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Eigenvalue conditions and asymptotic freedom of SO(N) gauge theories

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We report on the study of a class of SO(N) grand unified gauge theories that truly have only one coupling constant in the theory. The Yukawa and the different quartic self-coupling constants in the Higgs potential are totally fixed by the eigenvalue conditions required for asymptotic freedom. We discuss briefly the phenomenological implications of a particular SO(12) solution which, at the first stage of the hierarchy of spontaneous symmetry breakdown, exhibits a U(2) × SO(8) residual symmetry, which contains SU(2) × SU(4) × U(1) × U(1).

I. INTRODUCTION

In this paper we report on the study of a grand unification of strong, electromagnetic, and weak interactions that truly has only one coupling constant in the theory. This comes about through the imposition of eigenvalue conditions[1-4] on all the other coupling constants of the theory, i.e., Yukawa as well as the different quartic self-coupling constants in the theory. As a result of the eigenvalue conditions, the grand unified theory is asymptotically free.[5,6]

The new features of our grand unification scheme that emerge from our study are the following:

(i) The number of carbon copies of the basic ν, e family is limited. That is, the sequence of leptons $(e, \mu, \tau, \nu, \tau', \ldots)$ must end. The actual number depends on the particular SO(N) group desired.

(ii) Each fermion multiplet belonging to the spinorial representation of the SO(N) group unifies the light fermions $(u, d, \nu, e, \ldots)$ with superheavy fermions $(U, D, \ldots)$. The superheavy fermions have masses that are of the order of the superheavy gauge boson masses. Since the superheavy fermions are in the same representations as the leptons, the number of carbon copies of the superheavy fermions is similarly limited.

(iii) Even though SO(12) contains $SU(4)\times SU_L(2) \times SU_R(2)$, the structure of the vacuum that emerges from our study indicates a breakdown of manifest left-right invariance, already at superhigh energies. Our interest in SO(N) gauge groups stems from the previous attempts at an SO(10) grand unification scheme.[7,8] The assignment of fermions there to a spinorial representation unifies left- and right-handed fermions in the same multiplet. The resulting structure of Higgs scalars, necessary for phenomenology, is quite irregular,[9] and the asymptotic freedom of the theory is not clear.

II. SO(N) GAUGE THEORY

The gauge bosons, in the adjoint representation of the group, will be denoted by

$$A_{\mu ij} = -A_{\mu ji}, \quad i, j = 1, \ldots, N \tag{1}$$

and the gauge-invariant field strength is given by

$$G_{\mu\nu ij} = \partial_\mu A_{\nu ij} - \partial_\nu A_{\mu ij} - gA_{\mu il}A_{\nu lj} - gA_{\mu jl}A_{\nu il}. \tag{2}$$

The gauge-invariant pure Yang-Mills[10] Lagrangian is

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{8}(G_{\mu\nu ij})^2. \tag{3}$$

The Higgs system[11] for ϕ in the adjoint representation is given by

$$\mathcal{L}_{\rm Higgs} = -\tfrac{1}{4}(\partial_\mu \phi_{ij} - gA_{\mu il}\phi_{lj} - gA_{\mu jl}\phi_{il})^2 + \tfrac{1}{4}\mu^2(\phi_{ij})^2 - \tfrac{1}{4}\lambda(\phi_{ij}\phi_{ij})^2 - \tfrac{1}{4}\Lambda(\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}). \tag{4}$$

Finally, the fermions, in the spinorial representation, will couple to both gauge and Higgs bosons according to

$$\mathcal{L}_f = -\bar{\psi}\gamma_\mu[\partial_\mu - (i/2)g\sigma_{ij}A_{\mu ij}]\psi - h\bar{\psi}\sigma_{ij}\psi\phi_{ij}, \tag{5}$$

where σ_{ij} are the generators of SO(N) in the spinorial representation, satisfying the algebra

$$[\sigma_{ij}, \sigma_{kl}] = i(\delta_{ik}\sigma_{jl} - \delta_{il}\sigma_{jk} + \delta_{jl}\sigma_{ik} - \delta_{jk}\sigma_{il}). \tag{6}$$

The σ_{ij} can be very simply constructed in terms of the matrices, α_i, obeying the Clifford algebra[12]

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}. \tag{7}$$

For SO($2n$) groups, the α_i, in the irreducible representation, are 2^{n-1}-dimensional matrices, while for SO($2n+1$) groups, the α_i are 2^n-dimensional matrices.

III. RENORMALIZATION-GROUP EQUATIONS

For our analysis, we allow for n_F identical fermion multiplets. There are, in other words, n_F carbon copies of the basic family $(u, d, \nu_e, e, \ldots)$, so that we have $(c, s, \nu_\mu, \mu, \ldots)$ and possibly higher families involving even more heavy leptons. The fermions do not interact directly with each other.

We take a single Higgs boson multiplet which couples universally to all the n_F fermions universally. At superunification energies, all the $e, \mu, \ldots$ families are degenerate in mass. The mass difference between $e, \mu, \tau, \tau', \ldots$ of the order of several GeV will not be significant at superunification energies of the order of 10^{19} GeV.

With these qualifications, we are now ready to write down the renormalization-group equations.[5,13] For clarity, we list down first, in the Landau gauge, the wave-function renormalization constants for the gauge boson, fermion, and the Higgs bosons (n_D = dimension of the spinorial representation):

$$Z_{\text{gauge boson}} = 1 + \frac{g^2}{16\pi^2}\left[\tfrac{26}{3}(N-2)\frac{n_F n_D}{3} - \tfrac{1}{3}(N-2)\right]\ln\Lambda\,, \tag{8a}$$

$$Z_{\text{fermion}} = 1 - \frac{h^2}{16\pi^2}\left[\tfrac{1}{2}N(N-1)\right]\ln\Lambda\,, \tag{8b}$$

$$Z_{\text{Higgs}} = 1 + \frac{g^2}{16\pi^2}\left[12(N-2)\right]\ln\Lambda - \frac{h^2}{16\pi}(4n_F n_D)\ln\Lambda\,, \tag{8c}$$

$$16\pi^2\frac{dg}{dt} = -g^3\left[\tfrac{22}{3}(N-2) - \tfrac{1}{3}n_F n_D - \tfrac{1}{3}(N-2)\right] \equiv -\beta_0 g^3\,, \tag{9a}$$

$$16\pi^2\frac{dh}{dt} = h^3\left[2n_F n_D + \tfrac{1}{2}N(N-1) + (N-4)^2 - N\right] - 6g^2 h(N-2) - \tfrac{3}{4}g^2 h\left[(N-4)^2 - N\right]\,, \tag{9b}$$

$$16\pi^2\frac{d\lambda}{dt} = \lambda^2\left[4N(N-1) + 64\right] + \lambda\Lambda(16N-8) + 12\Lambda^2 - 24(N-2)g^2\lambda + 18g^4 - 6n_F n_D h^4 + 8n_F n_D\lambda h^2\,, \tag{9c}$$

$$16\pi^2\frac{d\Lambda}{dt} = \Lambda^2(8N-4) + 96\lambda\Lambda - 24(N-2)g^2\Lambda + 6(N-8)g^4 - 24n_F n_D h^4 + 8n_F n_D\Lambda h^2\,. \tag{9d}$$

As has been pointed out before, the presence of the h^4 term in the $d\lambda/dt$ and $d\Lambda/dt$ equations is absolutely crucial. It is a large negative term which helps bring about the needed negative contribution to the equation.

The strategy for the solution is as follows. We assume that eigenvalue conditions exist to the equations, viz.,

$$h(t) = \bar{h}g(t)\,,\quad \lambda(t) = \bar{\lambda}g^2(t)\,,\quad \Lambda(t) = \bar{\Lambda}g^2(t)\,. \tag{10}$$

With $\bar{h}, \bar{\lambda}, \bar{\Lambda}$ as numbers, these are special solutions to the coupled set of differential equations. This reduces them to a set of coupled algebraic equations, which can be solved. The results are summarized in Table I.

IV. STRUCTURE OF THE VACUUM

A unique feature of an asymptotically free grand unified theory is that all the coupling constants of the Higgs potential as well as the Yukawa coupling constants are predicted. It is at once a highly restrictive theory. The pseudomass term of the Higgs scalar is not restricted and is used to set the scale for the masses in the theory.

Another feature of this kind of asymptotically free grand unified theory is that the hierarchy of symmetry breaking is dictated by the theory. There is no longer room to declare a range of quartic self-couplings so as to choose one vacuum versus another. At the superunification energies the structure of the vacuum that develops spontaneously[14] are given in Table I. To follow the successive breaking of the symmetry down to energies of 100 GeV or so requires a study of the t dependence of the Higgs μ^2 parameter, which we have not done. It is, however, certainly a subject worthy of further investigation.

V. SO(12) GRAND UNIFICATION

In this section we explore the phenomenological implications for a particular SO(12) solution. It looks the most promising as a minimal candidate for grand unification. In any case, it has the prototype features common to all the higher SO(N) solutions.

The discussion is simplest in the explicit rep-

TABLE I. Structure of all the SO(N) gauge theories that are asymptotically free. The fermions are in the spinorial representation and the Higgs bosons are in the adjoint representation. For an explanation of the notation, see text. Here a^2 and k are the square of the vacuum expectation value (VEV) of the Higgs field and the number of them which develop a VEV, respectively, as in Ref. 14.

N	N_F	$\bar{h}^2$	$\bar{\lambda}$	$\bar{\Lambda}$	k	a^2	$V_{\min}$	Structure of vacuum
10	10	0.1658	−0.0329	0.9719	5	0.7778	−0.9722	U(5)
10	10	0.1658	−0.1991	1.0680	2	1.8425	−0.9213	U(2) × SO(6)
11	5	0.1764	0.0112	0.9340	5	0.4782	−0.5977	U(5)
11	5	0.1764	−0.2464	1.0671	2	6.1241	−3.0620	U(2) × SO(7)
12	6	0.1853	0.0591	0.9158	6	0.3077	−0.4615	U(6)
12	6	0.1853	−0.3378	1.0835	1	1.2257	−0.3064	U(1) × SO(10)
12	5	0.1880	0.0368	0.9168	6	0.3680	−0.5520	U(6)
12	5	0.1880	−0.2730	1.0606	1	0.9716	−0.2429	U(1) × SO(10)
12	4	0.1916	0.1313	−1.1184	5	2.5650	−3.2062	U(5) × U(1)
12	4	0.1916	−0.1911	1.0238	2	1.9263	−0.9632	U(2) × SO(8)
12	4	0.1916	−0.0007	0.9254	6	0.5450	−0.8175	U(6)
12	3	0.1968	0.1244	−0.8368	4	3.1603	−3.1603	U(4) × SO(4)
13	3	0.1962	0.1513	−1.6648	6	3.3278	−4.9917	U(6)
13	3	0.1962	−0.3446	1.0720	1	1.3061	−0.3265	U(1) × SO(11)
13	3	0.1962	0.0713	0.9127	6	0.2827	−0.4240	U(6)
13	2	0.2056	0.1795	−1.1750	4	1.9144	−1.9144	U(4) × SO(5)
13	2	0.2056	−0.2251	1.0319	2	3.8029	−1.9015	U(2) × SO(9)
13	2	0.2056	0.0293	0.9133	6	0.3952	−0.5928	U(6)
13	1	0.2238	0.1321	−0.5701	3	2.2453	−1.6840	U(3) × SO(7)
14	3	0.2077	0.1994	−1.7136	5	1.7819	−2.2273	U(5) × SO(4)
14	3	0.2077	−0.3488	1.0661	1	1.3566	−0.3391	U(1) × SO(12)
14	3	0.2077	0.0810	0.9159	7	0.2440	−0.4270	U(7)
14	2	0.2198	0.2045	−1.2185	3	56.9865	−42.7399	U(3) × SO(8)
14	2	0.2198	−0.2454	1.0380	2	8.8480	−4.4240	U(2) × SO(10)
14	2	0.2198	0.0480	0.9137	7	0.3154	−0.5520	U(7)
14	1	0.2421	0.1599	−0.6072	2	15.4098	−7.7049	U(2) × SO(10)
15	2	0.2100	0.2021	−2.1860	6	2.0861	−3.1292	U(6) × SO(3)
15	2	0.2100	−0.4264	1.0744	1	2.2571	−0.5643	U(1) × SO(13)
15	2	0.2100	0.1057	0.9267	7	0.2078	−0.3637	U(7)
15	1	0.2338	0.2209	−1.2548	3	7.0812	−5.3109	U(3) × SO(9)
15	1	0.2338	−0.2581	1.0449	2	40.0752	−20.0376	U(2) × SO(11)
15	1	0.2338	0.0609	0.9208	7	0.2819	−0.4933	U(7)
16	2	0.2202	0.2334	−2.2196	5	4.3659	−5.4574	U(5) × SO(6)
16	2	0.2202	−0.4204	1.0727	1	2.1563	−0.5391	U(1) × SO(14)
16	2	0.2202	0.1100	0.9361	8	0.1854	−0.3709	U(8)
16	1	0.2474	0.2321	−1.2841	3	4.6044	−3.4533	U(3) × SO(10)
16	1	0.2474	−0.2659	1.0526	1	0.9599	−0.2400	U(1) × SO(14)
16	1	0.2474	0.0704	0.9316	8	0.2430	−0.4860	U(8)
17	1	0.2304	0.2534	−2.2474	5	1.7471	−2.1839	U(5) × SO(7)
17	1	0.2304	−0.4145	1.0746	1	2.0348	−0.5087	U(1) × SO(15)
17	1	0.2304	0.1137	0.9477	8	0.1807	−0.3614	U(8)
18	1	0.2406	0.2671	−2.2711	5	1.2487	−1.5609	U(5) × SO(8)
18	1	0.2406	−0.4084	1.0792	1	1.9053	−0.4763	U(1) × SO(16)
18	1	0.2406	0.1168	0.9606	9	0.1632	−0.3673	U(9)

resentation of the Clifford algebra,

$$\begin{aligned}
&\alpha_1=\sigma_1\times\sigma_1\times1\times1\times1\times\sigma_2 \qquad \alpha_2=\sigma_1\times\sigma_2\times1\times1\times\sigma_3\times\sigma_2 \qquad \alpha_3=\sigma_1\times\sigma_1\times1\times1\times\sigma_2\times\sigma_3\\
&\alpha_4=\sigma_1\times\sigma_2\times1\times1\times\sigma_2\times1 \qquad \alpha_5=\sigma_1\times\sigma_1\times1\times1\times\sigma_2\times\sigma_1 \qquad \alpha_6=\sigma_1\times\sigma_2\times1\times1\times\sigma_1\times\sigma_2\\
&\alpha_7=\sigma_2\times1\times\sigma_1\times1\times1\times1 \qquad \alpha_8=\sigma_2\times1\times\sigma_2\times1\times1\times1 \qquad \alpha_9=\sigma_2\times1\times\sigma_3\times\sigma_3\times1\times1\\
&\alpha_{10}=\sigma_1\times\sigma_3\times1\times1\times1\times1 \qquad \alpha_{11}=\sigma_2\times1\times\sigma_3\times\sigma_1\times1\times1 \qquad \alpha_{12}=\sigma_2\times1\times\sigma_3\times\sigma_2\times1\times1 \qquad \chi=\sigma_3\times1\times1\times1\times1\times1\,.
\end{aligned} \tag{11}$$

χ is the chirality operator which splits the 64-dimensional representation into two irreducible 32-dimensional representations.

In this basis, the fermion representation reads

$$\psi \equiv \tfrac{1}{2}(1+\chi)\Psi = (u,d,U,D,B,T,b,t), \quad (11')$$

where u,d,U,D transform as the $\underline{4}$ representation of the SO(6) subgroup (i.e., under σ_{ab}, $a,b=1,\ldots,6$) while t,b,T,B transform as the $\underline{4}^*$ representation of SO(6). Here, we have in mind

$$u=(\nu_e, u^R, u^G, u^B), \quad d=(e, d^R, d^G, d^B), \quad (12)$$

while U,D will be superheavy fermions, and similarly for the t,b,T,B family, with τ replacing the electrons.

The SO(4) subgroup (i.e., σ_{AB}, with $A,B=9,\ldots,12$) splits into two commuting SU(2) subgroups, given in the 32-dimensional representation by

$$\mathrm{SU}(2)_\pm:\ \tfrac{1}{2}(1\times1\pm\sigma_3\times\sigma_3)\times\vec{\sigma}\times1\times1, \quad (13)$$

so that u,d,b,t transform under $\mathrm{SU}(2)_+$, $\mathrm{SU}(2)_-$ as (2, 1) while U,D,B,T transform as (1, 2). Under the SO(2) subgroup generated by σ_{78}, the u,d,T,B have $+Y$ charge, while t,b,U,D have $-Y$ charge.

The charge matrix reads

$$Q=-\tfrac{1}{3}(\sigma_{12}+\sigma_{34}+\sigma_{56})+\sigma_{11,12}. \quad (14)$$

The U,D fermions acquire a superheavy mass as a result of the spontaneous symmetry breakdown. For this discussion we take for consideration the SO(12) solution with

$$\begin{aligned} &n_F=4,\\ &h^2=0.1916g^2,\\ &\lambda=-0.1911g^2,\\ &\Lambda=1.0238g^2,\\ &\langle\phi_{9,10}\rangle^2=\langle\phi_{11,12}\rangle^2=1.9263\mu^2/g^2,\\ &\text{all other } \langle\phi_{ij}\rangle=0. \end{aligned} \quad (15)$$

The negative value for λ does not destabilize the structure of the vacuum around $k=2$, where k is the number of the vacuum expectation values that are nonzero in the sense of Ref. 14. The classical stability condition reads

$$2\lambda k+\Lambda>0, \quad (16)$$

and for $k=2$ the classical potential is certainly stable. Quantum loop corrections around this vacuum are positive and maintain the stability of this vacuum.[15] The classical value for the minimum of the Higgs potential is

$$\langle V\rangle=-0.9632\mu^4/g^2. \quad (17)$$

If we choose $\langle\phi_{9,10}\rangle=+\langle\phi_{11,12}\rangle$ the resulting mass operator for the fermion becomes proportional to (in the 32-dimensional space)

$$(1\times1-\sigma_3\times\sigma_3)\times\sigma_3\times1\times1 \quad (18)$$

and it is easy to see that at this initial stage of superheavy energy scales, u,d,t,b will remain massless, while U,D,T,B acquire superheavy masses.

The structure of the vacuum after spontaneous symmetry breakdown is, at this first stage,

$$\mathrm{U}(2)\times\mathrm{SO}(8).$$

This is encouraging since it includes as a subgroup $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SO}(6)\times\mathrm{SO}(2)\sim\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{U}(1)\times\mathrm{U}(1)$. Of course it is only with further study of the μ^2 dependence on energy scale through mass renormalization that we can make definitive statements about hierarchy of breakdown.

The mass spectrum for the gauge bosons can easily be derived. The massive bosons are

$$W_{\mu aA},\quad a=1,\ldots,8,\quad A=9,\ldots,12$$

$$Y_\mu=\tfrac{1}{2}[W_{\mu9,11}+W_{\mu10,12}+i(W_{\mu9,12}-W_{\mu10,11})], \quad (19)$$

with masses satisfying, respectively,

$$(\text{mass})^2 \ \text{for}\ W_{\mu aA}=g^2\langle\phi_{9,10}\rangle^2,$$

$$(\text{mass})^2 \ \text{for}\ Y_\mu=4g^2\langle\phi_{9,10}\rangle^2.$$

The remaining 32 bosons have, at these superunification energies, zero mass. They will acquire mass as we go down in energy. The precise mechanism awaits further study of the renormalization-group equations.

Note added in proof. The table circulated in preprint form was incorrect. The table in this published form is the corrected one.

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