Extended objects created by Goldstone bosons

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Extended objects which appear in relativistic quantum field theory with spontaneous breakdown of symmetries are studied. The extended objects are created through the condensation of the Goldstone bosons. The condensation is mathematically expressed by the boson transformation. The topological singularities are defined by means of the noncommutability condition among the derivatives acting on the boson transformation parameters which are functions of space and time coordinates. A systematic method for constructing the extended objects with topological singularities is presented. It is found that the extended objects can carry a full variety of topological singularities. Only some of the topological constants associated with these topological singularities are quantized. Though the choice of original symmetry does not restrict the choice of topological singularities, it does restrict the choice of quantized topological constants (topological quantum numbers). Special effort is devoted to the construction of the extended objects which are confined in a finite domain of the three-dimensional space. In particular, a spherical surface singularity (a "bag") is obtained explicitly. Furthermore, it is shown that instantaneous singularities and transient singularities are also acceptable to our scheme.

I. INTRODUCTION

Recently, study of the topological structure of extended objects has aroused much interest among high-energy physicists in connection with the problem of quark confinement and particle structure.[1] The present paper is devoted to a study of extended objects which appear in relativistic quantum field theory with spontaneous breakdown of symmetry.

In solid-state physics we find many phenomena in which certain macroscopic objects (extended objects) are created in various ordered states (systems with spontaneous breakdown of symmetry). Vortices in superconductors, dislocations in crystals, and magnetic domain walls in ferromagnets are some well-known examples of this kind. In these systems, classically behaving macroscopic objects coexist and interact with many quanta.

The extended objects in systems with spontaneous breakdown of symmetry are created through the condensation of certain bosons. To understand this intuitively, we denote by hN a quantum number carried by these bosons, and by $h\Delta N$ the quantum fluctuation. When the boson condensation makes N very large, then $h\Delta N/hN=\Delta N/N$ becomes so small that the system behaves classically. To establish such boson condensation we need certain bosons with gapless energy. According to the Goldstone theorem, spontaneous breakdown of symmetry is maintained by the presence of certain gapless bosons (i.e., the Goldstone bosons).[2] This is true even when the presence of certain gauge fields induces the Anderson-Higgs-Kibble mechanism[3,4] and the Goldstone levels become unobservable. Therefore, in systems with spontaneous breakdown of symmetry, one can create many kinds of extended objects through the condensation of the Goldstone bosons.

The above intuitive consideration has been put in a mathematical form which is called the boson method.[5] A brief account of the boson method can be summarized in the following way. One begins with a set of Heisenberg equations for Heisenberg fields (say, ψ). Then try to identify the in-fields. Whenever a certain symmetry is spontaneously broken, there appear among the in-fields certain Goldstone bosons $\chi_\alpha^{\text{in}}(x)$ $(\alpha=1,2,\ldots)$, which satisfy the massless equations

$$\partial^2\chi_\alpha^{\text{in}}(x)=0\,. \tag{1.1}$$

Let $\phi^{\text{in}}(x)$ stand for other in-fields. Calculating all the matrix elements of the Heisenberg field ψ, we can express ψ in terms of normal products of the in-fields χ_α^{in} and ϕ^{in}:

$$\psi(x)=\psi(x;\chi_\alpha^{\text{in}},\phi^{\text{in}})\,. \tag{1.2}$$

This expression is called *the dynamical map*.[6] The dynamical map of any Heisenberg operator $O_H[\psi]$ is obtained by use of (1.2):

$$O_H[\psi]=O_H(\chi_\alpha^{\text{in}},\phi^{\text{in}})\,. \tag{1.3}$$

The maps (1.2) and (1.3) determine the structure of Heisenberg operators when no extended objects are created in the system with spontaneous breakdown of symmetry. Now perform the boson transformation

$$\chi_\alpha^{in}(x) \to \chi_\alpha^{in}(x) + f_\alpha(x)\,, \tag{1.4}$$

where $f_\alpha(x)$ ($\alpha = 1, 2, \ldots$) are c-number functions which satisfy the same equations as $\chi_\alpha^{in}(x)$:

$$\partial^2 f_\alpha(x) = 0\,. \tag{1.5}$$

We define the boson-transformed Heisenberg field by

$$\psi^f(x) \equiv \psi(x; \chi_\alpha^{in} + f_\alpha, \phi^{in})\,. \tag{1.6}$$

It has been proved[7] that $\psi(x)$ *and* $\psi^f(x)$ *satisfy the same Heisenberg equations*. This is the boson-transformation theorem. The transformation (1.4) with the condition (1.5) shows that $f_\alpha(x)$ are created by the condensation of the Goldstone boson $\chi_\alpha^{in}(x)$. The $\psi^f(x)$ corresponds to the situation where the quantum system $(\chi_\alpha^{in}, \phi^{in})$ coexists with the extended objects created by $f_\alpha(x)$. The ground-state expectation value of $O_H(x)$ is now given by $\langle 0 | O_H(\chi_\alpha^{in} + f_\alpha, \phi^{in}) | 0\rangle$. In this way, we can calculate various observables such as energies, current, etc., associated with the extended objects. The boson-transformed S matrix $S[\chi_\alpha^{in} + f_\alpha, \phi^{in}]$ describes the reaction among the extended objects and quanta.

It has been widely known that to perform the transformation $\chi_\alpha^{in} \to \chi_\alpha^{in} + c$-number is one way to cover many unitarily inequivalent representations of canonical commutation relations.[8] However, when we speak of the boson transformation, we are concerned not only with the canonical commutation relations, but also with the Heisenberg equation; the boson transformation is conditioned by the requirement that the Heisenberg equation should not change. This condition is the one which leads us to (1.5). It should be noted that, when $f_\alpha(x)$ is singular at a certain point, the original Heisenberg equation should hold even at this singular point.

The situation becomes a little more involved when there exists a gauge field. This happens, for example, in the case of superconductivity models (i.e., the gauge-invariant Nambu or Goldstone model). There appears one Goldstone boson χ^{in} and the massive vector (plasmon) field U_μ^{in},[4] and the Anderson-Higgs-Kibble mechanism forces the boson transformation $\chi^{in} \to \chi^{in} + f$ to induce also the transformation $U_\mu^{in} \to U_\mu^{in} + a_\mu$.[7] Here a_μ is the c-number (and therfore classical) vector potential and is related to $f(x)$ through certain equation, which can be used as a basic equation for phenomenological analysis of macroscopic objects. In this case it has been shown[7,9] that the macroscopic current and electromagnetic field do not vanish when and only when

$$[\partial_\mu, \partial_\nu] f(x) \neq 0 \quad \text{for certain } \mu, \nu, x \tag{1.7}$$

under the condition

$$[\partial_\mu, \partial_\nu]\partial_\rho f(x) = 0\,, \tag{1.8}$$

where $[\partial_\mu, \partial_\nu] = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu$. Equation (1.7) implies that $f(x)$ is not single valued: $f(x)$ is path dependent and therefore has certain topological singularities. When there is no gauge field (i.e., no Anderson-Higgs-Kibble mechanism), both the extended objects without any topological singularity and the ones with topological singularities can appear. In a crystal model, for example, there appear three Goldstone bosons χ_α^{in} ($\alpha = 1, 2, 3$) (there crystal phonons), and $f_\alpha(x)$ ($\alpha = 1, 2, 3$) without any topological singularity create the classical sound wave, while $f_\alpha(x)$ with topological line singularities create dislocations.[10]

The main purpose of this article is to formulate a general method for constructing $f_\alpha(x)$ with topological singularities. Here, the topological singularities are defined by the condition

$$[\partial_\mu, \partial_\nu] f_\alpha(x) \neq 0 \quad \text{for certain } \mu, \nu, \alpha, x\,, \tag{1.9}$$

which is the differential expression of the multivaluedness of $f_\alpha(x)$. This definition of topological singularities is in sharp contrast to the commonly used definition: The topologically singular domains are usually defined as the domain where the order parameter vanishes. To see how our definition is related to the usual one, let us recall that the order parameter $\Delta(x)$ for a superconductor is related to $f(x)$ through the relation

$$\Delta(x) = |\Delta(x)| \exp[2if(x)]\,, \tag{1.10}$$

where the absolute value $|\Delta(x)|$ is a function of $\partial_\mu f(x)$. Since $\Delta(x)$ should be well defined everywhere, $|\Delta(x)|$ must vanish wherever $f(x)$ is singular.

In the following sections we shall show that use of the condition (1.9) leads to a systematic method for constructing extended objects with a variety of topological singularites.

In Sec. II we present a general method for constructing topological objects. In Sec. III we make a detailed study of strings. The significance of this study lies in the fact that, assembling and deforming many strings, one can obtain a variety of topological singularities. When a string is deformed in such a way that it has one end-point joint with a half-infinite line (the hairpin string), the end point acts as a monopole. It is obvious that the extended objects associated with an infinite-line singularity do not suit to the particle model. It will be shown that a finite-line singularity is obtained by deforming a closed string (the dihairpin string). In Sec. IV, the general theory of closed surfaces will be discussed. There it will be proved that any closed surface is acceptable to our scheme. Some of the examples which have wide applications both in elementary-particle physics and solid-state physics are presented. Of particular interest is a bag which is a spherical-surface singularity. It is a fascinating idea that the black hole in general relativity is an example of the closed-surface singularity. In Sec. V, extending the method of previous sections, we make a detailed analysis of those singularities, appearances of which are instantaneous in time (instantaneous singularity) or limited in a finite-time interval (transient singularities). There, we will show how the instantonlike objects can be constructed. The consideration in this section supplies us with a powerful method for study of macroscopic transient phenomena in systems with spontaneous breakdown of symmetry.

Finally, in Sec. VI we discuss the question of how the original group symmetry restricts the choice of topological objects. Although the number of Goldstone bosons depends on the original group symmetry, this number does not restrict the choice of topological singularities.[11] However, the original group symmetry strongly controls the structure of the dynamical map, through which *the topological quantum number is identified*. In other words, although the original symmetry does not restrict our choice of topological singularities, it severely influences the answer to the question of which topological constants should be quantized. We come to this conclusion because we do not treat the Heisenberg equation as a classical equation. We have considered how the extended objects are created in a *quantum* ordered state. The shape of the singularity depends on the boundary conditions: For example, the shape of the vortex line in a superconductor is conditioned by the requirement that no persistent current can cross the line singularity, and the shape of the closed surface depends on what is contained.

When we try to apply the theory of extended objects to particle physics, we meet a serious difficulty which arises from the fact that the extended objects behave classically, while we want to have quantum particles. To overcome this difficulty, we might need an enlarged Hilbert space which contains all the states with topological objects. A study along this line is in progress.

II. MACROSCOPIC OBJECTS WITH TOPOLOGICAL SINGULARITIES

We suppose that there appear n Goldstone bosons χ_α^{in} ($\alpha = 1, 2, \ldots, n$) in a system with breakdown of symmetry. The extended objects are created by the boson transformation $\chi_\alpha^{\text{in}} \to \chi_\alpha^{\text{in}} + f_\alpha$ with f_α satisfying the equation for χ_α^{in}:

$$\partial^2 f_\alpha(x) = 0\,. \tag{2.1}$$

The structure of the topological singularities is determined by the quantities $G_{\mu\nu}^{(\alpha)\dagger}$ which are defined by

$$G_{\mu\nu}^{(\alpha)\dagger}(x) = [\partial_\mu, \partial_\nu] f_\alpha(x)\,. \tag{2.2}$$

The domains in which some components of $G_{\mu\nu}^{(\alpha)\dagger}(x)$ do not vanish are the domains of topological singularity. In most cases known in solid-state physics, the spatial or temporal variation of the condensation is observable. We therefore require that the quantities $\partial_\rho f_\alpha$ are single valued:

$$[\partial_\mu, \partial_\nu]\partial_\rho f_\alpha(x) = 0\,. \tag{2.3}$$

By use of the Green's function $D(x)$ defined by

$$\partial^2 D(x) = \delta^{(4)}(x)\,, \tag{2.4}$$

we obtain from (2.2)

$$\partial_\mu f_\alpha(x) = \int d^4y D(x-y)\partial^\nu G_{\nu\mu}^{(\alpha)\dagger}(y)\,. \tag{2.5}$$

In order to construct $G_{\mu\nu}^{(\alpha)\dagger}$, it is useful to introduce

$$G_{\mu\nu}^{(\alpha)}(x) = \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\rho} G_{\lambda\rho}^{(\alpha)\dagger}(x)\,, \tag{2.6}$$

where our choice of the metric is given by $\epsilon_{0123} = 1$, $-g_{00} = g_{ii} = 1$ ($i = 1, 2, 3$). The relation (2.6) gives

$$G_{\mu\nu}^{(\alpha)\dagger}(x) = -\tfrac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\rho} G_{\lambda\rho}^{(\alpha)}(x)\,. \tag{2.7}$$

Making use of (2.3), we find that

$$\partial^\mu G_{\mu\nu}^{(\alpha)}(x) = 0\,. \tag{2.8}$$

It can be easily shown that the condition (2.8) is sufficient for the relation (2.5) to reproduce (2.2).

Making use of above relations, we can develop a systematic method for constructing $f_\alpha(x)$ with topological singularities. First, look for $G_{\mu\nu}^{(\alpha)}$ which satisfies the divergenceless condition (2.8), and then construct $G_{\mu\nu}^{(\alpha)\dagger}$ according to (2.7). Then, $\partial_\mu f_\alpha(x)$ can be calculated by means of (2.5). The multivalued function f_α is obtained through a path integral of $\partial_\mu f_\alpha$, the existence of which is guaranteed by (2.2) as long as the path does not cross the singularities. In fact, $G_{\mu\nu}^\dagger = 0$ is the integrability condition for f_α outside of the singular domain. Although $f_\alpha(x)$ is path dependent, an explicit expression of the function $f_\alpha(x)$ for x outside of the topologically singular domain, can be obtained.

When the number of Goldstone bosons performing the boson transformation is not larger than four, it is convenient to introduce the notations $G_{\mu\nu\lambda}^\dagger$ and G_μ which are defined as follows:

$$G_{\mu\nu\lambda}^\dagger(x) \equiv \tfrac{1}{3}[G_{\mu\nu}^{(\lambda)\dagger} + G_{\nu\lambda}^{(\mu)\dagger} + G_{\lambda\mu}^{(\nu)\dagger}]\,, \tag{2.9}$$

$$G_\mu(x) \equiv \tfrac{1}{2}\epsilon_\mu{}^{\nu\lambda\rho} G^\dagger_{\nu\lambda\rho}(x)\,. \tag{2.10}$$

We then have the following relations:

$$3G^\dagger_{\mu\nu\lambda}(x) = \epsilon_{\mu\nu\lambda}{}^{\rho} G_\rho(x)\,, \tag{2.11}$$

$$G^{(\rho)}_{\mu\rho}(x) = G_\mu(x)\,. \tag{2.12}$$

III. THE STRING

Let us consider a set of world sheets $y^{(a)}_\mu(\tau,\sigma)$ $(a=1,2,\ldots)$ which depend on two parameters τ and σ. Since the product of the Jacobian

$$\frac{\partial[y_\mu, y_\nu]}{\partial[\tau,\sigma]} = \frac{\partial y_\mu}{\partial\tau}\frac{\partial y_\nu}{\partial\sigma} - \frac{\partial y_\nu}{\partial\tau}\frac{\partial y_\mu}{\partial\sigma} \tag{3.1}$$

and $d\tau d\sigma$ is the surface element of the world sheets, we can put $G^{(\alpha)}_{\mu\nu}$ in the form

$$G^{(\alpha)}_{\mu\nu}(x) = \sum_a M^{\alpha a} \int d\tau \int d\sigma \frac{\partial[y^{(a)}_\mu, y^{(a)}_\nu]}{\partial[\tau,\sigma]} \times \delta^{(4)}(x - y^{(a)}(\tau,\sigma))\,. \tag{3.2}$$

This leads to

$$\partial^\mu G^{(\alpha)}_{\mu\nu}(x) = \sum_a M^{\alpha a} \int d\tau \int d\sigma \left(-\frac{\partial y^{(a)}_\nu}{\partial\sigma}\frac{\partial}{\partial\tau} + \frac{\partial y^{(a)}_\nu}{\partial\tau}\frac{\partial}{\partial\sigma}\right) \times \delta^{(4)}(x - y^{(a)}(\tau,\sigma))\,. \tag{3.3}$$

In the following part of this section we assume that τ is the timelike parameter. We then choose $y^{(a)}_0$ as follows:

$$y^{(a)}_0(\tau,\sigma) = \tau \quad \text{for all } a\,. \tag{3.4}$$

Then, $y^{(a)}_\mu(\tau,\sigma)$ appear to be lines at each instance τ. These lines are parametrized by the spatial parameter σ. In this case the extended objects are called the string. The vortices in superconductors and the dislocations in crystals are some examples of the string.

Uses of (3.3) and (3.4) lead to

$$\partial^\mu G^{(\alpha)}_{\mu\nu}(x) = \sum_a M^{\alpha a} \int d\tau \frac{\partial y^{(a)}_\nu}{\partial\tau} \delta^{(4)}(x - y^{(a)}) \Bigg|^{\sigma^{(a)}_1(\tau)}_{\sigma^{(a)}_2(\tau)}\,, \tag{3.5}$$

where $\sigma^{(a)}_1(\tau)$ and $\sigma^{(a)}_2(\tau)$ are the end points of the line $y^{(a)}(\tau,\sigma)$ at time τ. Using (3.4), we have

$$\partial^\mu G^{(\alpha)}_{\mu 0}(x) = \sum_a M^{\alpha a}[\delta^{(3)}(\vec{x} - \vec{y}^{(a)}(t,\sigma^{(a)}_1)) - \delta^{(3)}(\vec{x} - \vec{y}^{(a)}(t,\sigma^{(a)}_2))]\,. \tag{3.6}$$

We can see from (3.6) that, if a line has an isolated end point, the divergenceless condition (2.8) is violated. Therefore, assembly of all the lines should form a network which does not have any end point. When lines $y^{(b)}(\tau,\sigma)$ $(b=1,2,\ldots)$ join with each other at a point (called a vertex or node) $y(\tau)$, the divergenceless condition (2.8) requires the continuity relation

$$\sum_b (\pm M^{\alpha b}) = 0\,, \tag{3.7}$$

where the + (−) sign corresponds to the first (second) term in the right-hand side of (3.6). Conversely, the right-hand side in (3.5) for all ν vanishes when (3.7) holds at each vertex, implying that (3.7) is the complete condition for the divergenceless condition (2.8) to be satisfied.

Let us now consider a surface S_C which is enclosed by a closed path C and define

$$B^{(\alpha)}_C = \int_{S_C} dS n_j G^{(\alpha)}_{0j}(x)\,, \tag{3.8}$$

where n_j is the normal unit vector on the surface S_C. From $\partial_j G^{(\alpha)}_{0j} = 0$, we see that the constants $B^{(\alpha)}_C$ do not change when the surface S_C varies, as far as the path C is fixed, and that $B^{(\alpha)}_{C_1} = B^{(\alpha)}_{C_2}$ when two closed paths C_1 and C_2 can be deformed continuously onto each other without crossing any string. Therefore, $B^{(\alpha)}_C$ do not change when the closed paths belong to a same homotopy class. In this sense, $B^{(\alpha)}_C$ are topological constants.

Since (3.2) and (3.4) gave

$$G^{(\alpha)}_{0j}(x) = \sum_a M^{\alpha a} \int d\sigma \delta^{(3)}(\vec{x} - \vec{y}^{(a)}(t,\sigma)) \frac{\partial}{\partial\sigma} y^{(a)}_j(t,\sigma)\,, \tag{3.9}$$

we have

$$B^\alpha_C = \sum_a{}' M^{\alpha a}\,, \tag{3.10}$$

where $\sum'$ means that the summation is performed only over those strings which cross the surface S_C. The continuity relation (3.7) again shows that B^α_C depend only on the closed path C.

Uses of (2.2), (2.6), and (3.8) lead to the relation

$$B^{(\alpha)}_C = \int_C d\vec{s} \cdot \vec{\nabla} f_\alpha(x)\,. \tag{3.11}$$

In other words, the topological constants $B^{(\alpha)}_C$ are the closed-path integral of $\vec{\nabla} f_\alpha(x)$.

It has been frequently assumed that the ring string may be used as the Pomeron. On the other hand, the open string cannot be confined in a finite domain, and therefore it is hard to assume that they form particles. When we studied the relativistic superconductivity model in Ref. 9, we simply cut off both tails of an open straight string to obtain a straight string of finite length with two end points, which create the monopole fields in addition to certain short-range forces. Then we met a difficulty; the nonvanishing domain of $G^{\dagger}_{\mu\nu}$ extends to infinity (i.e., the topological singularity covers everywhere) as soon as we cut off the tails of the open string. The only excuse for using such a cutoff process was that all gauge-invariant quantities in the superconductivity model do not contain f_α without derivatives, while $\partial_\mu f_\alpha$ does not have any topological singularity according to (2.3). In this paper we do not employ the cutoff process mentioned above, and we are going to look for other possibilities of confining the topological singularities in a finite domain.

A significance of the following study of the string is that, by combining many strings and deforming them, we can construct a full variety of extended objects. This will be illustrated by several examples.

A half-infinite-line singularity with the end point $y(\tau)$ (the hairpin string) is obtained by folding a string $y(\tau,\sigma)$ around a point $y(\tau)$ with relative distance ϵl_μ;

$$y_\mu(\tau,\sigma)=z_\mu(\tau,|\sigma|)+\tfrac{1}{2}\epsilon l_\mu,\quad -\infty<\sigma\le-\tfrac{1}{2}\epsilon\,, \tag{3.12a}$$

$$=y_\mu(\tau)-\sigma l_\mu,\quad -\tfrac{1}{2}\epsilon<\sigma<\tfrac{1}{2}\epsilon\,, \tag{3.12b}$$

$$=z_\mu(\tau,\sigma)-\tfrac{1}{2}\epsilon l_\mu,\quad \tfrac{1}{2}\epsilon\le\sigma<\infty\,, \tag{3.12c}$$

where

$$z_\mu(\tau,|\tfrac{1}{2}\epsilon|)=y_\mu(\tau)\,. \tag{3.13}$$

Here we consider only one world sheet and therefore omit the superscript a and write $M^{\alpha a}$ simply as M^α. The limits $\epsilon\to 0$ and $M^\alpha\to\infty$ should be performed in such a way that $\eta^\alpha\equiv\epsilon M^\alpha$ stays finite. Feeding (3.12) into (3.2) and performing the above limit, we obtain

$$G^{(\alpha)}_{\mu\nu}(x)=\int d\tau\int_0^\infty d\sigma\frac{\partial[z_\mu,z_\nu]}{\partial[\tau,\sigma]}B^\alpha_\lambda\partial^\lambda_x\delta^{(4)}(x-z(\tau,\sigma))$$
$$-\int d\tau\left(B^\alpha_\nu\frac{\partial}{\partial\tau}y_\mu(\tau)-B^\alpha_\mu\frac{\partial}{\partial\tau}y_\nu(\tau)\right)$$
$$\times\delta^{(4)}(x-y(\tau))\,, \tag{3.14}$$

where $B^\alpha_\lambda=\eta^\alpha l_\lambda$. One may notice the similarity of the present construction to that of dipole in electromagnetism: Every time the line is folded, there appear more derivatives in front of the δ function. Therefore the behavior of f_α near the singular line becomes more singular than that of the string.

It is obvious from the structure of the hairpin string that

$$\int_C(d\vec{s}\cdot\vec{\nabla}f_\alpha)=0 \tag{3.15}$$

for any α and any closed path C which does not cross the singularities.

The right-hand side of (3.14) contains the contributions of the end point $y(t)$ and the line singularities. However, we can make certain combinations of $G^{(\alpha)}_{\mu\nu}$ which have only the end-point effect, when B^α_λ and $z_\mu(\tau,\sigma)$ satisfy the following relations:

$$\sum_\alpha B^\alpha_\lambda\frac{\partial z_\alpha(\tau,\sigma)}{\partial\tau}=C_{\tau\tau}\frac{\partial z_\lambda(\tau,\sigma)}{\partial\tau}+C_{\tau\sigma}\frac{\partial z_\lambda(\tau,\sigma)}{\partial\sigma}\,, \tag{3.16a}$$

$$\sum_\alpha B^\alpha_\lambda\frac{\partial z_\alpha(\tau,\sigma)}{\partial\sigma}=C_{\sigma\tau}\frac{\partial z_\lambda(\tau,\sigma)}{\partial\tau}+C_{\sigma\sigma}\frac{\partial z_\lambda(\tau,\sigma)}{\partial\sigma}\,. \tag{3.16b}$$

Here it was assumed that $\eta_\alpha=0$ for $\alpha\ne 0$, 1, 2, and 3, and the C's are certain constants. Equations (3.16) lead to the wave equation for $z_\lambda(\tau,\sigma)$;

$$C_{\sigma\tau}\frac{\partial^2 z_\lambda}{\partial\tau^2}+(C_{\sigma\sigma}-C_{\tau\tau})\frac{\partial^2 z_\lambda}{\partial\tau\partial\sigma}-C_{\tau\sigma}\frac{\partial^2 z_\lambda}{\partial\sigma^2}=0\,. \tag{3.17}$$

When (3.16a), and (3.16b) are assumed, (3.14) gives

$$G^{(\alpha)}_{\mu\alpha}(x)=-(B^\alpha_\alpha-C_{\tau\tau}-C_{\sigma\sigma})\frac{dy_\mu(t)}{dt}\delta^{(3)}(\vec{x}-\vec{y}(t))\,. \tag{3.18}$$

Thus $G^{(\alpha)}_{\mu\alpha}$ contains only the end-point effect.

When observables depend on $f^{(\alpha)}(x)$ only through the quantity

$$G_\mu=G^{(\alpha)}_{\mu\alpha}$$
$$=\tfrac{1}{6}\epsilon_\mu{}^{\nu\lambda\rho}\{[\partial_\nu,\partial_\lambda]f_\rho+[\partial_\lambda,\partial_\rho]f_\nu+[\partial_\rho,\partial_\nu]f_\lambda\}\,, \tag{3.19}$$

one observes only the end-point effect. The relations (3.19) and (3.18) lead to

$$G_0(x)=\vec{\nabla}\cdot\vec{\nabla}\times\vec{f}$$
$$=-(B^\alpha_\alpha-C_{\tau\tau}-C_{\sigma\sigma})\delta^{(3)}(\vec{x}-\vec{y}(t))\,, \tag{3.20}$$

where $\vec{f}$ means (f_1,f_2,f_3). This shows that the hairpin string behaves as a monopole in the calculation of $\vec{\nabla}\cdot\vec{\nabla}\times\vec{f}$. It is straightforward to see that a closed-surface integral of $\vec{\nabla}\times\vec{f}$,

$$B_S=\int_S d\vec{s}\cdot\vec{\nabla}\times\vec{f}\,, \tag{3.21}$$

is a topological constant. When the surface S encloses the point $y(\tau)$, B_S is given by

$$B_S=C_{\tau\tau}+C_{\sigma\sigma}-B^\alpha_\alpha\,. \tag{3.22}$$

The expression (3.21) can be further rewritten as

$$B_S=\int_C d\vec{s}\cdot\vec{f}\,, \tag{3.23}$$

where C is a closed path in surface S. The closed path C should be chosen to pick up all the contributions from the line singularities possessed by $\vec{\nabla}\times\vec{f}$.

Since the hairpin strings have half-infinite lines, they cannot be confined to a finite domain. As was pointed out previously, one way of getting rid of this difficulty is to assume that f_α in the observables appear only through G_μ, that is, in the form

$$G_0=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{f}),$$

$$G_i=\partial_0(\vec{\nabla}\times\vec{f})_i-(\vec{\nabla}\times\partial_0\vec{f})_i .$$

This assumption is satisfied when and only when the Goldstone bosons $\chi_\alpha^{\mathrm{in}}$ in the dynamical map (1.3) for any observable appear through the forms $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\chi}^{\mathrm{in}})$ or $\partial_0(\vec{\nabla}\times\vec{\chi}^{\mathrm{in}})-(\vec{\nabla}\times\partial_0\vec{\chi}^{\mathrm{in}})$. We have not yet studied the question of what kind of Lagrangian satisfies this assumption.

Let us now turn our attention to closed strings. For simplicity we assume that only one Goldstone boson performs the boson transformation. In the static case, the nonvanishing components of $G_{\mu\nu}(x)$ are

$$G_{0k}(x)=-G_{k0}(x)=M\int d\sigma\frac{dy_k(\sigma)}{d\sigma}\delta^{(3)}(\vec{x}-\vec{y}(\sigma)). \tag{3.24}$$

As can be seen from (3.5), $\partial^\mu G_{\mu\nu}(x)$ vanishes when $y_k(\sigma)$ is a closed line. Therefore a closed string of any shape is acceptable to us.

As an example, we consider the case

$$\vec{y}(\sigma)=(a\cos\sigma, a\sin\sigma, 0),\quad a>0,\ 0\le\sigma<2\pi . \tag{3.25}$$

Substituting (3.25) and (3.24), we obtain

$$G_{01}(x)=-G_{10}(x)=-2M\delta(x_3)x_2\delta(r^2-a^2),$$
$$G_{02}(x)=-G_{20}(x)=2M\delta(x_3)x_1\delta(r^2-a^2), \tag{3.26}$$

where $r^2=x_1{}^2+x_2{}^2+x_3{}^2$. Other components of $G_{\mu\nu}$ vanish. We shall call this type of closed string a loop or a ring. It is known that, in crystals, many loop dislocations are porduced, for instance, by the Frank-Read mechanism. As for a model of particles, an infinitesimal loop might be interesting. In the limit of $a\to 0$ and $M\to\infty$, (3.24) and (3.25) yields

$$G_{01}(x)=-G_{10}(x)=g\delta(x_1)\delta'(x_2)\delta(x_3), \tag{3.27a}$$

$$G_{02}(x)=-G_{20}(x)=-g\delta'(x_1)\delta(x_2)\delta(x_3), \tag{3.27b}$$

where $g=\pi Ma^2$. Other components of $G_{\mu\nu}$ vanish.

A line-shape object of finite length is given by a rectangular closed loop with an infinitesimal width ϵl_μ;

$$y_\mu(\tau,\sigma)=z_\mu(\tau,\sigma)-\tfrac{1}{2}\epsilon l_\mu \quad\text{for } \tfrac{1}{2}\epsilon\le\sigma<1, \tag{3.28a}$$

$$=z_\mu(\tau,\sigma)+\tfrac{1}{2}\epsilon l_\mu \quad\text{for } -1\le\sigma\le-\tfrac{1}{2}\epsilon, \tag{3.28b}$$

$$=y_\mu^1(\tau)-l_\mu\sigma \quad\text{for } -\tfrac{1}{2}\epsilon<\sigma<\tfrac{1}{2}\epsilon, \tag{3.28c}$$

$$=y_\mu^2(\tau)+l_\mu(\sigma-1-\tfrac{1}{2}\epsilon) \quad\text{for } 1\le\sigma\le 1+\epsilon, \tag{3.28d}$$

where

$$y_\mu^1(\tau)=z_\mu(\tau,0), \tag{3.29a}$$

$$y_\mu^2(\tau)=z_\mu(\tau,1). \tag{3.29b}$$

The two points, y_μ^1 and y_μ^2, are the end points of the line object. Since we are considering one closed string, we can omit the superscript (a) in (3.2) and denote $M^{\alpha a}$ simply by M^α. Then (3.2) leads to

$$G_{\mu\nu}^{(\alpha)}(x)=\int d\tau\int_0^1 d\sigma\frac{\partial[z_\mu,z_\nu]}{\partial[\tau,\sigma]}B_\lambda^\alpha\partial_x^\lambda\delta^{(4)}(x-z)-[B_\nu^\alpha\dot{y}_\mu^1(t)\delta^{(3)}(\vec{x}-\vec{y}^1(t))-B_\mu^\alpha\dot{y}_\nu^1(t)\delta^{(3)}(\vec{x}-\vec{y}^1(t))]$$
$$+[B_\nu^\alpha\dot{y}_\mu^2(t)\delta^{(3)}(\vec{x}-\vec{y}^2(t))-B_\mu^\alpha\dot{y}_\nu^2(t)\delta^{(3)}(\vec{x}-\vec{y}^2(t))], \tag{3.30}$$

where $B_\lambda^\alpha=\epsilon M^\alpha l_\lambda$. The limits $\epsilon\to 0$ and $M^\alpha\to\infty$ were performed in such a way that ϵM^α stays finite. It can be easily seen that the result (3.30) can be obtained also by superposing the two sets of the hairpin strings with opposite strength; one with end point $y_\mu^1(\tau)$ and the other with end point $y_\mu^2(\tau)$. The extended object under consideration will be called a dihairpin string.

As a particular case, let us consider the static dihairpin string, which is along the first axis:

$$z_\mu(\tau,\sigma)=(\tau,\sigma d,0,0). \tag{3.31}$$

This specifies the two end points, $y_\mu^{(1)}$ and $y_\mu^{(2)}$:

$$y_\mu^1(\tau)=(\tau,0,0,0),\quad y_\mu^2(\tau)=(\tau,d,0,0). \tag{3.32}$$

The symbol d denotes the distance between two end points. To study this static dihairpin string we choose

$$B_0^\alpha=0. \tag{3.33}$$

We now assume that not more than three Goldstone bosons ($\alpha=1,2,3$) perform the boson transformation. Then (3.30) gives

$$G_{ij}^{(\alpha)}(x)=0\,, \tag{3.34a}$$

$$G_{0j}^{(\alpha)}(x)=B_j^\alpha[\delta(x_1)-\delta(x_1-d)]\delta(x_2)\delta(x_3) + \delta_{j1}\sum_k B_k^\alpha \partial^k[\theta(x_1)-\theta(x_1-d)]\,\delta(x_2)\delta(x_3)\,. \tag{3.34b}$$

According to (3.34b), the topological singularity is the line of length d along the x_1 axis. It is an interesting question to ask whether we can construct, by means of the dihairpin strings, a model for particles which satisfies the duality condition.

When we choose

$$B_2^1=B_3^1=0\,, \tag{3.35}$$

we have, from (3.34b), the relation

$$\sum_j G_{0j}^{(j)}(x)=-(B_2^2+B_3^3)[\delta(\vec{x})-\delta(\vec{x}-\vec{d})]\,, \tag{3.36}$$

where $\vec{d}=(d,0,0)$. This gives

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{f}(x)=-(B_2^2+B_3^3)[\delta(\vec{x})-\delta(\vec{x}-\vec{d})]\,. \tag{3.37}$$

This relation shows that the dihairpin string acts as a dimonopole in the calculation of $\vec{\nabla}\cdot\vec{\nabla}\times\vec{f}(x)$. When some observables contain $\partial_\mu f^{(\alpha)}$ in other forms than $\vec{\nabla}\cdot\vec{\nabla}\times\vec{f}$, we observe the line effect in addition to the dimonopole effect.

IV. THE CLOSED-SURFACE SINGULARITIES

In the previous section, we have considered the extended objects with line singularities. We devote this section to a general study of closed objects with surface singularities which can be created by the boson transformation.

The straightforward extension to a surface from the strings is to pile up strings with certain weight $\rho(\eta)$;

$$G_{\mu\nu}(x)=\int d\tau d\sigma d\eta \frac{\partial[y_\mu,y_\nu]}{\partial[\tau,\sigma]}\rho(\eta)\delta^{(4)}(x-y(\tau,\sigma,\eta))\,. \tag{4.1}$$

For example, a static closed shell

$$y_\mu(\tau,\sigma,\eta)=(\tau,a\sin\eta\cos\sigma,a\sin\eta\sin\sigma,a\cos\eta)\,,\quad 0\leqslant\sigma<2\pi,\quad 0\leqslant\eta<\pi\,, \tag{4.2}$$

with the weight $\rho(\eta)$,

$$\rho(\eta)=M\sin\eta\,, \tag{4.3}$$

leads to

$$\begin{aligned} G_{01}(x)&=-\frac{2M}{a^2}x_2(x_1^{\,2}+x_2^{\,2})^{1/2}\delta(r^2-a^2)\,,\\ G_{02}(x)&=\frac{2M}{a^2}x_1(x_1^{\,2}+x_2^{\,2})^{1/2}\delta(r^2-a^2)\,, \end{aligned} \tag{4.4}$$

which implies that the singular surface is a sphere. In the case of an infinitesimal closed shell ($a\to 0$), (4.1)–(4.3) give

$$\begin{aligned} G_{01}(x)&=g\delta(x_1)\delta'(x_2)\delta(x_3)\,,\\ G_{02}(x)&=-g\delta'(x_1)\delta(x_2)\delta(x_3)\,, \end{aligned} \tag{4.5}$$

where $g=(4\pi/3)Ma^2$. It is to be remarked that an infinitesimal loop and an infinitesimal closed shell have the same type of topological singularities [cf. (3.27)].

The example presented above is made of closed rings. To look for a general expression for closed system, let us come back to the divergenceless condition $\partial^\mu G_{\mu\nu}(x)=0$. As will be shown later, this condition means the continuity of the singularity. Therefore, when an object has a domain of singularity without any end point, its $G_{\mu\nu}$ satisfies the divergenceless condition. The sphere in three dimensions is an example of this kind. Quite generally we can state that any closed surface is acceptable to our scheme.

To prove this, let us consider a closed system which has no end points. The surface singularity is expressed by $y_\mu(\tau,\xi,\sigma)$, in which τ is the time. For convenience, we use the parameter ξ to parametrize the flat planes which cut the object under consideration. The domain of this parameter is chosen according to $0\leqslant\xi\leqslant\pi$. The cross section of the closed surface and the plane ξ is a closed loop, which is parametrized by σ ($0\leqslant\sigma\leqslant 2\pi$). We then have

$$\begin{aligned} y_\mu(\tau,\xi,0)&=y_\mu(\tau,\xi,2\pi)\,,\\ y_\mu(\tau,0,\sigma)&=y_\mu(\tau,0,0)\,,\\ y_\mu(\tau,\pi,\sigma)&=y_\mu(\tau,\pi,0)\,, \end{aligned} \tag{4.6}$$

which imply that, at $\xi=0$ and π, y_μ does not depend on σ. It is easy to show that

$$G_{\mu\nu}^{(\alpha)}=M^{\alpha\beta}\int\int\int d\tau d\xi d\sigma\frac{\partial[y_\mu,y_\nu,y_\beta]}{\partial[\tau,\xi,\sigma]}\times\delta^{(4)}(x-y(\tau,\xi,\sigma)) \tag{4.7}$$

satisfies the divergenceless condition. Indeed we have

$$\partial^\mu G_{\mu\nu}^{(\alpha)}=-M^{\alpha\beta}\int\int\int d\tau d\xi d\sigma\left(\frac{\partial[y_\nu,y_\beta]}{\partial[\xi,\sigma]}\frac{\partial}{\partial\tau}-\frac{\partial[y_\nu,y_\beta]}{\partial[\tau,\sigma]}\frac{\partial}{\partial\xi}+\frac{\partial[y_\nu,y_\beta]}{\partial[\tau,\xi]}\frac{\partial}{\partial\sigma}\right)\delta^{(4)}(x-y(\tau,\xi,\sigma))\,. \tag{4.8}$$

The condition $|y_0(\tau,\xi,\sigma)|\to\infty$ for $|\tau|\to+\infty$ together with (4.6) leads to the result

$$\partial^\mu G^{(\alpha)}_{\mu\nu}=0\,,$$

implying that any closed surface in space is acceptable to ours scheme. As an example, we consider the static sphere:

$$y_\mu(\tau,\xi,\sigma)=(\tau, a\sin\xi\cos\sigma, a\sin\xi\sin\sigma, a\cos\xi)\,. \tag{4.9}$$

This leads to

$$G^{(\alpha)}_{0i}(x)=M^{\alpha\beta}\epsilon_{i\beta k}\frac{x_k}{r}\delta(r-a)\,,\qquad G^{(\alpha)}_{ij}(x)=0\,. \tag{4.10}$$

This singularity is on the surface of the sphere with radius a and will be called a bag. In the limit $a\to 0$, we have

$$G^{(\alpha)}_{0i}(x)=\frac{4\pi}{3}a^3M^{\alpha\beta}\epsilon_{i\beta k}\nabla_k\delta^{(3)}(\vec{x})\,. \tag{4.11}$$

The special case in which all $M^{\alpha\beta}$ except M^{13} vanish, this again leads to (4.5).

On the other hand, when we have $(4\pi/3)a^3M^{\alpha\beta}=g\delta_{\alpha\beta}$ $(\alpha,\beta=1,2,3)$, (2.5), (2.6), and (4.11) lead to

$$\vec{\nabla}\cdot\vec{f}(\vec{x})=-2g\delta^{(3)}(\vec{x})\,, \tag{4.12}$$

where $\vec{f}$ means (f_1,f_2,f_3). We thus see that

$$G_S=\int_S d\vec{S}\cdot\vec{f} \tag{4.13}$$

is a topological constant when S is a closed surface which does not cross the singular domain.

Closed surfaces have many applications. While infinitesimal closed surfaces might describe elementary particles, closed surfaces with finite size might be applicable to the bag for the quark-confinement model, the black holes in general relativity, the magnetic domains in ferromagnets, crystallites in solid, etc.

V. INSTANTANEOUS AND FINITE-LIFETIME SINGULARITIES

In previous sections, we have considered only those extended objects which exist at all times. However, there can be other kinds of extended objects created by the boson transformation. When a singular domain has no end points in the four-dimensional space-time world, the divergenceless condition (2.8) is still satisfied. Therefore, considering a world sheet confined in a certain time interval, we can deal with the extended objects which exist only for this time interval. In this section, we shall study this kind of objects, i.e., the instantaneous and finite-lifetime singularities.

Let us begin with the spatial surface singularity:

$$G_{\mu\nu}(x)=\int\int d\xi d\sigma\frac{\partial[y_\mu,y_\nu]}{\partial[\xi,\sigma]}\delta^{(4)}(x-y(\xi,\sigma))\,. \tag{5.1}$$

This is different from (3.2), because the parameters ξ and σ in (5.1) are both spacelike parameters; the world sheet $y_\mu(\xi,\sigma)$ extends in a spacelike direction. When we choose

$$y_\mu(\xi,\sigma)=(t_0,\vec{y}(\xi,\sigma))\,, \tag{5.2}$$

with t_0 being a constant, then (5.1) gives

$$G_{ij}(x)=\delta(x_0-t_0)M\int\int d\xi d\sigma\frac{\partial[y_i,y_j]}{\partial[\xi,\sigma]}\delta^{(3)}(\vec{x}-\vec{y}(\xi,\sigma))\,,\qquad G_{0i}(x)=0\,, \tag{5.3}$$

which implies that the surface singularity appears only at $x_0=t_0$. The divergenceless condition $\partial^\mu G_{\mu\nu}(x)=0$ is satisfied when $\vec{y}(\xi,\sigma)$ forms a surface without any end point (see Sec. IV). In order to obtain an intuitive picture of this type of singularity, let us consider an instantaneous plane which appears only at $t_0=0$:

$$y_\mu(\xi,\sigma)=(0,0,\sigma,\xi),\quad -\infty<\xi,\sigma<\infty\,. \tag{5.4}$$

In this case we find

$$-G_{23}(x)=G_{32}(x)=M\delta(x_0)\delta(x_1) \tag{5.5}$$

and the other $G_{\mu\nu}$ vanish. According to (2.5) the boson function $f(x)$ in this case is given by the relations

$$\partial_0 f(x)=\frac{-i}{(2\pi)^2}M\int dp_0dp_1\frac{p_1}{p_0^{\,2}-p_1^{\,2}}e^{-ip_0x_0+ip_1x_1}\,, \tag{5.6a}$$

$$\partial_1 f(x)=\frac{-i}{(2\pi)^2}M\int dp_0dp_1\frac{p_0}{p_0^{\,2}-p_1^{\,2}}e^{-ip_0x_0+ip_1x_1}\,. \tag{5.6b}$$

We use the retarded Green's function in (5.6a) and (5.6b), since we are interested in effects of the instantaneous plane after it is created. Replacing $p_0^{\,2}$ in (5.6a) and (5.6b) by $(p_0+i\epsilon)^2$, we get

$$\partial_0 f(x)=-\tfrac{1}{2}M\,\theta(x_0)[\delta(x_0-x_1)-\delta(x_0+x_1)]\,, \tag{5.7a}$$

$$\partial_1 f(x)=-\tfrac{1}{2}M\,\theta(x_0)[\delta(x_0-x_1)+\delta(x_0+x_1)]\,. \tag{5.7b}$$

Then, $f(x)$ is obtained by means of the path-integral calculation as follows:

$$f(x)=\int dx^\mu\partial_\mu f(x)=\int[dx_1\partial_1 f(x)-dx_0\partial_0 f(x)]\,. \tag{5.8}$$

This leads to

$$f(x)=\tfrac{1}{2}M[\theta(\phi(x)-\pi/4)+\theta(\phi(x)-3\pi/4)]+nM\,, \tag{5.9}$$

where $\tan\phi(x)=x_0/x_1$ and n is a certain integer. As is seen from (5.9), $f(x)$ jumps whenever x crosses the forward light cone. In the superconductivity model,[7,9] $f(x)$ is the phase of the order parameter and creates the electric field

$$E_1(x)=-\frac{m^2}{2e}\frac{1}{\partial^2-m^2}(\partial_0\partial_1-\partial_1\partial_0)f$$
$$=-\frac{m^2}{2e}\frac{M}{(2\pi)^2}\int dp_0dp_1\frac{1}{(p_0+i\epsilon)^2-p_1{}^2-m^2}\times e^{-ip_0x_0+ip_1x_1}$$
$$=\theta(x_0)\frac{m^2}{2e}\frac{M}{2}J_0(m(x_0{}^2-x_1{}^2)^{1/2})\quad(|x_1|<x_0) \tag{5.10}$$

while no magnetic field is created. This is the effect of instantaneous appearance of the singular plane at $x_0=0$ which propagates in the future light cone. A detailed analysis of this phenomenon in a superconductor has been made in Ref. 12.

Following the way in which we constructed the hairpin string, we can deform a singular world sheet in a variety of ways in order to make the domain of instantaneous singularities smaller. In the following, we present some examples.

By exchanging the role of τ and σ in the expression (3.14) for the hairpin string, we find an open spatial-line singularity which is born at $x_0=0$; i.e., a born distring. In this case we have

$$G_{\mu\nu}(x)=B\int_0^\infty d\tau\int_{-\infty}^\infty d\sigma\frac{\partial[y_\mu,y_\nu]}{\partial[\tau,\sigma]}l_\lambda\partial_x^\lambda\delta^{(4)}(x-y(\tau,\sigma))+B\int_{-\infty}^\infty d\sigma\left(\frac{\partial y_\mu(0,\sigma)}{\partial\sigma}l_\nu-\frac{\partial y_\nu(0,\sigma)}{\partial\sigma}l_\mu\right)\times\delta^{(4)}(x-y(0,\sigma)), \tag{5.11}$$

where $B=\epsilon M$. The choice

$$y_\mu(\tau,\sigma)=(\tau,\sigma,0,0), \tag{5.12a}$$
$$l_\mu=(0,0,1,0) \tag{5.12b}$$

leads to

$$G_{01}(x)=\theta(x_0)\delta'(x_2)\delta(x_3), \tag{5.13a}$$
$$G_{12}(x)=\delta(x_0)\delta(x_2)\delta(x_3). \tag{5.13b}$$

We consider two overlapping world sheets with a common boundary which is a timelike closed line (i.e., a flattened closed world sheet). The result is a dihairpin which exists for certain time intervals and is called the transient dihairpin string. In this case we have

$$\begin{aligned}G_{\mu\nu}(x)=&B\int_0^{t_0}d\tau\int_0^d d\sigma\frac{\partial[y_\mu,y_\nu]}{\partial[\tau,\sigma]}l_\lambda\partial_x^\lambda\delta^{(4)}(x-y(\tau,\sigma))+B\int_0^d\left(\frac{\partial y_\mu(0,\sigma)}{\partial\sigma}l_\nu-\frac{\partial y_\nu(0,\sigma)}{\partial\sigma}l_\mu\right)\delta^{(4)}(x-y(0,\sigma))\\&-B\int_0^d d\sigma\left(\frac{\partial y_\mu(t_0,\sigma)}{\partial\sigma}l_\nu-\frac{\partial y_\nu(t_0,\sigma)}{\partial\sigma}l_\mu\right)\delta^{(4)}(x-y(t_0,\sigma))\\&-B\int_0^{t_0}d\tau\left(\frac{\partial y_\mu(\tau,0)}{\partial\tau}l_\nu-\frac{\partial y_\nu(\tau,0)}{\partial\tau}l_\nu\right)\delta^{(4)}(x-y(\tau,0))\\&+B\int_0^{t_0}d\tau\left(\frac{\partial y_\mu(\tau,d)}{\partial\tau}l_\nu-\frac{\partial y_\nu(\tau,d)}{\partial\tau}l_\mu\right)\delta^{(4)}(x-y(\tau,d)).\end{aligned} \tag{5.14}$$

The choice (5.12a) and (5.12b) lead to

$$\begin{aligned}G_{01}(x)&=[\theta(x_0)-\theta(x_0-t_0)][\theta(x_1)-\theta(x_1-d)]\delta'(x_2)\delta(x_3),\\G_{02}(x)&=-[\theta(x_0)-\theta(x_0-t_0)][\delta(x_1)-\delta(x_1-d)]\delta(x_2)\delta(x_3),\\G_{12}(x)&=[\delta(x_0)-\delta(x_0-t_0)][\theta(x_1)-\theta(x_1-d)]\delta(x_2)\delta(x_3).\end{aligned} \tag{5.15}$$

Performing the limit t_0, $d\to 0$ in (5.14), we find a point singularity in the (3+1)-dimensional space (a kind of instanton):

$$\begin{aligned}G_{\mu\nu}(x)=(Bt_0d)\Bigg[&\frac{\partial[y_\mu,y_\nu]}{\partial[\tau,\sigma]}l_\lambda\partial_x^\lambda\delta^{(4)}(x-y(\tau,\sigma))+\left(\frac{\partial y_\mu}{\partial\sigma}l_\nu-\frac{\partial y_\nu}{\partial\sigma}l_\mu\right)\frac{\partial y_\lambda}{\partial\tau}\partial_x^\lambda\delta^{(4)}(x-y(\tau,\sigma))\\&-\left(\frac{\partial y_\mu}{\partial\tau}l_\nu-\frac{\partial y_\nu}{\partial\tau}l_\mu\right)\frac{\partial y_\lambda}{\partial\sigma}\partial_\lambda^x\delta^{(4)}(x-y(\tau,\sigma))\Bigg]_{\tau=\sigma=0}.\end{aligned} \tag{5.16}$$

For example the choice (5.12a) and (5.12b) give

$$G_{01}(x)=(Bt_0d)\delta(x_0)\delta(x_1)\delta'(x_2)\delta(x_3)\,,$$
$$G_{02}(x)=-(Bt_0d)\delta(x_0)\delta'(x_1)\delta(x_2)\delta(x_3)\,,\qquad(5.17)$$
$$G_{12}(x)=(Bt_0d)\delta'(x_0)\delta(x_1)\delta(x_2)\delta(x_3)\,.$$

Let us turn our attention to transient closed systems which are created at t_0 and disappear at t_1. The transient dihairpin studied above is an example of this kind. A transient loop is given by

$$G_{\mu\nu}(x)=M\int_{t_0}^{t_1}d\tau\int_0^{2\pi}d\sigma\frac{\partial[y_\mu,y_\nu]}{\partial[\tau,\sigma]}\delta^{(4)}(x-y(\tau,\sigma))\qquad(5.18)$$

with conditions

$$y_\mu(t_0,\sigma)=(t_0,\vec{y}(t_0))\,,\quad y_\mu(t_1,\sigma)=(t_1,\vec{y}(t_1))$$

and

$$y_\mu(\tau,0)=y_\mu(\tau,2\pi)\,.\qquad(5.19)$$

A transient ring is given by

$$y_\mu(\tau,\sigma)=(\tau,a(\tau)\cos\sigma,a(\tau)\sin\sigma,0)\qquad(5.20)$$

with

$$a(t_0)=a(t_1)=0\,.$$

Any closed surface which appears at t_0 and disappears at t_1 can be put in the form

$$G_{\mu\nu}(x)=M^\beta\int_{t_0}^{t_1}d\tau\int_0^{\pi}d\xi\int_0^{2\pi}d\sigma\frac{\partial[y_\mu,y_\nu,y_\beta]}{\partial[\tau,\xi,\sigma]}\times\delta^{(4)}(x-y(\tau,\xi,\sigma))\qquad(5.21)$$

with the conditions that $y_\mu(\tau,\xi,\sigma)$ forms a closed surface for any value of τ and that $y_\mu(t_0,\xi,\sigma)$ and $y_\mu(t_1,\xi,\sigma)$ are independent of ξ and σ. At the limit $t_0\to t_1$, this gives the instanton mentioned above.

Use of these instantaneous or transient singularities supplies us with a powerful method for study of time-dependent macroscopic phenomena in systems with spontaneously broken symmetry (i.e., the transient phenomena in systems with spontaneously broken symmetry).

VI. THE SYMMETRY REARRANGEMENT AND THE TOPOLOGICAL QUANTUM NUMBERS

In previous sections, we studied a variety of topological singularities associated with extended objects which appear in systems with spontaneous breakdown of symmetry. The constants such as

$$B_C^{(\alpha)}=\int_C d\vec{s}\cdot\vec{\nabla}f_\alpha(x)$$

in (3.11) and

$$B_S=\int_S d\vec{S}\cdot\vec{\nabla}\times\vec{f}(x)$$

in (3.21) are some examples of topological constants. When a topological constant is quantized, it is called the topological quantum number.

To find which topological numbers should be quantized, we must know the structure of the symmetry rearrangement which is induced by the spontaneous breakdown of symmetry.

In the first section we introduced the dynamical map [cf. (1.2)]

$$\psi(x)=\psi(x;\chi_\alpha^{\text{in}},\phi^{\text{in}})\,,\qquad(6.1)$$

in which the Heisenberg field ψ is expressed in terms of normal products of in-fields χ_α^{in} and ϕ^{in}. Suppose now that the Lagrangian of the system is invariant under certain transformation (say, Q transformation) of ψ:

$$\psi\to\psi'=Q[\psi]\,.\qquad(6.2)$$

When this transformation is induced by a certain in-field transformation (the q transformation)

$$\chi_\beta^{\text{in}}\to\chi_\beta^{\text{in}\prime}=q_\chi(\chi_\alpha^{\text{in}},\phi^{\text{in}})\,,\qquad(6.3\text{a})$$

$$\phi^{\text{in}}\to\phi^{\text{in}\prime}=q_\phi(\chi_\alpha^{\text{in}},\phi^{\text{in}})\,,\qquad(6.3\text{b})$$

we can write as

$$Q[\psi]=\psi(x;q_\chi,q_\phi)\,.\qquad(6.4)$$

We then state that the Q symmetry is dynamically rearranged into the q symmetry. The Q symmetry is the symmetry of basic equations, while the q symmetry is the observable symmetry. The symmetry groups associated Q and q transformations will be denoted by G_Q and G_q, respectively.

To see how the dynamical map selects the topological quantum number, we recall two examples: the superconductivity model and the crystal model.

In the case of the superconductivity model, the Lagrangian is invariant under the phase transformation $\psi\to\exp(i\theta)\psi$. In this case, there appears only one Goldstone boson χ^{in}, and the dynamical map for ψ takes the form

$$\psi(x)={:}\exp(iC\chi^{\text{in}})F(x;\partial\chi^{\text{in}},\phi^{\text{in}}){:}\,.\qquad(6.5)$$

Here C is a constant, ϕ^{in} stands for the in-fields other than χ^{in}, and F is certain function. The q transformation $\chi^{\text{in}}\to\chi^{\text{in}}+\theta/C$ induces the Q transformation $\psi\to e^{i\theta}\psi$. This is the dynamical rearrangement of phase symmetry.

The boson transformation $\chi^{\text{in}}\to\chi^{\text{in}}+(1/C)f$ (with f satisfying $\partial^2 f=0$) changes ψ into

$$\psi^f(x)=\exp(if){:}\exp(iC\chi^{\text{in}})F(x;\partial\chi^{\text{in}}+\partial f,\phi^{\text{in}}){:}\,.\qquad(6.6)$$

Let the position $\vec{x}$ move along a closed path C.

We see from (6.6) that $\partial_\mu f$ does not have any topological singularity. The requirement that the dynamical map for ψ^f in (6.6) is well defined at each point $\vec{x}$ leads to the condition

$$B_C \equiv \int_C d\vec{s}\cdot\vec{\nabla} f(x) = 2\pi n \quad (n = \text{integer}). \tag{6.7}$$

This implies that B_C is the topological quantum number. The relation (6.7) is called the flux quantization.

In the case of the crystal model, the Lagrangian is invariant under the spatial translation $\psi(\vec{x},t) \to \psi(\vec{x}+\vec{\alpha},t)$. There appear three Goldstone bosons χ^{in}_α ($\alpha = 1,2,3$) which are identified as acoustic phonons. We use the notation $\vec{\chi}^{\text{in}} = (\chi^{\text{in}}_1, \chi^{\text{in}}_2, \chi^{\text{in}}_3)$. The dynamical map takes the form[10]

$$\psi(x) = :\sum_\lambda \Phi_\lambda(\vec{x}+\vec{\chi}^{\text{in}}) F_\lambda(x;\partial\vec{\chi}^{\text{in}}, \phi^{\text{in}}):. \tag{6.8}$$

Here $\{\Phi_\lambda(\vec{x})\}$ is the orthonormalized complete set of periodic functions with the lattice vectors $\vec{a}_i$ ($i = 1,2,3$).

According to (6.8), the q transformation

$$\vec{\chi}^{\text{in}}(\vec{x}+t) \to \vec{\chi}^{\text{in}}(\vec{x}+\vec{\alpha},t) + \vec{\alpha}, \tag{6.9a}$$

$$\phi^{\text{in}}(\vec{x},t) \to \phi^{\text{in}}(\vec{x}+\vec{\alpha},t) \tag{6.9b}$$

induces the Q transformation

$$\psi(\vec{x},t) \to \psi(\vec{x}+\vec{\alpha},t), \tag{6.10}$$

that is, the spatial translation. This shows how the translational symmetry is rearranged.

The boson transformation $\vec{\chi}^{\text{in}} \to \vec{\chi}^{\text{in}} + \vec{f}$ ($\partial^2 f = 0$) changes ψ into

$$\psi^f(x) = :\sum_\lambda \Phi_\lambda(\vec{x}+\vec{f}+\vec{\chi}^{\text{in}}) F_\lambda(x;\partial\vec{\chi}^{\text{in}}+\partial\vec{f}, \phi^{\text{in}}):. \tag{6.11}$$

Let the position $\vec{x}$ move along a closed path C. Equation (6.11) shows that $\partial_\mu \vec{f}$ does not have any topological singularity. Since ψ^f is well defined at each point $\vec{x}$, we need the condition

$$\vec{B}_C = \int_C (d\vec{s}\cdot\vec{\nabla})\vec{f}(x) = \sum_i n_i \vec{a}_i, \tag{6.12}$$

where n_i are integers. This implies that $\vec{B}_C$ is the topological quantum number, which is usually called the Burgers vector in solid-state physics.

In general, when groups G_Q and G_q are known, the number of Goldstone bosons is determined and the structure of the dynamical map (and therefore, of the symmetry rearrangement) is identified. This structure of the dynamical map is the key to tell us which topological constants should be quantized. On the other hand, the choice of the singularities is not strongly restricted by the groups G_Q and G_q. As was mentioned in Sec. I. the shape of the singularities depends on boundary conditions.

Let us close this paper by a short comment on the Bohm-Aharonov phenomenon. According to the Gupta-Bleuler formalism, there exist a massless (Goldstone) boson χ^{in} and a ghost b^{in} in quantum electrodynamics. It has been shown[13] that the boson transformation

$$\begin{aligned} \chi^{\text{in}} &\to \chi^{\text{in}} + f, \\ b^{\text{in}} &\to b^{\text{in}} + f \end{aligned} \tag{6.13}$$

induces the gauge transformation. When f has a topological singularity, this boson transformation induces the following changes in the vector potential $a_\mu(x)$ and the field strength $F_{\mu\nu}(x)$:

$$\begin{aligned} \delta a_\mu(x) &= \partial_\mu f(x), \\ \delta F_{\mu\nu}(x) &= [\partial_\mu, \partial_\nu] f(x) = G^\dagger_{\mu\nu}(x). \end{aligned} \tag{6.14}$$

The latter relation shows that $\delta F_{\mu\nu} = 0$ outside of the singular domain.

The topological constant B_C is given by

$$\begin{aligned} B_C &= \int_C d\vec{s}\cdot\vec{\nabla} f(x) \\ &= \int_C d\vec{s}\cdot\delta\vec{a} \\ &= \int_S dS^{\mu\nu} G^\dagger_{\mu\nu}(x), \end{aligned} \tag{6.15}$$

where C is a closed path going around the singular domain and S is the surface enclosed by C. This result shows that the topological constant B_C is the electron phase difference in the Bohm-Aharonov phenomena. From a somewhat different viewpoint, the topological aspect of the Bohm-Aharonov phenomena in Abelian and non-Abelian gauges has been analyzed by Wu and Yang.[14]

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