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Structure of the gauge theory vacuum at finite temperatures

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The space of moduli, i.e., the space of field configurations modulo gauge equivalence is constructed for *calorons* (self-dual gauge fields on $\mathbb{R}^3 \times S^1$) with vanishing topological charge. Such calorons represent vacuum field configurations at a finite temperature. It is found that, unlike its zero-temperature counterpart, this space of moduli is nontrivial, being, in fact, a one-dimensional manifold.

I. INTRODUCTION

It is well known[1] that the space of moduli of all self-dual SU(2) gauge fields on $\mathbb{R}^4$ with topological number $q>0$ is an $(8q-3)$-dimensional manifold. This result has been obtained by the use of the Atiyah-Singer index theorem[2] on S^4, which is the unique conformal compactification of Euclidean spacetime $\mathbb{R}^4$. Moreover, all these pseudoparticle solutions have been explicitly constructed.[3]

Calorons have been introduced by Harrington and Shepard[4,5] as the analogs of pseudoparticles[6] at a finite temperature. The transition from zero to a finite temperature T is generally effected[7] by the substitution $x_0 \to -ix_4$, where x_0 is the real time and x_4 is an angular variable which ranges from 0 to $\beta = 1/kT$ (k is the Boltzmann constant) and covers simply an S^1. This substitution applied to the generating functional for Green's functions yields the partition function, which in the case of an SU(2) gauge theory is given by[4]

$$Z = \int (dA) \exp\left[-S(A) + \int_0^\beta dx_4 \int_{\mathbb{R}^3} d^3x \, \mathrm{Tr}(j_\mu A_\mu)\right]. \tag{1.1}$$

In this formula the fields are defined on $\mathbb{R}^3 \times S^1$, where one also performs the spacetime integrations. The metric is Euclidean, $\mu = 1, 2, 3, 4$, j_μ denotes an external current, and

$$S(A) = -\frac{1}{2g^2} \int_{\mathbb{R}^3 \times S^1} d^4x \, \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}), \tag{1.2}$$

where g is the coupling constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ with $A_\mu = (\sigma_\alpha / 2i) A_\mu^\alpha$ an SU(2) gauge field on $\mathbb{R}^3 \times S^1$ and σ_α ($\alpha = 1, 2, 3$) the Pauli matrices. Field configurations on $\mathbb{R}^3 \times S^1$ with finite action $S(A)$ can be classified[4] in terms of the topological number[6]

$$q = -\frac{1}{16\pi^2} \int_{\mathbb{R}^3 \times S^1} d^4x \, \mathrm{Tr}(^*F_{\mu\nu} F_{\mu\nu}), \tag{1.3}$$

where $^*F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}$ is the dual of $F_{\mu\nu}$. A caloron with charge $q \geq 0$ is a self-dual field configuration A_μ $(^*F_{\mu\nu} = F_{\mu\nu})$ on $\mathbb{R}^3 \times S^1$ which minimizes the action $S(A)$ in the sector of gauge fields with topological number q. Such configurations have been shown to exist[5] for every $q \geq 0$.

Pseudoparticle solutions are believed to be fundamental in understanding the problem of color confinement. Should this prove to be the case, calorons would possibly provide us with a mechanism of thermal color liberation. While a very restricted class of caloron solutions is known,[5] it is certain that any realistic calculation would necessitate a much deeper understanding, involving the study and construction of the spaces of moduli $M(q)$ for each q. In particular, the dimensionalities of these spaces must be found. However, one is faced here with a serious difficulty. The Atiyah-Singer index theorem, which worked well for the case of pseudoparticles, cannot be directly applied here, because the ***unique*** conformal compactification of $\mathbb{R}^3 \times S^1$ is singular[8] and, further, it cannot accommodate even the already known caloron solutions. In spite of this, the space of moduli $M(0)$ for $q=0$ *can* be explicitly constructed. It is found that, unlike its zero-temperature counterpart, it is nontrivial and, in fact, it is a one-dimensional manifold. This is a first indication of an essential difference between the zero- and finite-temperature cases.

II. THE VACUUM AT FINITE TEMPERATURES

In the following, SU(2)′ denotes the Lie algebra of SU(2). $A = A_\mu dx_\mu$ and $F = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu = dA + \frac{1}{2} A \wedge A$ are SU(2)′-valued forms on $\mathbb{R}^3 \times S^1$ with d the exterior differentiation and $\wedge$ the exterior multiplication defined as $A \wedge A = [A_\mu, A_\nu] dx_\mu \wedge dx_\nu$.

Any self-dual SU(2) gauge field configuration on $\mathbb{R}^3 \times S^1$ with vanishing topological number ($q=0$) must have vanishing action and thus be a vacuum configuration ($F=0$). We will now seek to construct the space of moduli for $F=0$, i.e., the space of all A's with $F=0$ modulo gauge equivalence. Consider the obvious vacuum configuration $A=0$ and any infinitesimal variation δA thereof, such that $F=d(\delta A)=0$. We consider these variations modulo the infinitesimal gauge transformations of $A=0$ which are given by $\delta A=d(\delta\theta)$, where $\delta\theta$ is an infinitesimal SU(2)′-valued 0-form on $\mathbf{R}^3\times S^1$. Thus δA belongs to $H^1_{\mathrm{SU}(2)'}(\mathbf{R}^3\times S^1)$, the first cohomology group of SU(2)′-valued forms on $\mathbf{R}^3\times S^1$. One can easily see that the space of δA's is three-dimensional. This follows from the fact that

$$\dim H^1_{\mathrm{SU}(2)'}(\mathbf{R}^3\times S^1)=3\dim H^1(\mathbf{R}^3\times S^1)=3\dim H^1(S^1)=3\,,$$

where H^1 denotes the first de Rham cohomology group.[9] Noticing further that dx_4 is a closed but *not* exact one-form on $\mathbf{R}^3\times S^1$ we can write $\delta A=i\delta\xi_\alpha\sigma_\alpha dx_4$. Moreover, it is not difficult to see that the fields

$$A=i\xi_\alpha\sigma_\alpha dx_4\equiv\xi dx_4 \tag{2.1}$$

are also vacuum configurations for any real constants ξ_α.

We will now prove that any vacuum configuration $A=A_\mu(x)dx_\mu$ on $\mathbf{R}^3\times S^1$ can be gauge transformed to a configuration of the form (2.1). To this end, we consider the gauge transformation

$$g(x)=P\exp\left[\int_{\{x_j\}}^0 A_i(y)dy_i\right], \tag{2.2}$$

where $i,j=1,2,3$ and the integral is path-ordered along any path in $\mathbf{R}^3$ connecting the point $\{x_j\}$ with 0. This is a well-defined gauge transformation on $\mathbf{R}^3\times S^1$ since $\mathbf{R}^3$ is simply connected and $F=0$. Now it is not hard to see that the gauge-transformed field $A'=g^{-1}Ag+g^{-1}dg$ is of the form $A'=A_4'(x_4)dx_4$. There remains, however, a freedom for gauge transformations, which depend only on x_4. Exploiting this, A' can be brought into the form of Eq. (2.1) by the well-defined gauge transformation on $\mathbf{R}^3\times S^1$,

$$g'(x_4)=P\exp\left[\int_{x_4}^0 A_4'(y_4)dy_4\right]\times\exp\left(-\frac{\xi x_4}{\beta}\right), \tag{2.3}$$

where $\xi\in\mathrm{SU}(2)'$ is given by

$$\exp(\xi)=P\exp\left[\int_\beta^0 A_4'(y_4)dy_4\right]. \tag{2.4}$$

We have, so far, established a homomorphism of SU(2)′ onto $M(0)$, i.e., to every element $\xi=i\xi_\alpha\sigma_\alpha$ of SU(2)′ we have associated an element of $M(0)$ represented by A in Eq. (2.1). However, this homomorphism is not one-to-one. A constant gauge transformation g applied on $A=\xi dx_4$ yields $A'=g^{-1}Ag=g^{-1}\xi g dx_4\equiv\xi' dx_4$ with $|\xi'|=|\xi|$, where

$$|\xi|^2=-\tfrac{1}{2}\mathrm{Tr}(\xi\cdot\xi),\ |\xi|\geq 0\,.$$

Clearly then every element ξ of SU(2)′ lying on the two-sphere $|\xi|=\text{const}$ corresponds to the same point of $M(0)$. It will thus suffice to consider only those ξ's which belong to a one-dimensional linear subspace of SU(2)′ and, without loss of generality, assume that

$$A=\xi dx_4=i\xi_1\sigma_1 dx_4\,. \tag{2.5}$$

Finally, to exhaust the remaining gauge freedom, let us consider the gauge transformations $g=g(x_4)$ which preserve the form of A in Eq. (2.5). From

$$A=\xi dx_4\to A'=g^{-1}Ag+g^{-1}dg=\xi' dx_4\equiv i\xi_1'\sigma_1 dx_4\,, \tag{2.6}$$

it follows that

$$\partial_4 g(x_4)=-\xi g+g\xi'\,, \tag{2.7}$$

whose general solution is

$$g(x_4)=e^{-\xi x_4}g_0 e^{\xi' x_4}\,, \tag{2.8}$$

where $g_0=\alpha_0+i\alpha_\alpha\sigma_\alpha$ ($\alpha_\mu\alpha_\mu=1$) is any element of SU(2). A necessary condition for $g(x_4)$ to be well defined on $\mathbf{R}^3\times S^1$ is $g(0)=g(\beta)$, which gives

$$g_0 e^{\xi'\beta}g_0^{-1}=e^{\xi\beta}\,. \tag{2.9}$$

For $e^{\xi'\beta}\neq\pm 1$, Eq. (2.9) implies

$$\alpha_0\alpha_2+\alpha_1\alpha_3=\alpha_1\alpha_2-\alpha_0\alpha_3=0\,, \tag{2.10}$$

which admits the following two solutions:

$$\alpha_0=\alpha_1=0 \tag{2.11}$$

and

$$\alpha_2=\alpha_3=0\,. \tag{2.12}$$

From Eq. (2.9) and each of (2.11) and (2.12) we have $\exp[\beta(\xi\pm\xi')]=1$. respectively, which implies

$$\xi\pm\xi'=n\rho,\quad n=0\pm 1,\ldots\,, \tag{2.13}$$

where $\rho=i\rho_1\sigma_1$ and $\rho_1=2\pi/\beta$. The gauge transformation in Eq. (2.8) reduces to

$$g(x_4)=e^{-(\xi\pm\xi')x_4}g_0=e^{-n\rho x_4}g_0\,, \tag{2.14}$$

which, by virtue of Eq. (2.13), is well defined on $\mathbf{R}^3\times S^1$. For $e^{\xi'\beta}=\pm 1$ we have $\xi'=n'\rho$ and $\xi'=(n'+\frac{1}{2})\rho$, respectively, and Eq. (2.9) implies $e^{\xi\beta}=\pm 1$, i.e., $\xi=n\rho$ and $\xi=(n+\frac{1}{2})\rho$, respectively. Then Eq. (2.8), with $g_0=1$, reduces to the well-defined gauge transformation

$$g(x_4)=\exp(\xi'-\xi)x_4=\exp(n'-n)\rho x_4\,. \tag{2.15}$$

We thus have found that the only gauge fields A of the form given in Eq. (2.5) which are gauge equivalent are the ones for which Eq. (2.13) holds. In other words, for every $A'=\xi' dx_4$ corresponding to an element $\xi'=i\xi_1'\sigma_1$ of the one-dimensional linear subspace of SU(2)$'$, there always exists a gauge equivalent field $A=i\xi_1\sigma_1 dx_4$ with ξ_1 restricted to the interval $[0,\pi/\beta]$.

To summarize, *the space of moduli* $M(0)$ *is isomorphic to the* D^1 *submanifold of* SU(2)$'$ *defined by* $\xi=i\xi_1\sigma_1$ *with* $0\le\xi_1\le\pi/\beta=\pi kT$. At the zero-temperature limit, $M(0)$ reduces to a point, as expected.

We would like to conclude with a comment, to which the above result has led us. Consider a general gauge field configuration A on $\mathbf{R}^3\times S^1$ with $F\to 0$ as $r^2=x_i^2\to\infty$. It has been assumed in the literature[4] that

$$A \underset{r\to\infty}{\sim} g^{-1}dg\,, \tag{2.16}$$

where, g is a map $S^2\times S^1\to$ SU(2). However, it should be clear now that this is not quite the case. Instead, we have

$$A \underset{r\to\infty}{\sim} g^{-1}(i\xi_1\sigma_1 dx_4)g+g^{-1}dg \tag{2.17}$$

with $0\le\xi_1\le\pi kT$ and, again, $g{:}S^2\times S^1\to$ SU(2). The topological number q of A can now be written[6,10] as

$$q=\frac{1}{24\pi^2}\int_{S^2\times S^1} d\sigma_\mu\epsilon^{\mu\nu\lambda\rho}\mathrm{Tr}(A_\nu A_\lambda A_\rho)$$
$$=\frac{1}{24\pi^2}\int_{S^2\times S^1} d\sigma_\mu\epsilon^{\mu\nu\lambda\rho}\mathrm{Tr}(g^{-1}\partial_\nu g g^{-1}\partial_\lambda g g^{-1}\partial_\rho g) \tag{2.18}$$

and the relation $q=-\,$degree(g) still holds. For the definition of the *degree* of a map see, e.g., Ref. 9.

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