Note on a classical solution for the 't Hooft monopole and the Julia-Zee Dyon

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A real stable solution for the SU(2) Yang-Mills field coupled to the Higgs field is constructed from a complex sourceless solution of the Yang-Mills equation. It has the same properties as the Prasad-Sommerfield solution except for a singularity at the origin.

I. INTRODUCTION

It has been noted[1] that complex solutions in the Minkowski space for the sourceless Yang-Mills (YM) equations can be understood as real solutions for the YM field coupled to the Higgs field in the limit that the self-interaction potential of the Higgs field vanishes. The converse is also valid. Thus one readily obtains the Prasad-Sommerfield real solution[2] from the Hsu-Mac complex solution[3] for the sourceless YM field. In this paper we present another exact solution for the 't Hooft–Polyakov–Julia–Zee dyon from a complex solution[4] for the sourceless YM equation. The solution has the same finite energy, same electric and magnetic charges as Ref. 2, and is stable. However, it has a pole at the origin.

II. SOLUTIONS

The Lagrangian density for the SU(2) YM field coupled to the triplet Higgs field is

$$L=-\tfrac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu}-\tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a+\tfrac{1}{2}\mu^2\phi^2-\tfrac{1}{4}\lambda\phi^4\,, \quad (1)$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu+e\epsilon^{abc}A^b_\mu A^c_\nu \quad (2)$$

and

$$D_\mu\phi^a=\partial_\mu\phi^a+e\epsilon^{abc}A^b_\mu\phi^c\,. \quad (3)$$

The ansatz for the solution takes the form[4]

$$A^a_i=\epsilon_{iab}\hat{r}_b\frac{K(r)-1}{er}+(\hat{r}_i\hat{r}_a-\delta_{ia})\frac{W(r)}{er}\,,$$
$$A^a_0=\hat{r}^aJ(r)/er\,,\quad \phi^a=\hat{r}^aH(r)/er \quad (4)$$

where $\hat{r}^a\hat{r}^a=1$, and the field equations are[5]

$$r^2K''=K(W^2+K^2-1)+K(H^2-J^2)\,, \quad (5a)$$

$$r^2W''=W(W^2+K^2-1)+W(H^2-J^2)\,, \quad (5b)$$

$$r^2J''=2J(K^2+W^2)\,, \quad (5c)$$

$$rH''=2H(K^2+W^2)+(\lambda/e^2)H(H^2-\mu^2e^2r^2/\lambda)\,. \quad (5d)$$

Comparing Eqs. (5a) and (5b), $K(r)$ is proportional to $W(r)$. From Ref. 4 and in the limit $\lambda\to 0$, we have

$$K=\frac{\pm\sin\theta pr}{\sinh p(r+r_0)}\,,\quad W=\frac{\pm\cos\theta pr}{\sinh p(r+r_0)}\,,$$
$$J=\pm\sinh\gamma[1-pr\coth p(r+r_0)]\,, \quad (6)$$
$$H=\pm\cosh\gamma[1-pr\coth p(r+r_0)]\,.$$

Here θ,γ are arbitrary parameters, and p, r_0 are constants. The solution of the 't Hooft model[6] corresponds to $\gamma=0$. The vanishing of $W(r)$ and r_0 in Eq. (6) reduces to the Prasad-Sommerfield solution. If $p\to 0$, Eq. (6) becomes

$$\frac{K}{\sin\theta}=\frac{W}{\cos\theta}=\frac{\pm r}{r+r_0}\,,$$
$$\frac{J}{\sinh\gamma}=\frac{H}{\cosh\gamma}=\frac{\pm r_0}{r+r_0}\,. \quad (7)$$

For arbitrary r_0, solutions (6) and (7) have singularities. For solution (6) with $r_0=0$, ϕ^a and A^a_0 are regular everywhere, but A^a_i is singular at $r=0$.

III. CHARGES AND MASS OF THE DYON

If we follow the 't Hooft definition[6] of the electromagnetic field $\mathfrak{F}_{\mu\nu}$, the field strengths are

$$\mathcal{B}_i=\tfrac{1}{2}\epsilon_{ijk}\mathfrak{F}_{jk}=-\hat{r}_i/er^2$$

and

$$\mathcal{E}_i=\mathfrak{F}_{0i}=-\hat{r}_i\frac{d}{dr}\left(\frac{J}{er}\right)$$
$$=\hat{r}_i\sinh\gamma[(1-K^2-W^2)/er^2]\,. \quad (8)$$

We now restrict our discussion to Eq. (6) with $r_0=0$ and call it Eq. (6′). Then $(K^2+W^2)\to 1+O(r^2)$ as $r\to 0$, and from Eqs. (8) and (6′) we readily see that the solution represents a point monopole with strength $4\pi g=-4\pi/e$ which is surrounded by a cloud of electric charge $(4\pi/e)\sinh\gamma$. The energy or mass is given by

$$M=\int d^3rT_{00}\,, \quad (9)$$

with

$$T_{00}=F^a_{0j}F^a_{0j}+D_0\phi^aD_0\phi^a-L\,. \quad (10)$$

Using expression (4) and after some calculation, we find

$$M=\frac{4\pi}{e^2}\int_0^\infty dr\left(K'^2+W'^2+\frac{(K^2+W^2-1)^2}{2r^2}+\frac{J^2(K^2+W^2)}{r^2}+\frac{(rJ'-J)^2}{2r^2}+\frac{H^2(J^2+W^2)}{r^2}+\frac{(rH'-H)^2}{2r^2}\right). \quad (11)$$

For solutions (6) and (7) M diverges; but for solution (6′) we obtain

$$M=(4\pi/e^2)p\cosh^2\gamma . \quad (12)$$

Note that M vanishes identically for complex sourceless solutions of the YM equation.[3,4]

IV. STABILITY AND CONCLUSIONS

From the above, solution (6′) has the same electric and magnetic charges and same mass M as those of Ref. 2 in spite of the fact that A_i^a here has a pole at $r=0$ whereas A_i^a in Ref. 2 is nonsingular. That the properties are the same is due to the fact that (K^2+W^2) here is same as the K^2 in Ref. 2. Furthermore, solution (6′) is also stable. Following Ref. 7, solution (6′) will be stable if $M^2=Q^2+\Phi^2$ where Q and Φ are respectively, the electric and magnetic charges as defined in Ref. 7. We have explicitly carried out the calculation for solution (6′) and find

$$Q=(4\pi/e^2)p\sinh\gamma\cosh\gamma ,$$
$$\Phi=(4\pi/e^2)p\cosh\gamma , \quad (13)$$

thus verifying the stability of solution (6′). Solutions (6) and (7) are unstable.

Note added. While this work was being completed, we received a report[8] containing a general solution from which Eq. (6) here could be derived.

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