{for } z \leq 0. \tag{3.25}$$

The $\bar{\chi}$ and $\bar{\psi}$ of (3 24) and (3.25) are smooth functions of z, ρ which satisfy

$$\begin{aligned} &\bar{\chi} = 0 \quad \text{for } 0 \leq \theta_+ \leq \delta, \\ &\bar{\chi} = \pi \quad \text{for either } \pi - \delta \leq \theta_+ < \pi \text{ or } 0 \leq z \leq \delta, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\bar{\psi} = -\pi \quad \text{for either } 0 \leq \theta_- \leq \delta \text{ or } -\delta \leq z \leq 0, \\ &\bar{\psi} = 0 \quad \text{for } \pi - \delta \leq \theta_- \leq \pi. \end{aligned} \tag{3.27}$$

Equations (3.24) and (3.25) define a smooth $\hat{\Phi}^a$. Using this $\hat{\Phi}^a$, the general form of A_1^a is still given by (3.7), and the field equations take the same form in terms of G_i as (3.11). The reduced problem, i.e., the boundary condition (3.20) is now changed into

$$\hat{r}_-^i \partial_i U = \epsilon^{ijk} r_{-i} \partial_j \left\{ \left[b(R, \theta_-) + \frac{1}{e} \frac{\cos\bar{\psi}}{\rho} \right] \hat{\phi}_k \right\} \text{ on } \partial B_- . \tag{3.28}$$

Once we have a solution U which satisfies the new boundary conditions, G_i is given by

$$G_i(z, \rho) = -\frac{1}{e} \frac{\cos\eta}{\rho} \hat{\phi}_i + f\hat{\phi}_i , \tag{3.29}$$

where $\eta = \bar{\chi}$ for $z \geq 0$ and $\eta = \bar{\psi}$ for $z < 0$ and f is defined in terms of U as before. Now the integrals of $\partial_i U$ over both spheres are $4\pi/e$. However, the stress-energy tensor of the non-Abelian system still depends on U and H as in (3.22). Therefore, the force law (3.23) is changed into the repulsive Coulomb force.

IV. INTERACTIONS IN THE PRASAD-SOMMERFIELD LIMIT

How does the interaction change in the Prasad-Sommerfield limit of the SO(3) model? This limit is defined by taking $\lambda = 0$ but keeping the asymptotic boundary condition $\Phi^a\Phi_a \to c^2$ as $r \to \infty$. In this limit one of the characteristic lengths, η, is infinite. For an isolated monopole (or antimonopole) the magnitude of the Higgs field approaches c as[7]

$$H \sim c + 1/er \quad \text{for } r \to \infty \tag{4.1}$$

instead of the exponential approach of the general case.

With just a few modifications *only in the magnitude of the Higgs field*, the argument of Sec. II can be used in this limit. Taking into account Eq. (4.1) for an isolated monopole, we have to choose the boundary condition on H over ∂B_+ and ∂B_- as

$$\int_{\partial B_+} \partial_i H \, da^i = \int_{\partial B_-} \partial_i H \, da^i = 4\pi/e \tag{4.2}$$

for both a monopole-monopole pair and a monopole-antimonopole pair. The equations for H and $T^{\mu\nu}$ of the system are still given by (3.9) and (3.22). Notice that the equation and the boundary conditions (4.2) for $(H - c)$ for both pairs are the same as those of U for the monopole-monopole pair; hence, *in the zeroth order of R/d the solutions for $(H - c)$ for both pairs are the same as U of the monopole-monopole pair*. Further, substituting $T^{\mu\nu}$ of (3.22) into dP^μ/dt, Eq. (2.10), notice that the resulting formula changes its sign if we interchange U and H. Hence, the contribution to the force from H is the attractive Coulomb force for both pairs. Therefore, when we add this force from H to the force from U the net effect is twice the attractive Coulomb force in the monopole-antimonopole case and zero force in the monopole-monopole case.

Using a different method, Manton has studied the interactions between monopoles in the Prasad-Sommerfield limit. His method consists in first choosing a specific time dependence of the fields such that the field configurations are unchanged under the motion generated by a unit vector field η^μ orthogonal to the $t = 0$ surface. Further, the vector field η_μ, on the $t = 0$ surface, has a constant acceleration along the z axis with as yet unspecified magnitude near the monopole. Near the other monopole (or antimonopole) η^μ also has a constant acceleration with the same magnitude, but in the opposite direction. Then Manton finds initial data which will have such time dependence. He works in the gauge where $A_\mu^a \eta^\mu = 0$, and specifies initial data only in the region where $D_\mu \hat{\Phi}^a = 0$. Because $\eta^\mu = \hat{t}^\mu$ on the initial surface and the time dependence of fields is already specified, the field equations on the initial surface reduce to equations involving only spacelike derivatives. We shall shortly display these equations. Manton claims that these equations have solutions only when the magnitude of the acceleration near each monopole has a unique value, and arrives at the same interactions as ours in the Prasad-Sommerfield limit.

However, his analysis does not support this conclusion.

In order to show this, we first write down the above-mentioned equations. Once we choose $\hat{\Phi}^a$ appropriately, the most general forms of $\hat{\Phi}^a$ and A^a_i on the initial surface are again given by (3.5) and (3.7). Because A^a_μ is unchanged under the motion generated by η^μ (i.e., A^a_μ is Lie-derived along η^μ), the Yang-Mills field also has only spacelike components on the initial surface. Hence, the form of F^a_{ij} is again given by (3.8). The Higgs field equation outside B_+ and B_- reduces to

$$\partial^2 H + a^i \partial_i H = 0, \qquad (4.3)$$

where a^i is the acceleration of η^μ at the initial moment. This can be shown as

$$\begin{aligned} 0 = D^\mu D_\mu \Phi^a &= (h^{\mu\nu} - t^\mu t^\nu) D_\mu D_\nu (H\hat{\Phi}^a) \\ &= \hat{\Phi}^a (h^{\mu\nu} - \eta^\mu \eta^\nu) \partial_\mu \partial_\nu H \\ &= \hat{\Phi}^a [\partial^i \partial_i H + (\eta^\mu \partial_\mu \eta^\nu) \partial_\nu H] \\ &= \hat{\Phi}^a (\partial^i \partial_i H + a^i \partial_i H), \end{aligned} \qquad (4.4)$$

where $h^{\mu\nu}$ is the Euclidean metric of the initial surface: The last step follows because the acceleration is orthogonal to $\hat{t}^\nu$. Using a similar technique as above, the Yang-Mills field equation can be reduced to

$$D^i F^a_{ij} + a^i F^a_{ij} = 0 \qquad (4.5)$$

on the initial surface. In terms of G_i, this can be rewritten as

$$\partial^i \partial_i G_j + a^i (\partial_i G_j - \partial_j G_i) = \frac{1}{e} \partial^i (\epsilon_{abc} \hat{\Phi}^a \partial_i \hat{\Phi}^b \partial_j \hat{\Phi}^c). \qquad (4.6)$$

Hence, the problem of finding initial data which will have the specified time dependence reduces to the problem of finding solutions of Eq. (4.3) and (4.6). From the theory of elliptic equations, it follows that (4.3) and (4.6) will have smooth solutions if a^i and boundary conditions for G_i and H are smooth. This conclusion is in a contradiction to Manton's result that a^i should take on only a specific value in order that (4.3) and (4.6) have solutions.

V. SUMMARY AND DISCUSSION

We have shown how the stress-energy tensor and the associated energy-momentum conservation that follows from the field equations can be used to study the interactions between classical field configurations. The field configurations we have considered are smooth solutions in an SO(3) gauge-invariant Yang-Mills-Higgs system. As initial data, we choose the configurations to approximate widely separated monopole or antimonopole cores of the 't Hooft-Polyakov type. To accomplish this we imposed three minimal conditions on the fields over the boundaries of the cores. One of the conditions combined with the static field equations implied that the magnitude of the Higgs field H is constant everywhere outside the cores. The static Yang-Mills field equations reduce to the Laplace equations for a scalar potential U with boundary conditions on the cores such that the total flux of the gradient of U over the cores equals the flux due to a unit charge inside each core. Furhtermore, the stress-energy tensor of the non-Abelian system depends only on U, and this dependence is formally identical to that of the conventional electromagnetic stress-energy tensor on the scalar electrostatic potential. Consequently, we can immediately deduce from the energy-momentum conservation equation that the force between the two cores is the expected Coulomb force.

In the case of the singular limit considered by Prasad and Sommerfield, the behavior of H is nontrivial outside the cores. It turns out that H is a solution of Laplace's equation, the flux boundary conditions being identical to those on U in the case of two monopoles. The additional contribution due to H to the stress-energy tensor is such that one obtains twice the Coulomb force between a monopole and an antimonopole and zero force between two monopoles.

Finally, it might be worthwhile to discuss the uniqueness of the interactions we have determined. Our result states that if appropriate initial data sets for a multimonopole system are indeed those which satisfy the usual asymptotic condition for $r \to \infty$ and the three minimal conditions on the boundaries of B_+ and B_-, then the interactions in the zeroth order of R/d are of the Coulomb type. The second and third of the minimal conditions are the critical ones for our argument; these conditions lead to the unique flux integral of $\partial_i U$ over the boundaries of B_+ and B_- and consequently leads to the Coulomb-type interactions. These were chosen to reflect the physical expectation that, for widely separated monopoles, the configuration near each monopole would not be very different from that of an isolated monopole. Although this expectation seems to be reasonable, it is by no means obvious that it is warranted. It should be noticed that the minimal conditions exclude vortex-type configurations ending in monopoles. Such configurations have been considered by many authors[8] because the emerging force from these configurations contains a term which does not depend on the separation distance. Clearly much further work is necessary to resolve this issue.

ACKNOWLEDGMENTS

The work of two of the authors (J.N.G. and P.S.J.) was supported in part by NSF under Grant No. MPS74-15246, and the work of S.Y.P. and K.C.W. was supported in part by the U. S. Department of Energy under Contract No. EY-76-S-02-3533.

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