Upper bound on the color-confining potential

Erhard Seiler
Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540
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Under rather general assumptions it is shown that the (suitably defined) potential between color charges cannot rise faster than linearly with distance. The main ingredients are the existence of a positive Hamiltonian and invariance under 90° Euclidean rotations. In order to be on firm ground mathematically, the discussion is carried out for lattice theories; the role of the lattice, however, is purely technical.

INTRODUCTION

We discuss two cases: pure Yang-Mills theory with "heavy external charges" and lattice quantum chromodynamics (QCD).

We use the setup of lattice gauge theories described in Refs. 1–4; the lattice constant is denoted by ϵ.. There is a transfer matrix T obeying $0<T\leq 1$ which we write as $e^{-\epsilon H}$.[3-5]

I. PURE YANG-MILLS THEORY

In this theory it is possible to define a Hilbert space containing "charged" states (see Ref. 4) by working in the axial gauge $A_0=0$. These states are not invariant under (time-independent) gauge transformations; this noninvariance corresponds to the presence of "external" charges. Let ψ_R denote any such state containing just two such charges separated by a distance R (total charge 0). We can define

$$V(R)\equiv \inf_{\psi_R} \inf \operatorname{spec} H\restriction\psi_R \tag{1}$$

(that is, the lowest energy present in any of the admissible trial states ψ_R). A particularly simple trial state is the "string"

$$\phi_R \equiv \operatorname{Tr}\left\{P \exp\left[i\int_0^R A_1(0,\vec{x})dx_1\right]\Omega\right\}, \tag{2}$$

where Ω is the vacuum, P is a path-ordering symbol, and $A_\mu=\sum A_\mu^a X^a$ where $\{X^a\}$ is a basis of the Lie algebra (in some faithful representation) of the gauge group.

The integral in (2) actually reduces to a sum on the lattice. By construction $\|\phi_R\|=1$. (The reader who wants to check this should use the version of the reconstruction given in Ref. 3, where $t=0$ is a lattice hyperplane.) Now let t be any integer multiple of ϵ. Clearly,

$$V(R)\leq -\frac{1}{t}\ln(\phi_R, e^{-tH}\phi_R). \tag{3}$$

The right-hand side can be rewritten in gauge-invariant fashion in terms of the Wilson loop:

$$(\phi_R, e^{-tH}\phi_R)=\left\langle P\exp\left(i\oint A_\mu dx^\mu\right)\right\rangle, \tag{4}$$

where the integral on the right is over a lattice rectangle of sides R and t and the angular brackets denote the average in the gauge-invariant measure (Refs. 3 and 4). By performing now a 90° rotation and going to the axial gauge $A_0=0$ again we obtain

$$(\phi_R, e^{-tH}\phi_R)=(\phi_t, e^{-RH}\phi_t) \tag{5}$$

(a relation of this type is known in constructive field theory as Nelson's symmetry).

Using now the spectral decomposition of H and Hölder's inequality, it is straightforward to see that

$$\begin{aligned} V(R) &\leq -\frac{1}{t}\ln(\phi_t, e^{-RH}\phi_t)\\ &\leq -\frac{R}{t\epsilon}\ln(\phi_t, e^{-\epsilon H}\phi_t)\\ &= -\frac{R}{t\epsilon}\ln(\phi_\epsilon, e^{-tH}\phi_\epsilon)\\ &\leq -\frac{R}{\epsilon^2}\ln(\phi_\epsilon, e^{-\epsilon H}\phi_\epsilon), \end{aligned} \tag{6}$$

which is the desired bound.

Remark. This argument also shows that the Wilson loop cannot have more than area decay, i.e.,

$$(\phi_R, e^{-tH}\phi_R)\geq e^{-Rt/\epsilon^2}(\phi_\epsilon, e^{-H}\phi_\epsilon) \tag{7}$$

(an upper bound of the same form was proved in Ref. 4 for strong coupling). For the Abelian case, such a bound is proved in the second paper of Ref. 2.

II. LATTICE QCD

Here it would be unsuitable to define the potential as in Sec. I (even though it would be easy to prove a bound for it), since the states considered usually have overlap with all sorts of low-lying states: This can be seen easily in the Schwinger model where, however, the average energy $(\psi_R, H\psi_R)$ shows the linear Coulomb potential.

On the lattice the closest thing to the expectation of H is

$$V(R) \equiv \inf_{\psi_R} \left[-\frac{1}{\epsilon} \ln(\psi_R, e^{-\epsilon H}\psi_R) \right]. \qquad (8)$$

Again we choose simple normalized trial states:

$$\phi_R = \frac{1}{N(R)} \psi(0) \times P \exp\left[i \int_0^R A_1(0, \vec{x}) dx_1 \right] \Gamma \psi^*(0, R, 0, 0)\Omega \qquad (9)$$

(Γ may be any Dirac matrix). $N(R)$ is a normalization which in the framework of Ref. 4 will not be simply a constant, but $0 < N(R) \leq$ const. As before, however, $(\phi_R, e^{-\epsilon H}\phi_R)N(R)^2$ can be interpreted as (a finite sum of) expectations of e^{-RH}. Proceeding as before, we obtain

$$\begin{aligned} V(R) &\leq -\frac{1}{\epsilon} \ln\left[(\phi_\epsilon, e^{-RH}\phi_\epsilon) \frac{N(\epsilon)^2}{N(R)^2} \right] \\ &\leq -\frac{R}{\epsilon} \ln(\phi_\epsilon, e^{-\epsilon H}\phi_\epsilon) + \frac{2}{\epsilon} \ln \frac{N(R)}{N(\epsilon)} \\ &\leq Rc_1 + c_2 \end{aligned}$$

with two constants c_1, c_2 $(c_1 > 0)$.

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