Late terms in the asymptotic expansion for the energy levels of a periodic potential

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We obtain a general formula for the late terms in the perturbation expansion of the energy levels of a periodic potential and compare it with the computed values for the first one hundred terms.

I. INTRODUCTION

The equations we shall discuss are

$$y'' + \left(p + \frac{k^2}{2\beta^2}\cos 2\beta x\right)y = 0, \quad M$$

and

$$y'' + \left(p' - \frac{k^2}{2\beta^2}\cosh 2\beta x\right)y = 0, \quad M'.$$

The former is Mathieu's equation and the latter is the modified Mathieu equation.[1] They have a number of applications in theoretical physics: Apart from being the separated form of Laplace's equation in elliptic coordinates, M is the Schrödinger equation for the simple pendulum or a particle in a periodic potential. The relation p as a function of β for the smallest periodic solution of M is also the exact solution of the statistical mechanics of the one-dimensional Coulomb gas.[2]

Our interest in the problem stems from the recent discovery of vacuum periodicity in Yang-Mills theory,[3] and we hoped to discover what connection there is between the "instanton" governing the band width and the high-order behavior of the perturbation theory.

The periodic-potential problem in quantum mechanics has many of the features of the Yang-Mills problem and we hope to gain some insight from its solution.

The values of p for which M has solutions of period $2\pi/\beta$ (i.e., the top and bottom of the "allowed" bands) are called the characteristic values of M, while M' can be discussed as a Sturm-Liouville problem (with $y=0$ as $x\to+\infty$ as the boundary conditions) defining the eigenvalues p'. When $\beta\to 0$ (or $k\to\infty$) both M and M' reduce to the harmonic oscillator, and one can use perturbation theory about $\beta=0$ to obtain a series expansion for p or p'. In the case of the ground state one can, for example, use the Feynman-diagram expansion for a one-dimensional field theory. For M, one obtains

$$p = -\frac{k^2}{2\beta^2} + k - \frac{\beta^2}{4} - \frac{\beta^4}{2^4k} - \frac{6\beta^6}{2^7k^2} - \frac{53\beta^8}{2^{10}k^3} - \frac{594\beta^{10}}{2^{13}k^4} - \frac{7922\beta^{12}}{2^{16}k^5} - \frac{121\,454\beta^{14}}{2^{19}k^6} - \frac{2\,095\,501\beta^{16}}{2^{22}k^7}$$
$$- \frac{40\,114\,410\beta^{18}}{2^{25}k^8} - \frac{843\,289\,718\beta^{20}}{2^{28}k^9} - \frac{19\,343\,816\,948\beta^{22}}{2^{31}k^{10}} - \frac{478\,935\,069\,186\beta^{24}}{2^{34}k^{11}} - \cdots, \quad (1.1)$$

and so on until the calculator runs out of digits or one tires of the labor. The series for the eigenvalues of M' is obtained by replacing $\beta^2\to-\beta^2$ throughout. The terms up to k^{-5} were obtained in the 1920's by the authors of Refs. 4 and 5 (although the published k^{-5} values differ from source to source). We believe we are the first to obtain new values since then.

Certain general features show up in these first few terms:

(1) All the coefficients consist of an integer divided by 2^n. This may easily be shown to be a general result (see Appendix A).

(2) Despite the factors of 2, the numerator blows up and the series is at best asymptotic.

(3) In the case M, all the late terms have the same sign, which means that the series is not Borel summable and is also an indication that we are on a Stokes line.

Since, in general, a power series has a radius of convergence equal to the distance of the nearest singularity, and our series is divergent, it must necessarily follow that $\beta=0$ is in some sense singular. The nature of this singularity is intimately connected with the late terms in the expansion which govern the divergence.[6] In the subsequent sections we will discuss the singularity and exhibit the limiting form of the coefficients for the general band (level). We show in Appendix A how the coefficients may be computed and compare our estimates with the first one hundred or so terms in the expansion.

II. ANALYTICITY PROPERTIES OF $p'(\beta^2)$

By letting $\beta \to \pm i\beta$, one obtains equation M from M' and vice versa. The eigenvalues and characteristic values, however, are defined by the boundary conditions or periodicity conditions imposed on the solution of these equations. So, while such a substitution alters the equation, it does not correctly transform the problem of computing eigenvalues of M' into that of finding the characteristic values of M, because the boundary conditions are radically different in the two cases.

It is a measure of the incompleteness of information in the perturbation expansion (it misses the band splitting) that such a substitution *does* convert the series for the two problems into each other.

We will make a choice and start from M' because the Sturm-Liouville problem is conceptually simpler than the periodicity problem. We will extend the boundary conditions so as to continue p' into a function analytic on the β plane cut from 0 to ∞. When $\arg \beta^2 = \pm\pi$, we find the equation M but with non-self-adjoint boundary conditions so that $p'(-\beta^2)$ has an imaginary part. This gives the discontinuity across the cut (Schwartz reflection principle). The methods we need are those of Bender and Wu[7] who were the first to obtain such high-order behavior in perturbation theory. In their case and that of other recent work in quantum mechanics and field theory[8] the imaginary part has been clearly interpretable in terms of a quantum-mechanical metastability due to the decay of the relevant state into a continuum. In the present case there is no metastability even though probability can leak away because the continuum is needed for *exponential* decay—not discrete degenerate levels.

When $x \to \pm\infty$, the appropriate solution of M is

$$y \sim e^{-(k/\beta^2)\cosh \beta x}. \tag{2.1}$$

In order to keep the same solution as β^2 becomes complex and therefore to extend p' analytically we must vary the boundary conditions with β^2 so that $y \to 0$ along curves in the x plane on which

$$1 - \frac{k}{\beta^2} \cosh \beta x \tag{2.2}$$

remains real and tends to ∞. We exhibit such curves for various values of $\arg \beta$ in Fig. 1.

When $\arg \beta^2 = \pm\pi$, the boundary conditions become $y \to 0$ at

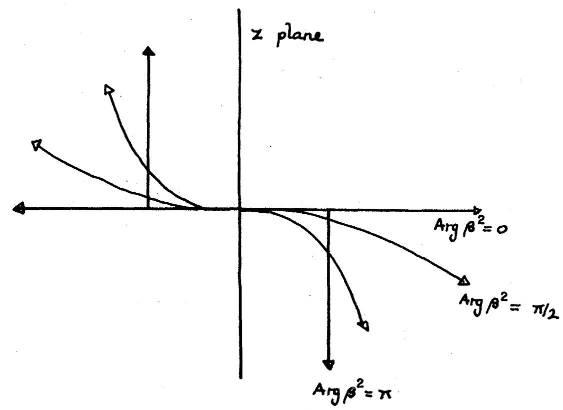

FIG. 1. A dual purpose diagram: It shows contours on which $(\alpha/\beta^2)(\cosh\beta z - 1) \in \mathbb{R}$ and tends to ∞. They are (i) the curves on which the boundary conditions (2.3) for the differential equals are to be applied and (ii) the contour on which the toy integrals of Appendix B are to be evaluated for various values of β.

$$\begin{aligned} x &= \pm\left(\frac{\pi}{\beta} + i\infty\right), \quad \arg \beta^2 = -\pi, \\ x &= \pm\left(\frac{\pi}{\beta} - i\infty\right), \quad \arg \beta^2 = +\pi. \end{aligned} \tag{2.3}$$

We now use the WKB expression and the method of matched asymptotic expansions to compute the imaginary part of p' for these boundary conditions.

Put ($|\beta| = 1$)

$$\begin{aligned} &y = \Phi_1 + i\,\Phi_2 \quad (\Phi_1, \Phi_2 \in \mathbb{R} \text{ and } \Phi_2 \ll \Phi_1), \\ &E = E_1 + i\,E_2 = p + k^2/2, \end{aligned} \tag{2.4}$$

so that M becomes

$$\begin{aligned} &\Phi_1'' + \left(E_1 + \frac{k^2}{2}(\cos 2x - 1)\right)\Phi_1 = E_2\,\Phi_2, \\ &\Phi_2'' + \left(E_1 + \frac{k^2}{2}(\cos 2x - 1)\right)\Phi_2 = -E_2\Phi_1; \end{aligned} \tag{2.5}$$

for x small and $E_2 \sim 0$ these take the form of Whittaker's equation[9]

$$\begin{aligned} &\Phi'' + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)\Phi = 0, \quad z = \sqrt{2k}\,x \\ &E/2k = n + \tfrac{1}{2}, \end{aligned} \tag{2.6}$$

with solutions $D_n(\pm z), D_{-n-1}(\pm iz)$.

For the harmonic oscillator the usual solution is $D_n(z), n \in \mathbb{Z}$, so that

$$E_1 = 2k(n + \tfrac{1}{2}). \tag{2.7}$$

We will now concentrate on the effect that the second valley at $x = \pi$ has on the ground state ($n = 0$) of this system. The situation is sketched in Fig. 2.

In B and C the WKB method yields[1]

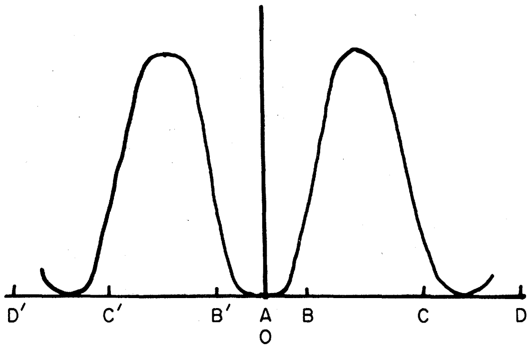

FIG. 2. The different regions for the matching of the asymptotic WKB formulas.

$$\Phi_1 \sim C_1(e^{+k\cos x})/\cos(x/2)\,,$$
$$\Phi_2 \sim C_2(e^{-k\cos x})/\sin(x/2)\,. \tag{2.8}$$

Matching at B to

$$\Phi_1 = D_0(\sqrt{2k}\,x) = e^{-kx^2/2}\,, \tag{2.9}$$

yields

$$C_1 = e^{-k}. \tag{2.10}$$

Near C but still in BC $(x' = x - \pi)$, we have

$$\Phi_1 \sim e^{-2k} e^{kx'^2/2}/\tfrac{1}{2}x'\,,$$
$$\Phi_2 \sim C_2 e^{k} e^{-kx'^2/2}. \tag{2.11}$$

In the region CD, the equation again takes the form (2.6), but the boundary condition (2.3) necessitates the solution

$$y_{CD} = A\,D_{-1}(i\sqrt{2k}\,x') \tag{2.12}$$

to be chosen as only this has the correct behavior as $x' \to i\infty$.

Now

$$D_{-1}(i\sqrt{2k}\,x) = \tfrac{1}{2}\left[D_{-1}(\sqrt{2k}\,ix') + D_{-1}(-\sqrt{2k}\,ix')\right] + \tfrac{1}{2}\left[D_{-1}(\sqrt{2k}\,ix') - D_{-1}(-\sqrt{2k}\,ix')\right];$$

the first two terms become[9]

$$\left(\frac{\pi}{2}\right)^{1/2} D_0(\sqrt{2k}\,x') \sim \left(\frac{\pi}{3}\right)^{1/2} e^{-kx'^2/2} \quad \text{(purely real)} \tag{2.13}$$

and the last two terms become

$$\sim -i\,\frac{e^{kx^2}}{\sqrt{2k}\,x} \quad \text{(purely imaginary)}. \tag{2.14}$$

Identifying these real and imaginary parts with Φ_1, Φ_2 at C gives

$$\Phi_1 \sim 2e^{-2k}\,\frac{e^{kx^2/2}}{x} \sim A\left(\frac{-i}{\sqrt{2k}}\right)\frac{e^{kx^2/2}}{x}\,,$$

so

$$A = 2i\sqrt{2k}\,e^{-2k}\,, \tag{2.15}$$

$$i\Phi_{2C} \sim iC_2 e^{k} e^{-kx'^2/2} \sim A\left(\frac{\pi}{2}\right)^{1/2} e^{-kx'^2/2}\,,$$

so

$$A = 2i\sqrt{2k}\,e^{-2k}\,,$$
$$C_2 = 2\sqrt{k}\,\sqrt{\pi}\,e^{-3k}. \tag{2.16}$$

Now we come to the difficult part—the chief achievement of Bender and Wu. This is the problem of handling the subdominant, imaginary term in $A\ B$ so as to find out how large E_2 has to be in order to produce C_2 and thus allow the boundary conditions to be satisfied.

Using (2.16) and matching at B, we obtain

$$\Phi_2 \sim 4\sqrt{k}\,\sqrt{\pi}\,e^{-4k}\,\frac{e^{kx^2/2}}{x} \quad \text{near } B; \tag{2.18}$$

E_2 is of the same order of magnitude (i.e., e^{-4k}), so Bender and Wu put

$$\chi = \Phi_1 + E_2\left(\frac{\Phi_2}{E_2}\right); \tag{2.19}$$

then, treating it as a perturbation expansion, we get to zeroth order

$$\Phi_1'' + (E_1 - k^2x^2)\Phi_1 = 0\,,$$

and to first order

$$\Phi_2'' + (E_1 - k^2x^1)\Phi_2 = -E_2\Phi_2\,, \tag{2.5}$$

provided χ satisfies

$$\chi'' + (E_1 + E_2 - kx^2)\chi = 0\,. \tag{2.20}$$

We seek a symmetric solution for the ground state, and this is

$$\tfrac{1}{2}\left[D_{E_2/2k}(\sqrt{2k}\,x) + D_{E_2/2k}(-\sqrt{2k}\,x)\right].$$

When $E_2 = 0$ this reduces to $\Phi_0 = D_0(\sqrt{2k}\,x)$, as it should according to (2.19). We extract Φ from it:

$$\Phi_2 = E_2\frac{\partial}{\partial E_2}\frac{1}{2}\left[D_{E_2/2k}(\sqrt{2k}\,x) + D_{E_2/2k}(-\sqrt{2k}\,x)\right]. \tag{2.22}$$

Near B the dominant contribution comes from the second term

$$D_{E_2/2k}(-\sqrt{2k}\,x) \sim e^{E\pi i/2k}(\sqrt{2k}\,\pi)^{-(1+E_2/2k)} e^{kx^{1/2}}\,\frac{\sqrt{2\pi}}{\Gamma(-E_2/2k)}\,, \tag{2.23}$$

so

$$\Phi_{2B} = -\frac{E_2}{2}\frac{1}{2k}\frac{\sqrt{2\pi}}{\sqrt{2k}}\frac{e^{kx^2/2}}{x}, \qquad (2.24)$$

and comparison with (2.16) gives

$$E_2^{(0)} = 16k^2 e^{-4k}. \qquad (2.25)$$

For the general level we would obtain

$$E_2^{(\nu)} = \frac{2^{6\nu+4}k^{2\nu+2}e^{-4k}}{(\nu!)^2}. \qquad (2.26)$$

Twice this is then the discontinuity across the cut. Restoring β shows it to have the form

$$\mathrm{Im}E \sim \frac{e^{-4k/\beta^2}2^{6\nu+4}}{(\nu!)^2}\left(\frac{k}{\beta^2}\right)^{2\nu+2}\beta^2,$$

so the discontintuity has an essential singularity at $\beta^2 = 0$.

Expressing $E(\beta^2)$ as a dispersion relation about the cut (with the requisite number of subtractions— finite) and locating the coefficient of $1/k^n$, we find that the coefficient for the νth level of M' is

$$\alpha_n^\nu = (-1)^n \frac{2^{2\nu}}{(\nu!)^2}\frac{1}{\pi}\left(\frac{1}{4}\right)^n (2\nu+1+n)!\left[1+O\left(\frac{1}{n}\right)\right].$$

This is our main result and may be compared with the data in Tables I and II. The good agreement indicates that we have made the correct analytic continuation.

III. DISCUSSION

We had hoped to be able to derive this result by applying the method of steepest descents to the functional-integral form of the quantum-mechanical problems. For the various $\lambda\varphi^4$ theories considered by previous authors this is very easy to do because the imaginary part is the result of an obvious instability of the theory (it is also a consequence of analytic continuation of the boundary conditions, as Bender and Wu's work shows). This instability shows up in the functional integral as a negative eigenvalue for the small oscillator sums requiring *one* of the integration variables to be continued into the complex plane.[10] The choice of continuation corresponds to the choice of branches of E for β^2 negative. This does not obviously happen for degenerate minima because the periodic potential is quite stable, and so the way to continue the domain of the functional integral to obtain nonstandard solutions to the associated Schrödinger equation is not obvious.[11] The basic philosophy behind this approach is illustrated by a toy integral in Appendix B.

We do, however, find empirically a relation between the imaginary part for $\beta^2 < 0$ and the splitting between the top and bottom of the bands

$$b_{n+1} - a_n \equiv E^n_{\text{top of band}} - E^n_{\text{bottom of band}} = 2\left(\frac{k}{\pi}\right)^{1/2}\frac{(8k)^{n+1}}{n!}e^{-2k}.$$

[This was proved by Goldstein in 1929 (Ref. 5). In the case $n=0$ it is most easily derived by using the method of steepest descents about the kink

TABLE I. The first seven coefficients, C_l, of the expansion

$$p = -k^2/2 + a_0 k - (a_0^2+1)/2^3 - k\sum_{l=2}^{\infty} C_l/k^2\, 2^{4l-2},$$

where $a_0 = 2\nu+1$, $\nu\in\mathbf{Z}$.

$C_2 = a_0(a_0^2+2)$
$C_3 = 5a_0^4 + 34a_0^2 + 9$
$C_4 = a_0(33a_0^4 + 410a_0^2 + 4057)$
$C_5 = 252a_0^6 + 5040a_0^4 + 11\,772a_0^2 + 1944$
$C_6 = a_0(2108a_0^6 + 62\,468a_0^4 + 276\,004a_0^2 + 166\,428)$
$C_7 = 18\,774a_0^8 + 77\,560a_0^6 + 5\,691\,796a_0^4 + 8\,043\,768a_0^2 + 1\,013\,958$

TABLE II. Values of $b_2 = -a_{l+1} \times 4^2/(l+1)!$, where the a_l are the coefficients in (A2b) for the ground state ($\nu = 0$).

b_l	l	b_l	l
0.125 000 00	1	0.308 147 46	78
0.125 000 00	2	0.308 275 68	79
0.138 020 83	3	0.308 400 70	80
0.154 687 50	4	0.308 522 64	81
0.171 918 40	5	0.308 641 62	82
0.188 262 64	6	0.308 757 74	83
0.203 014 65	7	0.308 871 09	84
0.215 907 34	8	0.308 981 79	85
0.226 941 44	9	0.309 089 93	86
0.236 268 01	10	0.309 195 58	87
0.244 106 71	11	0.309 298 83	88
0.250 693 07	12	0.309 399 78	89
0.256 248 32	13	0.309 498 48	90
0.260 965 23	14	0.309 595 02	91
0.306 990 19	70	0.309 689 47	92
0.307 149 07	71	0.309 781 89	93
0.307 303 55	72	0.309 872 35	94
0.307 453 81	73	0.309 960 91	95
0.307 600 02	74	0.310 047 63	96
0.307 742 34	75	0.310 132 56	97
0.307 880 92	76	0.310 215 77	98
0.308 015 91	77	0.310 297 30	99

solution interpolating between two potential minima.] The fact that up to a constant factor the discontinuity is the square of level splitting is highly suggestive.

It is possible that the explanation is to be found in the last reference cited in Ref. 8, but there appear to be ambiguities in the approach of that paper which we do not understand— regarding the integration over the separation of the two kinks.

We hope to find a convincing argument for this observation which can be generalized to give a formula for the high-order behavior of Yang-Mills theory using the computations of the steepest-descent integrals about the Polyakov instanton.[12]

Note added in proof. After the appearance of this article as a preprint Professor H. J. W. Müller kindly pointed out to us that he and Professor R. B. Dingle had earlier investigated the problem by linearizing the recurrence relations. The results of their work are published in J. Reine Angewandte Math. 216, 123 (1964). Another article by the same authors which is of considerable interest to those using Mathieu functions appears in the same journal, Vol. 211, 11 (1962).

APPENDIX A: GENERATION OF THE ASYMPTOTIC SERIES FOR THE CHARACTERISTIC VALUES OF MATHIEU'S EQUATION

Mathieu's equation M written in standard form is

$$y'' + (p - 2q\cos 2x)\,y = 0 \,. \tag{A1}$$

We seek a solution for the characteristic value valid for large q. Firstly we transform (A1) by substituting

$$y = \exp(k\cos\alpha)u(x)$$

and assuming the expansions

$$u(x) = \sum_{l=0}^{\infty} u_l(x)/k^l \tag{A2a}$$

and

$$p = k^2/2 + k\sum_{l=0}^{\infty} u_l/k^2 , \tag{A2b}$$

where $k^2 = -4q$. Then equating coefficients of k^{-l} results in the set of equations

$$2\sin x\, u_0' + (\cos x - a_0)u_0 = 0 , \tag{A3a}$$

$$2\sin x\, u_l' + (\cos x - a_0)u_l = u_{l-1} + \sum_{j=1}^{n} a_j u_{l-j} , \quad l = 1, 2, \ldots \,. \tag{A3b}$$

The condition that p be a characteristic value restricts the value of a_0 and the form $2\nu + 1$, where $\nu = 0, 1, 2, \ldots$. The solution of (A3a) is

$$u_0 = \sin^{\nu}(\tfrac{1}{2}x)/\cos^{\nu+1}(\tfrac{1}{2}x) \,. \tag{A4}$$

(Constants of integration can be set equal to zero without affecting the result.) Equation (A3b) is simplified by substituting

$$u_l = [\sin^{\nu}(\tfrac{1}{2}x)/\cos^{\nu+1}(\tfrac{1}{2}x)]\, v_l(x) ,$$

giving a set of equations for the v_l, viz,

$$2\sin x\, v_l' = v_{l-1}'' + (a_0 - \cos x)v_{l-1}'/\sin x + [(b_0 - a_0\cos x)/\sin^2 x - b_0/2]\, v_{l-1} + \sum_{j=2}^{l} a_j v_{l-j} , \tag{A5}$$

where $b_0 = \nu^2 + \nu + 1$. The v_l can be expressed as a finite series in $\sin^2 x$, i.e.,

$$v_l = \sum_{j=1}^{2}(A_j^l + B_j^l\cos x)/\sin^{2j} x \,. \tag{A6}$$

Putting the form (A6) for the v_l in equation (A5) and equating coefficients of $\sin^{-2l}x$ and $\cos x\sin^{-2l}x$ gives a coupled set of recursion relations for the A's and B's of (A6), namely,

$$4jA_j^l = (2j-1)a_0 A_{j-1}^{l-1} - [4j(j-1) + b_0]B_{j-1}^{l-1} + [(2j-1)^2 + (2j-1) + b_0/2]B_j^{l-1} - \sum_{n=2}^{l} a_n B_n^{l-n} , \tag{A7a}$$

$$4jB_j^l = 2(2j+1)B_{j+1}^l - 2a_0 j B_j^{l-1} + a_0(2j-1)B_{j-1}^{l-1} - [4j(j-1) + b_0]A_{j-1}^{l-1} + [(2j)^2 + 2j + b_0/2]A_j^{l-1} - \sum_{n=2}^{l} a_n A_j^{l-n} \,. \tag{A7b}$$

The initial conditions are $A_0^0 = 1$ and $B_0^0 = 0$, and the required coefficients a_n are obtained from (A7b) by setting $j = 0$ (giving $a_n = 2B_1^l$).

The form of the expansion (A6) is that obtained by Goldstein. Unfortunately equations (A7) are unstable and could reliably give only the first ten or so terms of (A2b). We were, however, able to find a general form for the first seven terms of (A2b) as shown in Table I. It is worth mentioning that A and B could be identified as simple fractions for $l < 7$ in general. This information could be used to reinitialize (A7), and so in principle arbitrarily high order in K could be reached in the expansions of $u(x)$ and p. The asymptotic form (2.27) for the values of ν considered ($\nu \leqslant 9$) had only just been reached by the 10th term, so the general analysis of Sec. II was verified to two significant figures only.

For $\nu = 0$ and $\nu = 1$, however, the form (A6) for v_l can be replaced by

$$v_l = \sum_{j=1}^{l} A_j^l / \cos^{2l}(x/2). \quad \text{(A8)}$$

The resulting recursion relations for the A_j^l are

$$4jA_j^l = 4(j+1)A_{j+1}^l + j(j+\nu-\tfrac{1}{2})A_{j-1}^{l-1} - [j(j+1)+b_0/2]A_j^{l-1} + \sum_{n=2}^{l} a_n A_j^{n-2}, \quad \text{(A9)}$$

and $-4A_1^l = a_l$. The advantage of this form is that the relations (A9) are stable for $\nu=0$.

The first one hundred terms n of (A2b) (suitably scaled) are presented in Table II. Following the analysis of Bender and Wu we find numerically the asymptotic forms for the late nth term of (A2b):

$$\nu=0:\ \text{coeff}\left(\frac{1}{k^n}\right) = -\frac{1}{\pi}\left(1.0 - \frac{5}{2n} + \frac{7}{8n^2} + \cdots\right) \times (n+1)!\,4^{-n},$$

$$\nu=1:\ \text{coeff}\left(\frac{1}{k^n}\right) = -\frac{4}{\pi}\left(1.0 - \frac{21}{2n} + \frac{125}{6n^2} + \cdots\right) \times (n+3)!\,4^{-n}.$$

APPENDIX B: TOY INTEGRAL

We give here an elementary discussion of a toy model for the way one might expect a functional-integral approach to this problem to go.

Consider

$$\mathfrak{F}(\beta^2) = \int_{-\infty}^{+\infty} e^{-(z/\beta^2)(\cos\beta x - 1)}dx = \frac{2}{\beta}K_0\left(\frac{z}{\beta^2}\right)e^{z/\beta^2}.$$

(K_0 is the Bessel function of the third kind.[13]) By expanding about the saddle point $x=0$ we can compute the usual asymptotic expansion

$$\mathfrak{F}(\beta^2) = \left(\frac{\pi^2}{z}\right)^{1/2}\left[1 - \frac{\beta^2}{z}\frac{1}{8} + \frac{1^2 3^2 \beta^4}{2!\,(8z)^2} - \frac{1^2 3^2 5^2 \beta^6}{3!\,(8z)^3} + \cdots\right]$$

$$= \left(\frac{2}{\pi z}\right)^{1/2}\sum \frac{(-\beta^2)^n}{n!\,(2z)^n}[\Gamma(n+\tfrac{1}{2})]^2.$$

By distorting the contour into the complex plane we can continue $\mathfrak{F}$ to a function analytic in the cut plane. (K_0 is well known to have a logarithmic branch cut.) The integrals for $\arg\beta^2 = \pm\pi$ are

$$\mathfrak{F}(-\beta^2) = \left(\int_{-\pi/\beta \pm i\infty}^{-\pi/\beta} + \int_{-\pi/\beta}^{+\pi/\beta} + \int_{\pi/\beta}^{\pi/\beta \mp i\infty}\right) e^{(z/\beta^2)(\cos\beta x - 1)}dx,$$

and the contour has hit two additional saddle points at $x = \pm\pi/\beta$, each of which contributes *one half* a Gaussian integral—this is the genesis of the Stokes phenomenon and is a good counterexample to the common belief that one must always add the contributions from all saddle points in sight. An excellent discussion of this type of calculation is given in Ref. 14.

The discontinuity

$$\frac{4i}{\beta}K_0(z/\beta^2)e^{-z/\beta^2} \sim 2ie^{-2z/\beta^2}\left(\frac{2\pi}{z}\right)^{1/2}$$

is exponentially suppressed as is the "instanton" contribution. For the path integral, the formula

$$\text{coeff}\ \beta^{2n}\ \text{in}\ \mathfrak{F}(\beta) = \frac{1}{\pi}\int_0^{\infty} 2\left(\frac{2\pi}{2}\right)^{1/2} \frac{e^{-2z/\beta'^2}}{(\beta'^2)^{n+1}}d\beta \times [1 + O(1/n)]$$

may be checked by Stirling's formula.

[1] R. Campbell, *Theorie Generale de l'equation de Mathieu* (Mason, Paris, 1955).

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