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Forms of gauge fields and pure gauges

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An *Ansatz* of the form $eA_\mu^a = -k\epsilon^{abc}\, v^b(x)\partial_\mu v^c(x)$ for the SU(2) Yang-Mills field can be treated as the gauge transform of the vacuum for $k = 2$.

Solutions of the classical Yang-Mills (YM) fields can take different forms if different *Ansätze* are used.[1] For the sourceless YM field A_μ^a (with compact gauge groups) Coleman[2] has shown that in three spatial dimensions the only nonsingular finite-energy solutions are gauge transforms of $A_\mu^a=0$. The solution obtained by Hsu and Mac[3] is nonsingular as well as of finite energy. This does not contradict Coleman's result since the classical gauge field must be real, whereas their solution is complex. In fact Hsu's complex solution for SU(2) can be converted into real gauge fields for the noncompact group SL(2, C).[4] Alternatively, we can regard Hsu's solution for sourceless YM fields as a real solution for the SU(2) YM field A_μ^a coupled to an SU(2) Higgs field ϕ^a with the Lagrangian

$$L=-\tfrac{1}{4}F_{\mu\nu}^aF^{a\mu\nu}-\tfrac{1}{2}D_\mu\phi^aD^\mu\phi^a\,, \tag{1}$$

such that

$$A_0^a=0,\quad A_i^a=B_i^a,\quad \phi^a=iB_0^a\,, \tag{2}$$

where the B_μ^a are as given by Ref. 3. In other words, Hsu's complex solution for the sourceless YM field is actually a real solution for the YM field and the Higgs field with the complex B_0^a traded for real ϕ^a.

Recently another form of solutions was proposed[5] for the SU(2) sourceless gauge field

$$\begin{aligned}A_\mu^a&=-e^{-1}\epsilon^{abc}|u(x)|^{-2}u^b(x)\partial_\mu u^c(x)\\&=-e^{-1}\epsilon^{abc}v^b(x)\partial_\mu v^c(x)\,,\end{aligned} \tag{3}$$

where $u^b(x)$ is real and single valued, $u^b(x)u^b(x)=|u(x)|^2$, and $v^b(x)=u^b(x)|u(x)|^{-1}$. Expression (3) could be understood as a result of requiring the isotopic magnetic field to be given by the covariant derivative of the pseudoscalar field $u^a(x)$.[6] We now point out that solutions of the form similar to Eq. (3) could be treated as pure gauge terms.

Under the SU(2) gauge transformation $\omega(x)$, the gauge field A_μ^a transforms as follows:

$$A_\mu^{\prime a}\frac{\tau_a}{2}=\omega(x)A_\mu^a\frac{\tau_a}{2}\omega(x)^{-1}+\frac{i}{e}\partial_\mu\omega(x)\omega(x)^{-1}\,, \tag{4}$$

where the τ_a's ($a=1,2,3$) are the Pauli matrices. Choosing $\omega(x)=v^a(x)\tau_a$, we see that ω is equal to its inverse and is self-adjoint. Substituting this expression for $\omega(x)$ and putting $A_\mu^a=0$ in Eq. (4), we have

$$\begin{aligned}\tfrac{1}{2}A_\mu^{\prime a}\tau_a&=ie^{-1}\tau_a\tau_bv^a\partial_\mu v^b\\&=-e^{-1}\epsilon^{abc}\tau_av^b\partial_\mu v^c\,,\end{aligned} \tag{5}$$

which is of the same form as Eq. (3). In fact, if $v^a(x)=x^a/r$, and one substitutes $A_\mu^a=-ke^{-1}\epsilon^{abc}v^b\partial_\mu v^c$ into the YM field equation, then k can take values 1 and 2 only.[7] Thus solutions in the form such as (5) are gauge transforms of $A_\mu^a=0$. Complications arise if the gauge transformation $\omega(x)$ is such that the last term on the right-hand side of Eq. (4) has a singularity. As an example, when $v^a(x)=x^a/r$, then expression (5) has a singularity at the origin in comparison with the vanishing gauge field $A_\mu^a=0$. The same situation occurs also in Ref. 8, where the singular gauge transformation when applied to the 't Hooft nonsingular monopole solution results in the introduction of a string-type singularity for A_μ^a. This means that one has to be careful in dealing with singular gauge transformations as they may be unacceptable physically. We remark that (i) if $v^a(x)=x^a/r$, then

$$\begin{aligned}\omega&=\exp(-i\pi/2)\exp[i(\pi-\phi)\tau_3/2]\exp(i\theta\tau_2)\\&\quad\times\exp(i\phi\tau_3/2)\,,\end{aligned} \tag{6}$$

where $\phi=\arctan(x^2/x^1)$, $\theta=\arccos(x^3/r)$; (ii) as expression (5) is a pure gauge term in the nonsingular case, one immediately has the result that $\exp(ie\oint A_\mu^a\tau_a/2dx^\mu)=1$ without going through the argument in Ref. 5; (iii) Hsu[5] has shown that solutions of the form (3) have magnetic charges. Actually the magnetic charge has nothing to do with expression (3) but is due to the topological structure of $v^a(x)$. Following Ref. 8, one can define the 't Hooft "electromagnetic" field tensor $F_{\mu\nu}$ as

$$F_{\mu\nu}=M_{\mu\nu}+H_{\mu\nu}\,,$$

with

$$M_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu\,,$$

$$b_\mu = A^a_\mu v^a ,$$

and

$$H_{\mu\nu} = -e^{-1}\epsilon^{abc} v^a \partial_\mu v^b \partial_\nu v^c .$$

Provided A^a_μ is nonsingular, then whatever its form may be, the magnetic charge density arises from the term $H_{\mu\nu}$ only. With A^a_μ taking the form (3) or (5), $M_{\mu\nu}$ vanishes identically, and $F_{\mu\nu}$ is the same as the $\bar{f}_{\mu\nu}$ of Ref. 5. Thus $M_{\mu\nu}$ does not contribute to the magnetic charge density.

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