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Consequences of gauge-fixing ambiguities in non-Abelian gauge theories in axial and Coulomb gauges

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We consider a pure SU(2) non-Abelian gauge theory in a first-order canonical quantization scheme, and show how the proper treatment of zero modes in solving the constraint equations for the dependent variables leads to certain conditions on the physical states. In the axial gauge this leads to a result of Schwinger's. In the Coulomb gauge this leads to previously unknown conditions on the physical states, and to the conclusion that the theory does not describe a free radiation field under circumstances for which Gribov has shown that the vector potential becomes non-unique.

I. INTRODUCTION

The reasons why non-Abelian gauge theory is considered to be the leading candidate for the theory of strong interactions are so well known as to not need repeating here. Recently, Gribov[1] observed that in the Coulomb gauge the gauge-fixing condition does not necessarily lead to a unique potential. In the present work we are principally concerned with a related problem. Namely, we consider a pure SU(2) non-Abelian gauge theory in a first-order canonical quantization formulation in both the axial and Coulomb gauges, and examine under what conditions the constraint equations may be solved for the dependent variables. These conditions turn out to be nontrivial owing to the existence of zero modes of the operators one must invert. Indeed, we show that in the Coulomb gauge the physical states cannot be those of a free radiation field for sufficiently large fields. Rather, the physical states must obey certain constraints. We do not show how one may construct states which obey these constraints, but we conjecture that the construction of such states may be related to the question of confinement. Our results are of a nonperturbative albeit formal nature.

We first consider, in Sec. II, the $A_3=0$ gauge. Here we rederive a condition Schwinger[2] found some time ago from considerations of the finiteness of the energy density, but our method of derivation allows us to understand how similar results may be obtained in other gauges. In Sec. III we discuss the Coulomb gauge and derive the fact, mentioned in the foregoing, that the physical states must obey certain conditions whenever the fields are sufficiently large so as to lead to the existence of a zero mode of a particular operator. We also consider the relationship between the canonical quantization formulation and the path-integral formulation. Then in Sec. IV we make some comments about the connection between the Lorentz transformation properties of the theory and the existence of gauge-fixing ambiguities in axial gauges. Finally, in Sec. V we summarize our results and speculate on their significance.

Throughout we use both the three-component notation and the two-by-two matrix notation. Given a three-component object Θ^a, we define

$$\Theta \equiv g\frac{\tau^a}{2i}\Theta^a ,$$

where the τ^a are the usual Pauli matrices. Note then that

$$\Theta^a\Theta^a = -\frac{2}{g^2}\mathrm{Tr}[\Theta\Theta] .$$

II. AXIAL GAUGE

In matrix notation the pure Yang-Mills field Lagrangian, in a first-order formulation, has the form

$$\mathcal{L} = -\frac{2}{g^2}\mathrm{Tr}\{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu])\} , \tag{2.1}$$

and the derived Euler-Lagrange equations are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] . \tag{2.2}$$

and

$$\partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 . \tag{2.3}$$

We now wish to implement the axial gauge condition[3]

$$A_3 = 0 \tag{2.4}$$

to eliminate dependent degrees of freedom. Applying (2.4) to (2.2) and (2.3) yields the constraint equations

$$\partial_3 F_{03} = -D_i F_{0i} \tag{2.5}$$

and

$$\partial_3 A_0 = -F_{03}\,, \tag{2.6}$$

as well as the equations of motion

$$\dot{A}_i = F_{0i} + D_i A_0 \tag{2.7}$$

and

$$\dot{F}_{0i} = D_j F_{ji} + \partial_3 F_{3i} - [A_0, F_{0i}]\,, \tag{2.8}$$

where D_i is the covariant derivative, and in this section indices i,j run from 1 to 2.

With these two equations we can, in principle, eliminate the dependent variables F_{03} and A_0. However, we must carefully consider the conditions under which we can solve (2.5) and (2.6) in terms of well-defined Green's functions. For this purpose it is convenient to put the system in a box of length $2L$, so that $-L \leq x_3 \leq L$. Then the spectrum of the operator ∂_3 is discrete, and the orthonormal functions

$$u_n = \frac{1}{\sqrt{2L}} e^{i\omega_n x_3}\,, \tag{2.9}$$

with

$$\omega_n = \frac{\pi n}{L} \quad (n \text{ integer})\,, \tag{2.10}$$

obey

$$\partial_3 u_n = i\omega_n u_n \tag{2.11}$$

and the periodic boundary conditions

$$u_n(L) = u_n(-L)\,. \tag{2.12}$$

We then define the function

$$\tfrac{1}{2}\epsilon_L(x_3 - x_3') = \sum_n{}' \frac{u_n(x_3)u_n^*(x_3')}{i\omega_n}\,, \tag{2.13}$$

where the prime on the summation symbol indicates that the term $n=0$ is omitted.[4] Since $\omega_0 = 0$, omitting $n=0$ is obviously necessary to define a finite inverse to the operator ∂_3. This function obeys

$$\partial_3 \tfrac{1}{2}\epsilon_L(x_3 - x_3') = \delta(x_3 - x_3') - u_0 u_0^*\,, \tag{2.14}$$

owing to the completeness of the eigenfunctions. Then the solutions of (2.5) and (2.6) for F_{03} and A_0 are

$$F_{03}(x_3) = -\int_{-L}^{L} dx_3' \tfrac{1}{2}\epsilon_L(x_3 - x_3') D_i F_{0i}(x_3') \tag{2.15}$$

and

$$A_0(x_3) = -\int_{-L}^{L} dx_3' \tfrac{1}{2}\epsilon_L(x_3 - x_3') F_{03}(x_3')\,, \tag{2.16}$$

provided that

$$0 = \int_{-L}^{L} dx_3 u_0^* D_i F_{0i}(x_3) \tag{2.17}$$

and

$$0 = \int_{-L}^{L} dx_3 u_0^* F_{03}(x_3')\,. \tag{2.18}$$

That is, the "source terms" in (2.5) and (2.6) must be orthogonal to the zero mode of ∂_3, since otherwise no solutions for F_{03} and A_0 will exist in terms of the finite Green's function $\frac{1}{2}\epsilon_L$.

Now we want to take the continuum limit $L \to \infty$. It is easily seen that

$$\begin{aligned}\lim_{L\to\infty} \tfrac{1}{2}\epsilon_L(x_3 - x_3') &= \frac{1}{2\pi i} \mathrm{P}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(x_3 - x_3')}}{\omega} \\ &= \tfrac{1}{2}\epsilon(x_3 - x_3') \\ &= \tfrac{1}{2}[\theta(x_3 - x_3') - \theta(x_3' - x_3)]\,,\end{aligned} \tag{2.19}$$

where the Cauchy principal value is indicated. Then the infinite-space versions of (2.15)–(2.18) are

$$F_{03}(x_3) = -\int_{-\infty}^{\infty} dx_3' \tfrac{1}{2}\epsilon(x_3 - x_3') D_i F_{0i}(x_3')\,, \tag{2.20}$$

$$A_0(x_3) = -\int_{-\infty}^{\infty} dx_3' \tfrac{1}{2}\epsilon(x_3 - x_3') F_{03}(x_3')\,, \tag{2.21}$$

$$0 = \int_{-\infty}^{\infty} dx_3 D_i F_{0i}(x_3)\,, \tag{2.22}$$

and

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} dx_3 F_{03}(x_3) \\ &= -\int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_3' \tfrac{1}{2}\epsilon(x_3 - x_3') D_i F_{0i}(x_3') \\ &= \int_{-\infty}^{\infty} dx_3' x_3' D_i F_{0i}(x_3')\,.\end{aligned} \tag{2.23}$$

We now recognize the constraint (2.22) as just the condition that Schwinger found long ago by requiring that no spurious infinity occur in the equations of motion.[2] The present derivation of this condition allows us to see its fundamental origin. When solving constraint equations for dependent variables, one must properly treat the zero modes of the operators one is trying to invert. In the $A_3 = 0$ gauge, this automatically produces the Schwinger condition (2.22).

Note also that the inverse derivative $\frac{1}{2}\epsilon(x_3 - x_3')$ is not unique, but that we may make the replacement

$$\tfrac{1}{2}\epsilon(x_3 - x_3') \to \tfrac{1}{2}\epsilon(x_3 - x_3') + \Lambda(x_0, x_1, x_2)\,, \tag{2.24}$$

where Λ is an arbitrary function independent of x_3. But if the conditions (2.22) and (2.23) hold, then the values of A_0 and F_{03} given by (2.20) and (2.21) remain unchanged by the substitution (2.24). Thus, the uniqueness of the solutions for A_0 and F_{03} is guaranteed by the conditions (2.22) and

(2.23).

Let us now introduce the equal-time commutation relation

$$\{A_i^a(\vec{x},t), F_{0j}^b(\vec{x}',t)\}_- = i\delta_{ab}\delta^3(\vec{x}-\vec{x}')\,. \tag{2.25}$$

Here the slightly unorthodox notation $\{,\}_-$ is being used for an operator commutator to distinguish it from a matrix commutator $[,]$. Then, as Schwinger pointed out, the operator

$$Q = \int dx_1 dx_2 \Lambda^a(x_0,x_1,x_2) \int dx_3 D_j^{ab} F_{0j}^b\,, \tag{2.26}$$

Λ arbitrary, is the generator of infinitesimal gauge transformations which stay within the $A_3 = 0$ gauge, as illustrated by

$$\delta A_i^a(\vec{x},t) = i\{A_i^a(\vec{x},t), \epsilon Q(t)\}_- = \epsilon D_i^{ab}\Lambda^b\,, \tag{2.27}$$

where we have dropped a surface term. Thus, the condition (2.22) arises because of the gauge arbitrariness inherent in the $A_3 = 0$ gauge. The reader will recall, though, that we also derived a second constraint (2.23). To see the meaning of this second condition we define the operator R by

$$R = \int dx_1 dx_2 \Lambda^a(x_0,x_1,x_2) \int dx_3 x_3 D_j^{ab} F_{0j}^b\,. \tag{2.28}$$

Then we see that R generates transformations

$$\delta A_i^a(\vec{x},t) = i\{A_i^a(\vec{x},t), \epsilon R(t)\}_- = \epsilon x_3 D_i^{ab}\Lambda^b\,. \tag{2.29}$$

This is also an infinitesimal transformation which respects the $A_3 = 0$ gauge condition, though not a gauge transformation. Notice, however, that this is a transformation which is singular at $x_3 = \pm\infty$. Hence, by a choice of finite boundary conditions at $x_3 = \pm\infty$, we may eliminate the possibility of making such transformations, and, therefore, eliminate the necessity of enforcing the condition (2.23). Another way of saying the same thing is to note that taking the derivative of (2.6) we obtain

$$\partial_3{}^2 A_0 = D_i F_{0i}\,. \tag{2.30}$$

But the inverse of $\partial_3{}^2$ is not well defined since $\partial_3{}^2$ has zero modes of the form

$$u_0 = \Lambda_1(x_0,x_1,x_2) + x_3\Lambda_2(x_0,x_1,x_2)\,. \tag{2.31}$$

However, we may impose the boundary condition of finiteness at $x_3 = \pm\infty$ on the eigenmodes of $\partial_3{}^2$ so that $\Lambda_2 = 0$. Then a unique solution of (2.30) exists if and only if (2.22) holds.

As is well known, the condition (2.22), or its equivalent $Q = 0$, is not consistent with the canonical commutation relations, as shown by (2.27). So A_i and F_{0i} are not the independent dynamical variables. One can try to deal with this in two ways. The condition $Q = 0$ may be interpreted as a constraint on the physical states

$$Q\,|\psi\rangle = 0\,, \tag{2.32}$$

as recently attempted by Mandelstam.[5] Or, on the other hand, one may attempt to use the large gauge arbitrariness inherent in the $A_3 = 0$ gauge to enforce supplemental gauge conditions until all gauge arbitrariness is eliminated, and, hence, all necessity for additional conditions such as $Q = 0$, as recently attempted by Chodos.[6]

It is our point of view, though, that the very large gauge arbitrariness of the $A_3 = 0$ gauge, which is present even in QED, makes it difficult to see the essential differences between Abelian and non-Abelian theories. The Coulomb gauge, however, has the advantage that it is the gauge in which, in QED, the independent physical variables can be most readily identified. For this reason, we now turn our attention to the Coulomb gauge, and try to see whether conditions analogous to (2.22) and (2.23) may arise owing to zero-mode problems in solving for dependent variables. We will, however, have a little more to say about axial gauges in Sec. IV.

III. COULOMB GAUGE

In this section we will consider the canonical quantization scheme in the Coulomb gauge, and its relationship to a path-integral formulation. As Gribov[1] has pointed out, the Coulomb gauge condition

$$\partial_i A_i = 0 \tag{3.1}$$

does not fix the gauge uniquely. (In this section, indices i,j run from 1 to 3.) For, if we consider a gauge transformation

$$A_\mu \to A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U$$

such that U obeys

$$D_i \partial_i U U^{-1} = 0\,, \tag{3.2}$$

then $\partial_i A'_i = 0$. What will be of interest to us will be the case in which U is an infinitesimal transformation

$$U = e^{\epsilon h} \simeq 1 + \epsilon h\,, \tag{3.3a}$$

$$A'_i = A_i + \epsilon D_i h\,. \tag{3.3b}$$

Under such a transformation the Coulomb gauge condition is maintained if

$$D_i \partial_i h = 0\,. \tag{3.4}$$

The existence of normalizable zero modes of the operator $D_i\partial_i$ for sufficiently large A_i has been discussed by Gribov. In the present paper, we are interested in the formal consequences of such zero modes, and, hence, we will simply assume the existence of such normalizable solutions of (3.4). Pictorially, the condition (3.4) means that the hypersurface defined by the Coulomb gauge condition intersects the gauge orbit with zero "angle."

As in the usual Coulomb gauge formulation,[7] we introduce the transverse and longitudinal components of the canonical momentum F_{0i} according to

$$F_{0i}=E_i-\partial_i f\,, \tag{3.5}$$

where

$$\partial_i E_i=0\,. \tag{3.6}$$

Now considering $\partial_i F_{0i}$, and using (3.1) in (2.2), we obtain the constraint equation for A_0,

$$D_i\partial_i A_0=\nabla^2 f\,. \tag{3.7}$$

Substitution of the condition (3.1) into (2.3) yields the constraint equation for f,

$$\begin{aligned} D_i\partial_i f &= D_j E_j \\ &= [A_j, E_j]\,. \end{aligned} \tag{3.8}$$

To solve for the dependent variables A_0 and f, we must invert the operator $D_i\partial_i$. Since, by assumption, this operator has a normalizable zero mode, we must introduce a generalized Green's function $\tilde{G}$,[4] which obeys

$$D_i^{ab}\partial_i \tilde{G}^{bc}(\vec{x}-\vec{x}')=\delta_{ac}\delta^3(\vec{x}-\vec{x}')-h^a(\vec{x})h^c(\vec{x}')\,. \tag{3.9}$$

If the operator $D_i\partial_i$ has more than one zero mode, we must sum over them in (3.9). For the sake of notational convenience, we will assume the existence of only one such mode. We also assume that it is real, but this is inessential. The solutions of (3.7) and (3.8) are then given, symbolically, by

$$A_0=\tilde{G}\nabla^2 f \tag{3.10}$$

and

$$f=\tilde{G}D_iE_i\,. \tag{3.11}$$

However, this can only be true if the "source terms" lie in the subspace of functions orthogonal to the zero mode. That is we must have

$$\begin{aligned} Q &\equiv -\frac{2}{g^2}\mathrm{Tr}\int d^3x\, h(\vec{x})D_iE_i(\vec{x}) \\ &=0 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} R &\equiv -\frac{2}{g^2}\mathrm{Tr}\int d^3x\, h(\vec{x})\nabla^2 f(\vec{x}) \\ &=0\,. \end{aligned} \tag{3.13}$$

Otherwise, (3.7) and (3.8) cannot be solved in terms of the nonsingular Green's function $\tilde{G}$.

Notice, also, that $\tilde{G}$ is not uniquely defined, but may be replaced by $\tilde{G}+chh$, where c is an arbitrary constant. But the conditions (3.12) and (3.13) ensure that the solutions for A_0 and f are unique.

Using (3.11) we may rewrite (3.13) as a restriction on the independent variable E_i, viz.,

$$\begin{aligned} R &= -\frac{2}{g^2}\mathrm{Tr}\int d^3x\, h(\vec{x})\nabla^2\tilde{G}D_iE_i \\ &=0\,, \end{aligned} \tag{3.14}$$

or, less symbolically, and in component notation,

$$\begin{aligned} R &= \int d^3x\, h^a(\vec{x})\nabla^2\int d^3x'\,\tilde{G}^{ab}(\vec{x},\vec{x}')D_i'^{bc}E_i^c(\vec{x}') \\ &=0\,. \end{aligned} \tag{3.15}$$

An alternative way of deriving the restrictions (3.12) and (3.14), is to first solve (3.7) for f,

$$f=\frac{1}{\nabla^2}D_i\partial_i A_0\,. \tag{3.16}$$

The operator ∇^2, of course, has no normalizable zero mode. Then, using (3.16) in (3.8), we obtain

$$D_j\partial_j\frac{1}{\nabla^2}D_i\partial_i A_0=D_iE_i\,, \tag{3.17}$$

which defines A_0. To solve this last equation, we can use (3.10) and (3.11) to write

$$A_0=\tilde{G}\nabla^2\tilde{G}D_iE_i\,, \tag{3.18}$$

or, less symbolically, and in component notation,

$$A_0^a(\vec{x})=\int d^3x'\,\tilde{\mathfrak{D}}^{ab}(\vec{x},\vec{x}')D_j'^{bc}E_j^c(\vec{x}')\,, \tag{3.19}$$

where

$$\tilde{\mathfrak{D}}^{ab}(\vec{x},\vec{x}')=\int d^3x''\,\tilde{G}^{ac}(\vec{x},\vec{x}'')\nabla^{2\prime\prime}\tilde{G}^{cb}(\vec{x}'',\vec{x}')\,. \tag{3.20}$$

Now let us compute

$$D_j\partial_j\frac{1}{\nabla^2}D_i\partial_i\tilde{\mathfrak{D}}\,.$$

We find, recalling (3.9), that

$$\begin{aligned} -D_j^{ac}\partial_j\int d^3x''\frac{1}{4\pi|\vec{x}-\vec{x}''|}D_i''^{cd}\partial_i''\tilde{\mathfrak{D}}^{db}(\vec{x}'',\vec{x}') &= \delta_{ab}\delta^3(\vec{x}-\vec{x}')-h^a(\vec{x})h^b(\vec{x}') \\ &\quad + D_j^{ac}\partial_j\int d^3x''\frac{h^c(\vec{x}'')}{4\pi|\vec{x}-\vec{x}''|} \\ &\quad \times\int d^3x'''h^d(\vec{x}''')\nabla^{2\prime\prime\prime}\tilde{G}^{db}(\vec{x}''',\vec{x}')\,. \end{aligned} \tag{3.21}$$

So for

$$A_0=\tilde{\mathfrak{D}}D_iE_i \tag{3.22}$$

to be the solution of (3.17), we see again that the conditions (3.12) and (3.14) must hold. Here it might be added that in the presence of an external source J_μ (3.12) and (3.14) are modified by replacing D_iE_i by $D_iE_i + J_0$.

To see the connection between the restrictions (3.12) and (3.14) on E_i and the possibility of making an infinitesimal gauge transformation of the kind indicated in equations (3.3), let us first introduce the canonical commutation relation,

$$\begin{aligned}\{A_i^a(\vec{x},t), E_j^b(\vec{x}',t)\}_- &= i\delta_{ab}[\delta_{ij}\delta^3(\vec{x}-\vec{x}')]^T \\ &= i\delta_{ab}[\delta_{ij}\delta^3(\vec{x}-\vec{x}') - \partial_i\partial_j'(4\pi|\vec{x}-\vec{x}'|)^{-1}]. \quad (3.23)\end{aligned}$$

We see then that the operator Q is just the generator of infinitesimal gauge transformations which stay within the Coulomb gauge, as illustrated by,

$$\begin{aligned}\delta_Q A_i^a(\vec{x},t) &= i\{A_i^a(\vec{x},t), \epsilon Q(t)\}_- \\ &= \epsilon D_i^{ac}h^c. \quad (3.24)\end{aligned}$$

Clearly, the c-number transcription of $\delta_Q A_i^a$ obeys

$$\partial_i \delta_Q A_i^a = 0. \quad (3.25)$$

The operator R generates infinitesimal field transformations of the form

$$\begin{aligned}\delta_R A_i^a(\vec{x},t) &= i\{A_i^a(\vec{x},t), \epsilon R(t)\}_- \\ &= \epsilon D_i^{ab}\int d^3x' h^c(\vec{x}')\nabla^{2\prime}\tilde{G}^{bc}(x,x') - \epsilon\partial_i h^a \\ &\quad -\epsilon\partial_i \int d^3x' \frac{h^a(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\int d^3x'' h^b(\vec{x}'')\nabla^{2\prime\prime} \\ &\quad \times h^b(\vec{x}''). \quad (3.26)\end{aligned}$$

This is not a gauge transformation because of the gradient terms. However, as is readily seen, it obeys, if considered as a c number,

$$\partial_i \delta_R A_i^a = 0. \quad (3.27)$$

If we now compare the conditions (3.12) and (3.14) with the results (3.24) and (3.26), we see that, with an integration by parts, (3.12) and (3.14) state that E_i is orthogonal to $\delta_Q A_i$ and $\delta_R A_i$.

Since $\delta_Q A_i$ and $\delta_R A_i$ are not identically zero, it is clear from the foregoing that the conditions $Q = 0$ and $R = 0$ are not in general compatible with the canonical commutation relations. It follows that (3.23) must be false. A_i and E_i are not canonically conjugate variables. We see then that, when the field strengths become large enough so that zero modes of the operator $D_i\partial_i$ exist, the independent dynamical variables cannot be those of a free radiation field. This is the central result of this paper.

Of course, one may, paralleling the treatment of the electromagnetic field in Lorentz gauge,[8] treat A_i and E_i as if they were canonically conjugate, and interpret (3.12) and (3.14) as constraints on the physical states

$$Q|\psi\rangle = 0 \quad (3.28a)$$

and

$$R|\psi\rangle = 0. \quad (3.28b)$$

It is beyond the scope of this paper to attempt to actually construct such states $|\psi\rangle$.

Next we wish to construct the Feynman path integral for the vacuum-to-vacuum transition amplitude. The Hamiltonain density is

$$\mathcal{H} = E_i^a \frac{\partial A_i^a}{\partial t} - \mathcal{L}, \quad (3.29)$$

with

$$\mathcal{L} = \tfrac{1}{2}(E_i^a - \partial_i f^a)^2 - \tfrac{1}{2}(B_i^a)^2, \quad (3.30)$$

where

$$B_i^a = \tfrac{1}{2}\epsilon_{ijk}F_{jk}^a. \quad (3.31)$$

From (2.2), (3.5), and (3.10) we have

$$\frac{\partial A_i}{\partial t} = E_i - (\partial_i - D_i\tilde{G}\nabla^2)f. \quad (3.32)$$

Because $\tilde{G}$ obeys equation (3.9), the right-hand side of (3.32) is not explicitly transverse unless $R = 0$. Then we find that

$$\begin{aligned}\int d^3x E_i^a \frac{\partial A_i^a}{\partial t} &= \int d^3x[(E_i^a)^2 + E_i^a D_i^{ab}\tilde{G}^{bc}\nabla^2 f^2] \\ &= \int d^3x[(E_i^a)^2 - f^a\nabla^2 f^a]. \quad (3.33)\end{aligned}$$

To obtain the second line, we have used integration by parts, (3.8), (3.9), and the condition $R = 0$. We then obtain the usual result for the Hamiltonian

$$H = \tfrac{1}{2}\int d^3x[(E_i^a)^2 + (B_i^a)^2 + (\partial_i f^a)^2]. \quad (3.34)$$

In order to now write down the vacuum-to-vacuum transition amplitude,[9] we must restrict the set of admissible functions in the functional integrals to those which satisfy the constraints $Q = 0$ and $R = 0$. So compared to the usual case we should make the replacement

$$(\text{measure}) \to (\text{measure})\delta(Q)\delta(R). \quad (3.35)$$

Then the transition amplitude is

$$\begin{aligned}W = \int d[E_i^a]d[A_i^a]\delta(\partial_i E_i^a)\delta(\partial_i A_i^a)\delta(Q)\delta(R) \\ \times \exp\left\{i\int d^4x[E_i^a\dot{A}_i^a - \tfrac{1}{2}(E_i^a)^2 - \tfrac{1}{2}(B_i^a)^2 - \tfrac{1}{2}(\partial_i f^a)^2]\right\}. \\ (3.36)\end{aligned}$$

Now we want to change variables from E_i to F_{0i}, so we insert

$$1=\Delta\int d[f^b]\delta(D_i^{ab}\partial_i f^b - D_j^{ac}E_j^c) \qquad (3.37)$$

into the functional integral. Since in the functional integral Δ will multiply the factor $\delta(D_i\partial_i f - D_j E_j)$, we need only evaluate Δ for fields E_i which satisfy (3.8). So we write

$$\frac{1}{\Delta}=\int d[f^{b\prime}]\delta(D_i^{ab}\partial_i f^{b\prime} - D_j^{ac}E_j^c)\,,$$

and let

$$f' = f+\delta f\,, \qquad (3.38)$$

where

$$D_i\partial_i f = D_j E_j\,.$$

The key point then is that, since D_iE_i must be orthogonal to h, i.e., $Q=0$, f must lie in the subspace orthogonal to h. Hence, the integration variable f', or δf, may be taken to lie in this same subspace. So we expand δf in terms of the orthonormal eigenfunctions u_n which satisfy

$$D_i\partial_i u_n=\lambda_n u_n\,, \qquad (3.39)$$

viz.,

$$\delta f = \sum_n{}' c_n u_n\,. \qquad (3.40)$$

The prime on the summation symbol indicates that the zero mode is excluded. Then we find

$$\begin{aligned}\frac{1}{\Delta}&=\lim_{\beta\to 0} N\int d[\delta f^a]\exp\left[\frac{i}{\beta}\int d^3x (D_i^{ab}\partial_i \delta f^b)^2\right]\\ &=\lim_{\beta\to 0} N\int \prod_n{}' dc_n \exp\left(\frac{i}{\beta}\sum_n{}' c_n{}^2\lambda_n{}^2\right)\\ &=1/\prod_n{}'\lambda_n\\ &=[\det{}'(D_i\partial_i)]^{-1}\,. \end{aligned} \qquad (3.41)$$

That is, Δ is the product of the nonzero eigenvalues of $D_i\partial_i$. In the absence of the condition $Q=0$, the integral expression for $1/\Delta$ would be ill defined. Then it is easy to change variables from E_i to F_{0i}, do the f integration, and introduce A_0 and F_{ij} as dummy variables in the usual way, to obtain, finally,

$$W=\int d[A_\mu^a]d[F_{\mu\nu}^a](\partial_i A_i^a)\delta\Big)\int d^3x h^a D_i^{ab}F_{0i}^b\Big)\times\delta\left(\int d^3x h^a\partial_i F_{0i}^a\right)\exp\left(i\int d^4x \mathcal{L}\right), \qquad (3.42)$$

where $\mathcal{L}$ is given by (2.1).

If it were not for the two δ functions involving h, one could perform the integration over $F_{\mu\nu}$ to obtain the standard expression for W in the second-order formulation. Evidently, the canonical and the path-integral formulations are not equivalent.

As is well known,[10] if we start from the expression

$$W=\int d[A_\mu^a]\exp\left\{i\int d^4x[-\tfrac{1}{4}(F_{\mu\nu}^a)^2]\right\}, \qquad (3.43)$$

and insert

$$1=\Delta\int d[g]\delta(\partial_i A_i^{(g)})\,, \qquad (3.44)$$

where

$$A_\mu^{(g)}=U^{-1}A_\mu U+U^{-1}\partial_\mu U\,, \qquad (3.45)$$

with $U=U(g)$, then

$$W=\int d[A_\mu]e^{iS}\delta(\partial_i A_i)\Big[\sum_l \det{}^{-1}(D_i^{(l)}\partial_i)\Big]^{-1}. \qquad (3.46)$$

Here

$$D_i^{ab(l)}=\delta_{ab}\partial_i+g\epsilon_{acb}A_i^{c(g_l)}\,, \qquad (3.47)$$

and $U_l=U(g_l)$ obeys

$$D_i(\partial_i U_l U_l^{-1})=0\,.$$

On the one hand, starting from the canonical formulation it is not easy to see how one can obtain this factor in (3.46) which is the inverse of a sum of inverse determinants, while, on the other hand, in the case in which one of the operators $D_i^{(l)}\partial_i$ has a zero mode, it is not clear how one is to make the expression for W well defined in the path-integral formulation. It may be that there exists some overview which allows one to handle both the multiple intersections of the gauge-fixing condition with the gauge orbits and the zero-mode problem discussed in this paper. If so, the author does not know of it. Alternatively, it may just be possible that the multiplicity of gauge equivalent fields which satisfy the Coulomb gauge condition is an irrelevant mathematical pathology, and that the construction of states which satisfy the conditions (3.28) yields the relevant physics. Whether or not any of this has anything to do with the question of confinement is an open question.

IV. MORE ABOUT AXIAL GAUGES

As everyone knows, the gauge condition

$$A_3=0 \qquad (4.1)$$

does not fix the gauge completely, but one can make additional gauge transformations

$$A_\mu\to A'_\mu=U^{-1}A_\mu U+U^{-1}\partial_\mu U\,,$$

so long as U is independent of x_3.

It would appear then that by appending additional gauge-fixing conditions such as

$$A_2|_{x_3=0}=0\,, \tag{4.2a}$$

$$A_1|_{x_2,x_3=0}=0\,, \tag{4.2b}$$

and

$$A_0|_{x_1,x_2,x_3=0}=0\,, \tag{4.2c}$$

that one fixes the gauge completely *without* any Gribov type gauge ambiguities arising. However, as Yao[11] pointed out long ago in QED, the axial gauge condition is not compatible with Lorentz invariance if we assume reasonable boundary conditions. To rephrase his argument, consider an infinitesimal Lorentz transformation,

$$x\rightarrow x'=lx=x+\epsilon x\,, \tag{4.3}$$

along with a simultaneous compensating gauge transformation,

$$A_\mu(x)\rightarrow A'_\mu(x')=U^{-1}A_\mu(x)U+\epsilon_\mu^\nu U^{-1}A_\nu(x)U+U^{-1}\partial_\mu U\,. \tag{4.4}$$

Then we should have

$$\begin{aligned}0&=A'_3(x')\\&=\epsilon_3{}^\nu U^{-1}A_\nu(x)U+U^{-1}\partial_3 U\,.\end{aligned} \tag{4.5}$$

If we assume the boundary condition that A_μ be finite at $|x_3|=\infty$, then in order that the transverse components ($\mu=1,2$) of A_μ be well defined at $|x_3|=\infty$, we must have

$$U^{-1}\partial_3 U\xrightarrow[|x_3|\to\infty]{}0\,. \tag{4.6}$$

In this case, at $|x_3|=\infty$, (4.5) reads

$$0=\epsilon_3{}^\nu U^{-1}A_\nu U|_{|x_3|=\infty}\,. \tag{4.7}$$

But, since ϵ is arbitrary, either $A_\nu=0$ at $|x_3|=\infty$, which is too restrictive in general, or $A_3\neq 0$ when $|x_3|=\infty$. This contradiction is not rectified by imposing further axial-like gauge conditions such as equations (4.2). However, as Yao pointed out, one can fix the gauge, without having a conflict with Lorentz invariance at $|x_3|=\infty$, if one imposes the condition, for example,

$$\partial_i A_i=0\ (i=1,2) \tag{4.8}$$

at $|x_3|=\infty$ to replace $A_3=0$ there. In QED this eliminates all the residue gauge freedom, but in a non-Abelian theory we can make additional gauge transformations with transformation matrices which obey the two-dimensional Gribov condition,

$$D_i(\partial_i UU^{-1})=0\ \ (i=1,2)\,. \tag{4.9}$$

For example, in the case $A_\mu=0$ we can try the ansatz

$$U=e^{i\hat\rho\cdot\vec\tau\gamma(\rho)}\,, \tag{4.10}$$

where ρ is the two-dimensional radius vector, and $\hat\rho=\vec\rho/\rho$. This leads to the equation

$$\rho^2\gamma''+\rho\gamma'=\tfrac{1}{2}\sin(2\gamma)\,, \tag{4.11}$$

which upon introducing the variable $t=\ln\rho$ becomes the equation for an undamped pendulum

$$\ddot\gamma=\tfrac{1}{2}\sin(2\gamma)\,. \tag{4.12}$$

There exist solutions of this equation with the properties

$$\gamma\xrightarrow[\rho\to 0]{}0 \tag{4.13}$$

and

$$\gamma\xrightarrow[\rho\to\infty]{}\pi\,. \tag{4.14}$$

Such a solution will have the properties

$$A_i\xrightarrow[\rho\to 0]{}0 \tag{4.15}$$

and

$$A_i\xrightarrow[\rho\to\infty]{}i\tau^a(2\hat\rho^a\hat\rho^i-\delta_{ai})\rho^{-2}\,, \tag{4.16}$$

where $i=1,2$.

Hence, we see that in axial gauges one must be careful about concluding too hastily that no gauge ambiguities exist.

V. SUMMARY AND CONCLUSIONS

In the present work we have concentrated on the problems which arise, in a first-order canonical quantization procedure, owing to the existence of normalizable zero modes of the operators that must be inverted to eliminate the dependent variables in the axial gauge and in the Coulomb gauge.

We first rederived a condition that Schwinger found to be necessary for the finiteness of the energy density in the axial gauge. Our derivation of this condition shows that its origin lies in the existence of zero modes of the operator ∂_3.

Then we showed how in the Coulomb gauge similar conditions (3.12) and (3.14), arise owing to the fact that, as Gribov showed, there can exist zero modes of the operator $D_i\partial_i$. In both gauges these conditions are connected with the possibility of making infinitesimal field transformations which stay within the gauge. However, it should be pointed out that in the axial gauge this possibility of making infinitesimal field transformations arises trivially from the fact that one can make a noninfinitesimal transformation which respects the gauge condition, viz.,

$$A_\mu\rightarrow A'_\mu=U^{-1}A_\mu U+U^{-1}\partial_\mu U\,,$$

with U any unitary matrix independent of x_3. In the Coulomb gauge the situation is much less trivial, and, therefore, we speculate that the Coulomb gauge may be more physically relevant.

The principal conclusion of this paper is that when field strengths become large enough so that normalizable zero modes of the operator $D_i\partial_i$ exist the physical states are not those of a free radiation field. The situation is reminiscent of an order-disorder transition in a crystal, where when the temperature rises to a certain point there is a discontinuous change in the symmetry properties of the crystal.

Our results, obviously, have been of a formal nature. The problem then is to construct states that obey the conditions (3.28), which is seemingly a quite nontrivial task, and which may or may not be related to the question of confinement.

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