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Some remarks on Adler's classical algebraic chromodynamics

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A simple method for constructing Adler's algebra with N quark source charges is presented. As a byproduct one obtains some interesting properties of direct products of U(n) λ matrices. It is shown that already for $N=3$ Adler's "trace" condition is not satisfied.

I. INTRODUCTION

In a very interesting paper[1] Adler has presented an extension of the equations of U(n) chromodynamics introducing *classical, noncommuting* source charges (if the charges are regarded as c numbers, in color-singlet states they lead to a Coulomb force law[1]). In this approach the algebraic properties of the theory depend on the number and type of the sources, and Adler has illustrated in detail the quark-quark and quark-antiquark cases.

In this note we first reformulate the algebraic structure of Adler's theory thereby clarifying, at least in our view, its physical content. We next consider the problem of N noncommuting quark sources. It turns out that finding Adler's algebra for N quarks amounts to solving an amusing mathematical problem concerning the properties of tensor products of $N+1$ λ matrices. This problem is formulated in Sec. III and solved for the $N=1$, 2, and 3 cases.

II. OVERLYING ALGEBRAS

In our understanding the essence of Adler's idea can be rephrased as follows. Let us assume that we have a system with P sources $Q^a_{(1)}, Q^a_{(2)}, \ldots, Q^a_{(P)}$ ($a=0,1,\ldots,n^2-1$) corresponding to various representations of U(n) [U(n) is called the *underlying algebra*],

$$[Q^a_{(\alpha)}, Q^b_{(\alpha)}] = if^{abc}Q^c_{(\alpha)}\,,$$
$$[Q^a_{(\alpha)}, Q^b_{(\beta)}] = 0\,, \qquad (1)$$
$$\alpha,\beta = 1,\ldots,P\,,$$

where f^{abc} is totally antisymmetric and $f^{0bc}=0$. The Eqs. (1) define a U(n)$\oplus$U(n)$\oplus\cdots\oplus$U(n) (P times) algebra. In the U(3) case, for example, $Q^a_{(1)}$ may correspond to the three-dimensional (quark) representation, $Q^a_{(2)}$ to the three-dimensional (antiquark) representation, $Q^a_{(3)}$ to the nine-dimensional (adjoint) representation, etc. A system with N quarks and $\bar{N}$ antiquarks only ($N+\bar{N}=P$) will be denoted by $(N,\bar{N})$.

We now define a new algebra called the *overlying algebra*. Consider two sets of matrices u^a and v^a ($a=0,1,\ldots,n^2-1$). Their product is another set of matrices w^a defined as

$$w^a = P^a(u,v) = -P^a(v,u) = g^{abc}(u^bv^c - v^bu^c)\,, \qquad (2)$$

where

$$g^{abc} = d^{abc} + if^{abc} \qquad (3)$$

[d^{abc} is a totally symmetric tensor with $d^{0bc}=(2/n)^{1/2}\delta^{bc}$ and equal to the usual SU(n) d symbol otherwise]. The product defined by Eq. (2) satisfies a Jacobi-type identity:

$$P^a(u,P(v,w)) + P^a(w,P(u,v)) + P^a(v,P(w,u)) = 0\,. \qquad (4)$$

Assuming that the overlying algebra closes, let us take a basis z^a_i ($i=1,2,\ldots,q$; $a=0,1,\ldots,n^2-1$) for it (q denotes the number of generators of the algebra, $q\geq P$) and write

$$P^a(z_i,z_j) = C^k_{ij}z^a_k\,. \qquad (5)$$

If the sources $Q^a_{(\alpha)}$ can be expressed as a linear combination of the generators z^a_i, the structure constants C^k_{ij} define the overlying algebra corresponding to the sources $Q^a_{(\alpha)}$.

Let us observe that the *overlying algebra is a Lie algebra*.[2] In order to show it, we take into account that the U(n) λ^a matrices satisfy the identity

$$\lambda^a\lambda^b = g^{abc}\lambda^c\,, \qquad (6)$$

and multiply Eq. (2) by λ^a. With the notation (repetition of an index implies a summation over it)

$$W=\lambda^a\cdot w^a\,,\quad U=\lambda^b\cdot u^b\,,\quad V=\lambda^c\cdot v^c \qquad (7)$$

($A\cdot B$ denotes the direct product of the matrices A and B), we have

$$W = UV - VU = [U,V]\,, \qquad (8)$$

and thus the product $P^a(u,v)$ is replaced by the usual commutator, and the substitution of Eq. (5) into Eq. (4), gives the standard Jacobi identity for the structure constants of a Lie algebra.

We denote the minimal overlying Lie algebra corresponding to the sources (1) by L_Q. For different sources $Q^a_{(\alpha)}$ one obtains, of course, different algebras L_Q. Since the matrices $Q^a_{(\alpha)}$ are given once we specify the sources, by definition one deals with a certain representation R of L_Q. As shown in Ref. 1 a classical gauge theory with noncommuting sources with gauge fields corresponding to the *adjoint representation of* L_Q [*not* $U(n)$] can be constructed if for the representation R of L_Q the generators $Z_i = \lambda^a \cdot z^a_i$ ($[Z_i, Z_j] = C^k_{ij} Z_k$) satisfy a "trace" condition. Taking for the definition of the "trace"[1]

$$S(u,v) = \tfrac{1}{2}(u^a v^a + v^a u^a)\,, \tag{9}$$

one requires

$$S(u, P(v,w)) = S(P(u,v), w)\,. \tag{10}$$

Notice that the "trace" S is not a c number, but a matrix. In order to cast the "trace" condition in the language of the generators (7), we define the $\mathcal{S}$ operation on a matrix $W = \lambda^a \cdot w^a$ as

$$\mathcal{S}(W) = \mathcal{S}(\lambda^a \cdot w^a) = (\mathrm{tr}\lambda^a) w^a\,. \tag{11}$$

We then have

$$S(u,v) = \tfrac{1}{2}\mathcal{S}(\{U, V\})\,, \tag{12}$$

where $\{U, V\} = UV + VU$, and the condition (10) reads

$$\mathcal{S}(\{U, [V,W]\} - \{[U,V], W\}) = 0\,. \tag{13}$$

In the special case (this applies to the examples of the next section) in which the matrices Z_i corresponding to the R representation of L_Q close not only under the Lie product but also under the Jordan product,

$$[Z_i, Z_j] = C^k_{ij} Z_k\,, \quad \{Z_i, Z_j\} = A^k_{ij} Z_k\,, \tag{14}$$

Eq. (13) is equivalent to

$$(C^l_{jk} A^p_{li} - C^l_{ij} A^p_{lk}) \mathcal{S}(Z_p) = 0\,. \tag{15}$$

In the next section we present the method of finding the L_Q algebras in the case of N quark sources [this is the $(N,0)$ case in Adler's language].

III. ADLER'S ALGEBRA FOR N QUARKS

A. Formulation of the problem

We now specialize to the case in which the sources in Eq. (1) are N quarks ($Q^a_{(\alpha)} = \frac{1}{2}\lambda^a_{(\alpha)}$; $a = 0, 1, \ldots, n^2-1$):

$$[\lambda^a_{(\alpha)}, \lambda^b_{(\alpha)}] = 2if^{abc}\lambda^c_{(\alpha)}\,, \qquad [\lambda^a_{(\alpha)}, \lambda^b_{(\beta)}] = 0 \quad (\alpha, \beta = 1, \ldots, N)\,, \tag{16}$$

where the $U(n)$ λ matrices close under the usual multiplication of matrices:

$$\lambda^a_{(\alpha)}\lambda^b_{(\alpha)} = g^{abc}\lambda^c_{(\alpha)}\,. \tag{17}$$

We also take another set of λ matrices that we denote by $\lambda^a_{(0)}$.

In order to construct the overlying algebra, we consider the matrices

$$Z_1 = \lambda^a_{(0)} \cdot \lambda^a_{(1)}\,, \quad Z_2 = \lambda^a_{(0)} \cdot \lambda^a_{(2)}, \ldots, Z_N = \lambda^a_{(0)} \cdot \lambda^a_{(N)}\,, \tag{18}$$

and look for the matrices $Z_{n+1}, \ldots, Z_q$ such that the matrices Z_i ($i = 1, \ldots, q$) close under the product defined by the commutator:

$$[Z_i, Z_j] = C^k_{ij} Z_k\,. \tag{19}$$

This is a straightforward but tedious exercise which implies repeated use of Eq. (17) and of known identities for the g^{abc} symbols.[3] The Lie algebra (19) depends not only on the number of quarks N, but also on the choice of the underlying $U(n)$. Once the structure constants C^k_{ij} are computed, a second and even more laborious calculation is needed to find to which specific Lie algebra they correspond (the Lie algebras one obtains are reductive, not simple).

We have found that an enormous simplification of the calculations is obtained from the following two observations.

(a) The matrices

$$X^a = \lambda^a_{(0)} + \lambda^a_{(1)} + \cdots + \lambda^a_{(N)} \quad (a = 0, 1, \ldots, n^2-1) \tag{20}$$

which generate a $U(n)$ algebra

$$[X^a, X^b] = if^{abc}X^c \tag{21}$$

commute with the matrices Z_i:

$$[X^a, Z_i] = 0\,. \tag{22}$$

(b) The $\lambda^a_{(0)}$ matrices play the same role as the matrices $\lambda^a_{(\alpha)}$ ($\alpha = 1, 2, \ldots, N$). Thus the generators Z_i [which are *scalars* under the $U(n)$ algebra (21)] can be arranged into multiplets which correspond to different representations of the permutation group of $N+1$ objects, S_{N+1}.

We are now in the position to formulate our mathematical problem:

Take $N+1$ sets of $U(n)$ λ matrices $\lambda^a_{(\alpha)}$ ($\alpha = 0, 1, \ldots, N$; $a = 0, 1, \ldots, n^2-1$) and n^2 matrices X^a defined by Eq. (20); consider the matrices Z_i constructed out of direct products of $\lambda^a_{(\alpha)}$ matrices, which commute with X^a; and find the Lie and Jordan algebras generated by the Z_i matrices.

As one can notice we have asked also to find the Jordan algebra (the product in this case is defined by the anticommutator $\{A,B\}=AB+BA$) generated by the matrices Z_i. We have done so because the λ matrices close under both the Lie product and the Jordan product and the same remains valid for the Z_i matrices.

We certainly did not solve the problem in its full generality, although we guess that the result would be very neat. For $N=1,2$ we treat the U(n) case, while for $N=3$ we present in detail the U(2) case. For U(2) the number of independent tensor products which are scalar under the U(2) algebra given by the X^a generators ($a=0,1,2,3$) is smaller and the overlying Lie algebra may have fewer generators.

The coefficients g^{abc} in Eq. (17) are, for U(2),

$$g^{abc}=\delta^{a0}\delta^{bc}+\delta^{b0}\delta^{ac}+\delta^{c0}\delta^{ab}-2\delta^{a0}\delta^{b0}\delta^{c0}+i\epsilon^{abc}\,, \tag{23}$$

where ϵ^{abc} is totally antisymmetric and

$$\epsilon^{0bc}=0\,,\quad \epsilon^{123}=1\,. \tag{24}$$

At this point we introduce a definition which will be useful in further developments. Let us assume that we have a set of matrices A^r ($r=0,1,\ldots,m^2-1$) and B^s ($s=0,1,\ldots,n^2-1$) satisfying the relations

$$\begin{aligned}&[A^{r_1},A^{r_2}]=2if^{r_1r_2r_3}A^{r_3}\,,\\ &[B^{s_1},B^{s_2}]=2if^{s_1s_2s_3}B^{s_3}\,,\\ &[A^r,B^s]=0\,,\end{aligned} \tag{25a}$$

$$\begin{aligned}&\{A^{r_1},A^{r_2}\}=2d^{r_1r_2r_3}A^{r_3}\,,\\ &\{B^{s_1},B^{s_2}\}=2d^{s_1s_2s_3}B^{s_3}\,,\\ &\{A^r,B^s\}=0\,.\end{aligned} \tag{25b}$$

The matrices A^r and B^s are a representation of the Lie algebra U(m)⊕U(n) given by (25a) *and* of the Jordan algebra (25b). We will denote the algebra (25a) and (25b) by $\Lambda(m)\oplus\Lambda(n)$.

We now present the solution of our problem for $N=1$, 2, and 3. The "trace" condition (10), (15) is discussed in each case separately.

B. One-quark algebra

In this case

$$X^a=\lambda^a_{(0)}+\lambda^a_{(1)}\,, \tag{26}$$

and there are two independent matrices Z_1 and Z_2 which satisfy the condition

$$[X^a,Z_i]=0\,. \tag{27}$$

We have

$$Z_1=\lambda^a_{(0)}\cdot\lambda^a_{(1)}\,,\quad Z_2=1\,. \tag{28}$$

The Lie algebra is

$$[Z_1,Z_2]=0\,, \tag{29}$$

and the Jordan algebra is

$$\{Z_1,Z_1\}=8Z_2\,,\quad \{Z_1,Z_2\}=2Z_1\,,\quad \{Z_2,Z_2\}=2Z_1\,. \tag{30}$$

If we make the transformation

$$A=\tfrac14 Z_1+\tfrac12 Z_2\,,\quad B=-\tfrac14 Z_1+\tfrac12 Z_2\,, \tag{31}$$

we have

$$\begin{aligned}&[A,B]=0\,,\quad \{A,B\}=0\,,\\ &\{A,A\}=2A\,,\quad \{B,B\}=2B\,.\end{aligned} \tag{32}$$

Thus, the Lie algebra is U(1)⊕U(1) and the Jordan algebra splits into the sum of two one-dimensional algebras. With the definition given by Eq. (25) this is a $\Lambda(1)\oplus\Lambda(1)$ algebra. Notice that the "trace" condition (15) is satisfied since the structure constants C^k_{ij} vanish.

C. Two-quark algebra

We have

$$X^a=\lambda^a_{(0)}+\lambda^a_{(1)}+\lambda^a_{(2)}\,, \tag{33}$$

and the matrices which commute with X^a are

$$\begin{aligned}&Z_1=\lambda^a_{(0)}\cdot\lambda^a_{(0)}\,,\quad Z_2=\lambda^a_{(0)}\cdot\lambda^a_{(2)}\,,\quad Z_3=\lambda^a_{(0)}\cdot\lambda^a_{(3)}\,,\\ &Z_4=f^{abc}\lambda^a_{(0)}\cdot\lambda^b_{(0)}\cdot\lambda^c_{(0)}\,,\quad Z_5=1\,.\end{aligned} \tag{34}$$

At this point it is useful to arrange the matrices Z_i into multiplets corresponding to irreducible representations[4] of S_3 (the group of permutations of three objects):

$$\begin{aligned}&\text{representation }(2,1)\\ &U_1=\frac{1}{\sqrt{12}}(Z_1-Z_2)\,,\\ &U_2=\tfrac16(Z_1+Z_2-2Z_3)\,,\\ &\text{representation }(3)\\ &V=\tfrac16(Z_1+Z_2+Z_3)\,.\\ &\text{representation }(1^3)\\ &W=\frac{1}{\sqrt{12}}Z_4\,,\\ &\text{representation }(3)\,,\\ &Y=\tfrac16 Z_5\,.\end{aligned} \tag{35}$$

One uses now the properties of the Clebsch-Gordan series for S_3 to find the Lie algebra in a transparent form. For instance, from the branching rules

$$(2,1)\otimes(2,1)=1^3\oplus\cdots,$$
$$1^3\otimes(2,1)=(2,1), \tag{36}$$

where the discarded terms do not appear in the antisymmetric form (a commutator gives only antisymmetric combinations), we have

$$[U_2,U_1]=2iW,\quad [U_1,W]=2iU_2,\quad [W,U_2]=2iU_1,$$
$$[V,U_1]=[V,U_2]=[V,W]=[Y,U_1]=[Y,U_2]=[Y,W],=[Y,V]=0. \tag{37}$$

It is now convenient to make the transformations

$$A^0=Y-V,\quad A^1=U_1,\quad A^2=W,\quad A^3=U_2,$$
$$B=V,$$

and one obtains

$$[A^a,A^b]=2i\epsilon^{abc}A^c,\quad [A^a,B]=0,$$
$$\{A^a,A^b\}=2d^{abc}A^c,\quad \{B,B\}=2B,\quad \{A^a,B\}=0. \tag{38}$$

The Lie algebra is U(1)⊕ U(2), and using the definition given by Eqs. (25), it is a Λ(1)⊕ Λ(2) algebra. The "trace" condition (15) is satisfied since

$$\mathcal{S}(A^1)=\frac{1}{\sqrt{12}}[\mathcal{S}(Z_1)-\mathcal{S}(Z_2)]$$
$$=\frac{1}{\sqrt{12}}[(\mathrm{tr}\lambda^a_{(0)})\lambda^a_{(1)}-(\mathrm{tr}\lambda^a_{(0)})\lambda^a_{(2)}]=0,$$
$$\mathcal{S}(A^2)=\mathcal{S}(A^3)=0.$$

D. Three-quark algebra

In the three-quark case

$$X^a=\lambda^a_{(0)}+\lambda^a_{(1)}+\lambda^a_{(2)}+\lambda^a_{(3)}. \tag{39}$$

The matrices which commute with X^a are[5]

$$H_1=\lambda^a_{(0)}\cdot\lambda^a_{(1)},\quad H_2=\lambda^a_{(0)}\cdot\lambda^a_{(2)},\quad H_3=\lambda^a_{(0)}\cdot\lambda^a_{(3)},$$
$$H_4=\lambda^a_{(1)}\cdot\lambda^a_{(2)},\quad H_5=\lambda^a_{(1)}\cdot\lambda^a_{(3)},\quad H_6=\lambda^a_{(2)}\circ\lambda^a_{(3)},$$
$$I_1=f^{abc}\lambda^a_{(0)}\circ\lambda^b_{(1)}\cdot\lambda^c_{(2)},\quad I_2=f^{abc}\lambda^a_{(0)}\cdot\lambda^b_{(1)}\cdot\lambda^c_{(3)},$$
$$I_3=f^{abc}\lambda^a_{(0)}\cdot\lambda^b_{(2)}\cdot\lambda^c_{(3)},\quad I_4=f^{abc}\lambda^a_{(1)}\cdot\lambda^b_{(2)}\cdot\lambda^c_{(3)}, \tag{40}$$
$$K_1=\lambda^a_{(0)}\circ\lambda^a_{(1)}\cdot\lambda^b_{(2)}\cdot\lambda^b_{(3)},\quad K_2=\lambda^a_{(0)}\cdot\lambda^a_{(2)}\cdot\lambda^b_{(1)}\cdot\lambda^b_{(3)},$$
$$K_3=\lambda^a_{(0)}\cdot\lambda^a_{(3)}\cdot\lambda^b_{(1)}\cdot\lambda^b_{(2)},\quad L=1.$$

These matrices can be arranged into S_4 multiplets as follows:

representation (2^2)

$$\mathcal{A}_1=\frac{1}{\sqrt{12}}(2H_1+2H_6-H_2-H_3-H_4-H_5),$$
$$\mathcal{A}_2=\tfrac{1}{2}(H_2+H_5-H_3-H_4);$$

representation $(3,1)$

$$\mathcal{B}_1=\frac{1}{4\sqrt{2}}(H_4+H_5-H_2-H_3),$$
$$\mathcal{B}_2=\frac{1}{4\sqrt{6}}(2H_1-2H_6+H_3+H_5-H_2-H_4),$$
$$\mathcal{B}_3=\frac{1}{4\sqrt{3}}(H_1-H_6+H_2+H_4-H_3-H_5);$$

representation $(2,1^2)$

$$\mathcal{C}_1=\frac{-1}{4\sqrt{2}}(I_3+I_4),$$
$$\mathcal{C}_2=\frac{1}{4\sqrt{6}}(I_4-2I_2-I_3), \tag{41}$$
$$\mathcal{C}_3=\frac{1}{8\sqrt{3}}(I_3-3I_1-I_2-I_4);$$

representation (2^2)

$$\mathcal{D}_1=\frac{1}{4\sqrt{3}}(2K_1-K_2-K_3),$$
$$\mathcal{D}_2=\tfrac{1}{4}(K_2-K_3);$$

representation (1^4)

$$\mathcal{E}=\tfrac{1}{2}(I_1-I_2+I_3-I_4);$$

representation (4)

$$\mathcal{F}=(H_1+H_2+H_3+H_4+H_5+H_6);$$

representation (4)

$$\mathcal{G}=(K_1+K_2+K_3).$$

The commutation relations of the matrices (41) are

$$[\mathfrak{A}_1, \mathfrak{A}_2] = 2i\sqrt{3}\,\mathcal{E}, \quad [\mathfrak{D}_1, \mathfrak{D}_2] = 0, \quad [\mathfrak{D}_\alpha, \mathfrak{A}_\beta] = 0 \quad (\alpha, \beta = 1, 2),$$

$$[\mathcal{E}, \mathfrak{D}_\alpha] = 0 \quad (\alpha = 1, 2), \quad [\mathcal{E}, \mathfrak{A}_1] = 8i\sqrt{3}(\mathfrak{A}_2 - \mathfrak{D}_2),$$

$$[\mathcal{E}, \mathfrak{A}_2] = 8i\sqrt{3}\,(\mathfrak{D}_1 - \mathfrak{A}_1), \quad [\mathcal{E}, \mathfrak{C}_i] = [\mathcal{E}, \mathfrak{B}_i] = 0 \quad (i = 1, 2, 3),$$

$$[\mathfrak{C}_i, \mathfrak{C}_j] = i\epsilon_{ijk}\mathfrak{C}_k \quad (i = 1, 2, 3), \quad [\mathfrak{B}_1, \mathfrak{B}_2] = -i\mathfrak{C}_3,$$

$$[\mathfrak{B}_3, \mathfrak{B}_1] = i\mathfrak{C}_2, \quad [\mathfrak{B}_2, \mathfrak{B}_3] = i\mathfrak{C}_1,$$

$$[\mathfrak{A}_1, \mathfrak{B}_1] = 2i(\sqrt{2}\,\mathfrak{C}_3 + \mathfrak{C}_2),$$

$$[\mathfrak{A}_1, \mathfrak{B}_2] = 2i\mathfrak{C}_1, \quad [\mathfrak{A}_1, \mathfrak{B}_3] = -2i\sqrt{2}\,\mathfrak{C}_1,$$

$$[\mathfrak{A}_2, \mathfrak{B}_1] = 2i\mathfrak{C}_1,$$

$$[\mathfrak{A}_2, \mathfrak{B}_2] = 2i\sqrt{2}\,\mathfrak{C}_3 - \mathfrak{C}_2), \quad [\mathfrak{A}_2, \mathfrak{B}_3] = -2i\sqrt{2}\,\mathfrak{C}_2, \tag{42}$$

$$[\mathfrak{D}_\alpha, \mathfrak{B}_i] = [\mathfrak{A}_\alpha, \mathfrak{B}_i], \quad [\mathfrak{D}_\alpha, \mathfrak{C}_i] = [\mathfrak{A}_\alpha, \mathfrak{C}_i], \quad (\alpha = 1, 2; \; i = 1, 2, 3),$$

$$[\mathfrak{C}_1, \mathfrak{B}_1] = -\tfrac{1}{2}i\mathfrak{D}_2, \quad [\mathfrak{C}_2, \mathfrak{B}_2] = \tfrac{1}{2}i\mathfrak{D}_2,$$

$$[\mathfrak{C}_3, \mathfrak{B}_3] = 0, \quad [\mathfrak{C}_1, \mathfrak{B}_2] = i\mathfrak{B}_3 - \tfrac{1}{2}i\mathfrak{D}_1,$$

$$[\mathfrak{C}_1, \mathfrak{B}_3] = -i\mathfrak{B}_2 + \frac{i}{\sqrt{2}}\mathfrak{D}_1, \quad [\mathfrak{C}_2, \mathfrak{B}_1] = -i\mathfrak{B}_3 - \tfrac{1}{2}i\mathfrak{D}_1,$$

$$[\mathfrak{C}_2, \mathfrak{B}_3] = i\mathfrak{B}_1 + \frac{i}{\sqrt{2}}\mathfrak{D}_2, \quad [\mathfrak{C}_3, \mathfrak{B}_1] = -i\mathfrak{B}_2 - \frac{i}{\sqrt{2}}\mathfrak{D}_1,$$

$$[\mathfrak{C}_3, \mathfrak{B}_2] = i\mathfrak{B}_1 - \frac{i}{\sqrt{2}}\mathfrak{D}_2,$$

$$[\mathfrak{C}_1, \mathfrak{A}_1] = 2i(\mathfrak{B}_2 - \sqrt{2}\,\mathfrak{B}_3), \quad [\mathfrak{C}_2, \mathfrak{A}_1] = 2i\mathfrak{B}_1,$$

$$[\mathfrak{C}_3, \mathfrak{A}_1] = 2i\sqrt{2}\,\mathfrak{B}_1, \quad [\mathfrak{C}_1, \mathfrak{A}_2] = 2i\mathfrak{B}_1,$$

$$[\mathfrak{C}_2, \mathfrak{A}_2] = -2i(\sqrt{2}\,\mathfrak{B}_3 + \mathfrak{B}_2), \quad [\mathfrak{C}_3, \mathfrak{A}_2] = 2i\sqrt{2}\,\mathfrak{B}_2.$$

$\mathfrak{F}$, $\mathfrak{G}$, and L commute with all the other generators. An inspection of the commutation relations (42) suggest the following transformations:

$$A^0 = \frac{\sqrt{2}}{4\sqrt{3}}(3L - \tfrac{1}{4}\mathfrak{G}), \quad A^1 = \frac{1}{\sqrt{6}}\left(\mathfrak{B}_1 - \frac{1}{2\sqrt{2}}\mathfrak{D}_2\right),$$

$$A^2 = -\frac{1}{\sqrt{6}}\left(\mathfrak{B}_2 + \frac{1}{2\sqrt{2}}\mathfrak{D}_2\right),$$

$$A^3 = \mathfrak{C}_3, \quad A^4 = \frac{\mathfrak{C}_1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}(\sqrt{2}\,\mathfrak{B}_2 - \mathfrak{D}_1),$$

$$A^5 = \frac{1}{\sqrt{2}}\mathfrak{C}_2 - \frac{1}{2\sqrt{3}}(\sqrt{2}\,\mathfrak{B}_1 - \mathfrak{D}_2),$$

$$A^6 = \frac{1}{\sqrt{2}}\mathfrak{C}_1 - \frac{1}{2\sqrt{2}}(\sqrt{2}\,\mathfrak{B}_2 - \mathfrak{D}_1), \tag{43}$$

$$A^7 = -\frac{1}{\sqrt{2}}\mathfrak{C}_2 - \frac{1}{2\sqrt{3}}(\sqrt{2}\,\mathfrak{B}_1 - \mathfrak{D}_2), \quad A^8 = \mathfrak{B}_3,$$

$$B^0 = \tfrac{1}{12}\mathfrak{G} + \tfrac{1}{2}L - \tfrac{1}{12}\mathfrak{F}, \quad B^1 = \frac{1}{2\sqrt{3}}(\mathfrak{A}_1 - \mathfrak{D}_1),$$

$$B^2 = \frac{1}{2\sqrt{3}}(\mathfrak{A}_2 - \mathfrak{D}_2), \quad B^3 = \frac{1}{4\sqrt{3}}\mathcal{E},$$

$$C = \tfrac{1}{12}\mathfrak{F} + \tfrac{1}{48}\mathfrak{G} - \tfrac{1}{4}L.$$

The matrices A^a $(a = 0, \ldots, 8)$, $B^b (b = 0, \ldots, 3)$, and C close under both the Lie product and the Jordan product:

$$[A^a, A^b] = 2if^{abc}A^c, \quad [B^a, B^b] = 2if^{abc}B^c,$$
$$[A^a, B^b] = [A^a, C] = [B^b, C] = 0, \tag{44}$$
$$\{A^a, A^b\} = 2d^{abc}A^c, \quad \{B^a, B^b\} = 2d^{abc}B^c, \quad \{C, C\} = 2C,$$
$$\{A^a, B^b\} = \{A^a, C\} = \{B^b, C\} = 0,$$

where the f and d symbols are the U(n) symbols ($n = 3$ for the A^a matrices, $n = 2$ for the B^b matrices). The Lie algebra in Eq. (44) is U(3) $\oplus$ U(2) $\oplus$ U(1), and in the language of Eq. (25) we have a $\Lambda(3) \oplus \Lambda(2) \oplus \Lambda(1)$ algebra.

We now show through a counterexample that the "trace" condition (15) is not satisfied. Using Eq. (44), the trace condition (15) for the matrices A^a reads

$$(f^{abc}d^{cde} - f^{dac}d^{cbe})\mathcal{S}(A_e) = 0. \tag{45}$$

Since

$$\mathcal{S}(H_1) = (\mathrm{tr}\lambda^a_{(0)})\lambda^a_{(1)} = 2 \quad \text{etc.}, \tag{46}$$

we have

$$\mathcal{S}(A^5) = \frac{1}{\sqrt{6}}[1 - \lambda^a_{(2)} \cdot \lambda^a_{(3)} + \tfrac{1}{2}\lambda^a_{(1)} \cdot \lambda^a_{(3)} + \tfrac{1}{2}\lambda^a_{(1)} \cdot \lambda^a_{(2)}] \neq 0. \tag{47}$$

If we take $a = 1$, $b = 2$, and $d = 5$ in Eq. (45), the left-hand side does not vanish.

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[1]Stephen L. Adler, Phys. Rev. D 17, 3213 (1978).

[2]This observation clarifies the note (1) added in proof of Ref. 1 and makes unnecessary the discussion (6) Sec. II of the same reference.

[3]For the SU(3) case, see V. I. Ogievetskii and I. V. Polubarinov, Yad. Fiz. 4, 853 (1966) [Sov. J. Nucl. Phys. 4, 605 (1967)].

[4]We use the notations of M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley, New York, 1962).

[5]These are the independent matrices only in the U(2) case. For U(n) ($n > 2$) there are in general more generators.

[6]P. Cvitanović, R. J. Gonsalves, and D. E. Neville, Phys. Rev. D 18, 3881 (1978).