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Effect of instantons on the short-distance structure of hadronic currents

Nathan Andrei
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

David J. Gross*
Institute for Advanced Study, Princeton, New Jersey 08540
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The influence of the nonperturbative structure of the quantum-chromodynamic vacuum on the short-distance behavior of hadronic currents is discussed. The dilute-gas approximation is systematically used. We show how to calculate in this approximation aribtrary Green's functions and that the effects of tunneling (instantons) are summarized by an appropriate effective Lagrangian. These methods are applied to the two-point current correlation function, which is explicitly calculated in the dilute-gas approximation. We estimate the numerical size of the instanton effects for e^+e^- annihilation and find them to be strongly momentum dependent and large. The qualitative features of these corrections suggest an explanation of precocious scaling.

I. INTRODUCTION

One of the unique features of non-Abelian gauge theories is asymptotic freedom. The vanishing of the effective coupling in these theories at short distances leads to an explanation of scaling in deep-inelastic scattering and has provided much of the motivation for proposing quantum chromodynamics (QCD) as a theory of the strong interactions.[1] The standard analysis of the short-distance behavior of hadronic currents in QCD has relied on the use of the renormalization group and perturbation theory, and has not taken into account the nonperturbative dynamics of QCD which must be the ultimate source of chiral-symmetry breaking and quark confinement. Recently it has been proposed[2] that the dynamics of QCD can be understood in terms of the nonperturbative structure of the QCD vacuum which arises due to tunneling between classically degenerate vacuums. In this paper we shall examine the effects of the vacuum structure of QCD on the short-distance behavior of hadronic currents.

Until now the study of the short-distance behavior of hadronic currents has relied solely on the use of the renormalization group and perturbation theory. Consider, for example, the analysis of e^+e^- annihilation to hadrons.[3] Here one is interested in the large-momentum behavior of the two-point function of electromagnetic currents $J_\mu = \bar{\psi}\gamma_\mu Q\psi$,

$$\Pi_{\mu\nu}(p) = \int d^4x\, e^{ipx}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle = (g_{\mu\nu}P^2 - P_\mu P_\nu)\Pi(P, g, m_i, \mu)\,, \quad (1)$$

where Π is related to the total annihilation cross section by

$$\sigma^{\rm tot}_{e^+e^-\to\rm hadrons}(P^2) = \frac{32\pi^3\alpha^2}{p^2}\,{\rm Im}\,\Pi(p^2)\,, \quad (2)$$

g is the QCD coupling, m_i are the quark mass parameters, and μ is the renormalization scale parameter. By using the renormalization group equation[4]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \sum_i m_i\gamma_i(g)\frac{\partial}{\partial m_i}\right]\Pi = D(g)\,, \quad (3)$$

one deduces that

$$\Pi(\lambda P, g, m, \mu) = \Pi(p, \bar{g}(\lambda), \bar{m}(\lambda), \mu) - \int_g^{\bar{g}(\lambda)} \frac{D(x)dx}{\beta(x)}\,, \quad (4)$$

where $\bar{g}(\lambda)$ $[\bar{m}(\lambda)]$ is the effective coupling constant [mass]. Now as $\lambda\to\infty$, $\bar{g}^2(\lambda) = O((\ln\lambda)^{-1})$, $\bar{m}(\lambda) \sim \lambda^{-1}(\ln\lambda)^P$. Thus the large-momentum behavior of Π can be calculated from the weak-coupling zero-mass theory. For spacelike P^2, Π should have no mass singularities. It is therefore reasonable to assume that the perturbative expansion of Π is asymptotic. Thus,

$$\Pi(P^2) \underset{P^2\to\infty}{\sim} -D_0 \ln\frac{P^2}{\mu^2} - \frac{D_1}{b_0}\ln\left(\frac{g^2}{\bar{g}(P^2)}\right) + \sum_{i=0}^{\infty} \Pi^{(i)}(\mu^2)[\bar{g}^2(P^2)]^i + O\left(\frac{1}{P^2}\right). \quad (5)$$

This then leads to an asymptotic expansion for

$$R = \sigma^{\rm tot}_{e^+e^-\to\rm hadrons}/\sigma_{e^+e^-\to\mu^+\mu^-}\,,$$

$$R(P^2) \underset{P^2\to\infty}{\sim} \sum Q_i^2\left[1 + \frac{\bar{g}^2(P^2)}{4\pi^2} + O(\bar{g}^4)\right] + O\left(\frac{1}{P^2}\right), \quad (6)$$

which allows us to measure the sum of the squared quark charges, $\sum Q_i^2$, as well as the effective QCD coupling $\bar{g}(-p^2)$.

In the perturbative treatment outlined above the power corrections to the leading asymptotic behavior arise solely from the quark mass parameters and are of order $\bar{m}(p) \sim (m/p)(\ln P)^a$ for large p. If the quarks are massless then the perturbative expansion of $\Pi_{\mu\nu}$ can produce no such power corrections. For this reason alone it is clear that in QCD there must be important nonperturbative contributions to $\Pi_{\mu\nu}$, at least for moderate values of P.[5] This is because one can, to a good approximation, regard the light quarks (up, down, and perhaps strange quarks) as massless, so that chiral SU(3)×SU(3) is an exact symmetry of the strong interactions. The quark masses then arise by a nonperturbative mechanism due to the dynamical symmetry breaking of chiral symmetry. These dynamical masses are renormalization group invariants[6] of the form

$$M(g,\mu)=\mu\exp\left[-\int^{g}\frac{dx}{\beta(x)}\right]\underset{g\sim 0}{\sim}\mu\exp\left(\frac{-1}{b_0 g^2}\right). \quad (7)$$

Thus one expects that there will be "mass corrections" in QCD of the form $[M(g,\mu)/P]^a$. This is clearly consistent with the renormalization group equation, Eq. (3), due to Eq. (7). However, it is clear from the above form of $M(g,\mu)$ that such terms will not appear in an asymptotic expansion about $g=0$. Indeed if one examines Eq. (4) it is clear that inverse powers of λ can only appear in $\Pi(\lambda P, g, m=0, \mu)$ if $\Pi(P, \bar{g}(\lambda), \bar{m}=0, \mu)$ contains terms of the form

$$\exp\left[-\frac{c}{\bar{g}(\lambda)^2}\right]\underset{\lambda\to\infty}{\sim}\left(\frac{1}{\lambda}\right)^{b_0 c}.$$

Recently it has been discovered that the structure of the QCD vacuum itself cannot be described perturbatively. This is because of the existence of an infinite number of classically degenerate vacuum states. The QCD vacuum consists of a coherent superposition of these states labeled by a continuous parameter θ.[7,8] The properties of this vacuum can be easily studied, for weak coupling, in the semiclassical or WKB approximation by taking into account the tunneling between the classically degenerate vacuums. Here one uses the instanton solutions of Yang-Mills theory[9] which give the dominant contributions to the sum over path histories that tunnel from one classical vacuum to another.

The consequences of this vacuum structure are far-reaching. In addition to providing a solution of the U(1) problem[7,8,10] it has been argued that instantons provide a source for dynamical chiral-symmetry breaking.[7] Furthermore when one considers the interaction between quarks one finds that the effects of tunneling are very large. They give rise to a substantial coupling-constant renormalization[2] and have an important effect on the heavy-quark potential[2,11,12] Finally, it has been argued that quark confinement itself is a consequence of the structure of the vacuum due to tunneling via "meron" configurations, which become important at distances of the order of hadronic size.[2]

What is remarkable is that all of the above mechanisms are believed to set in when the effective coupling is still relatively small. The effects of instantons of a size ρ, or tunneling that takes place in a space-time region of size ρ^4, is of order

$$\text{const}\times\left(\frac{8\pi^2}{\bar{g}^2(\rho)}\right)^6\exp\left(-\frac{8\pi^2}{\bar{g}^2(\rho)}\right).$$

This is a typical tunneling amplitude. The reason that it can be substantial for small values of $g^2/8\pi^2$ is that there exist many distinct tunneling paths (degrees of freedom for instantons) in QCD. This is the origin of the term $(8\pi^2/g^2)^6$. Consequently the effects of instantons and ultimately of merons are important even when $g^2/8\pi^2 \sim \frac{1}{20}-\frac{1}{10}$. This ensures that even on a hadronic scale the effective coupling of QCD remains small and semiclassical methods may be employed.

In this paper we shall explore the consequences of this nonperturbative vacuum structure of the short-distance behavior of hadronic currents. We expect that at sufficiently small distances only small-size instantons will be relevant. Asymptotic freedom guarantees that the effective coupling is small enough so that the density or the probability of tunneling of such instantons vanishes rapidly, in which case the "dilute-gas approximation" can be used. The instantons will then lead to calculable power corrections to the standard high-energy behavior.

These corrections are calculated explicitly for e^+e^- annihilation in the dilute-gas approximation (DGA). Unfortunately we find that this approximation is not totally adequate for quantitative purposes. It turns out that the instantons responsible for these corrections are sufficiently dense that one cannot neglect their interactions. However, we do believe that the DGA does provide a qualitative, order-of-magnitude, description of the nonperturbative effects of instantons as well as providing the starting point for a more quantitative study of these effects.

What is clear already from the DGA approximation is that the instanton effects are very substantial, even for values of the momentum for which the effective coupling constant is very small. This we believe provides a qualitative explanation, together with the picture developed in Ref.

2, of precocious scaling. For large momentum one has almost free field behavior with small perturbative corrections and even smaller nonperturbative instanton corrections. As we decrease P the effective coupling remains small all the way down to hadronic momentum, M_H, whereas the nonperturbative instanton-generated effects turn on rapidly, dominating the perturbative corrections at $P\sim M_H$. Thus the logarithmic perturbative corrections to asymptotic freedom are always small, whereas the powerlike nonperturbative corrections turn off rapidly for $P>M_H$ leading to "precocious scaling."

We start, in Sec. II, by discussing in some detail the dilute-gas approximation. We show how one can calculate in the DGA the vacuum expectation values of products of local operators, and that the effects of instantons can always be summarized by an effective, nonlocal Lagrangian. As specific examples we consider the quark propagator and four-point functions, and derive an effective quark self-energy and four-fermion interaction vertex due to instantons.

In Sec. III we apply these methods to calculating the instanton contributions to e^+e^- annihilation to hadrons in the DGA. Here we make use of the explicit form of the massless quark propagator in a background instanton field[13,14] to derive an explicit analytic expression for $\Pi_{\mu\nu}$ for light quarks.

In Sec. IV we discuss the implications of the above calculation. We present some estimates of the magnitude of the power corrections for e^+e^- annihilation, emphasizing however the limitations of the DGA. In addition we discuss some of the possible theoretical implications of these results, noting that they may be used as input into Migdal's scheme[18] for calculating hadronic masses.

Section V contains some concluding remarks and in the Appendix we discuss the dependence of a fermion determinant (which enters into the instanton density) on the quark mass.

II. THE DILUTE-GAS APPROXIMATION

In this section we shall describe how one calculates the expectation value of local operators in the dilute-gas approximation. This approximation is valid as long as the analog gas of instantons and anti-instantons is sufficiently dilute. We shall see that in this approximation the effects of multiple-instanton–anti-instanton configurations can be easily calculated from the properties of the single-instanton configuration. We shall show that the effects of the instantons can be summarized by an effective Lagrangian.

Consider the vacuum expectation value of an operator $O(A,\psi,\overline{\Psi})$, which is a product of gauge or quark fields. In the $\theta=0$ vacuum this is written as[2]

$$\langle \mathrm{vac}|O|\mathrm{vac}\rangle=\frac{\sum_{n=-\infty}^{\infty}\int[DA_\mu D\psi D\overline{\psi}]_n e^{-S(A,\psi,\overline{\psi})}O(A,\psi,\overline{\psi})}{\sum_{n=-\infty}^{\infty}\int[DA_\mu D\psi D\overline{\psi}]_n e^{-S(A,\psi,\overline{\psi})}}. \tag{8}$$

The sum in Eq. (8) runs over the discrete sectors of function space corresponding to different values of the topological charge $Q=n$. In the semiclassical approximation, valid for small coupling, one evaluates the functional integral by performing a Gaussian integration about the saddle point in each topological sector. These saddle points can be taken to be superpositions of well-separated instantons and anti-instantons since the instanton "gas" for small coupling will turn out to be very dilute. Thus in the topological sector with $Q=n$, we expand the gauge field about $A_\mu^{n_+,n_-}$ which is taken to be a superposition of n_+ (n_-) instantons (anti-instantons) where $n=n_+-n_-$:

$$A_\mu^{n_+,n_-}=\sum_{i=1}^{n_+}A_\mu^{(i)}(x_i,R_i,\rho_i)+\sum_{i=1}^{n_-}\overline{A}_\mu^{(i)}(x_i,R_i,\rho_i). \tag{9}$$

The instanton fields must be superimposed in the singular gauge where

$$A_\mu^{(i)}(x_i,R_i,\rho_i)=\frac{R^i_{ab}\eta_{a\mu\nu}(x-x_i)_\nu\rho_i^2}{(x-x_i)^2[(x-x_i)^2+\rho_i^2]}\tau^b. \tag{10}$$

These configurations are parametrized by their position x_i, scale size ρ, and group orientation R^i.

Such a superposition of instantons is only an approximate saddle point of the classical action. A careful treatment requires introduction of constraints in order to fix the collective coordinates of each instanton. In addition one must take into account the interaction energy between an instanton and anti-instanton, $S_{\mathrm{int}}(x_i^\pm,R_i^\pm,\rho_i^\pm)$, which represents the difference between the action of the field given in Eq. (9) and $(n^++n^-)(8\pi^2/g^2)$. It has the structure of a dipole interaction which falls off as the fourth power of the instanton–anti-instanton separation.[2] As long as the instanton gas is sufficiently dilute the dominant contributions to the functional integral will come from well-separated instantons and S_{int} can be ignored. Furthermore in such an approximation the quantum fluctuations about a superposition of well-separated instantons is easily calculated. Thus in calculating the vacuum-to-vacuum amplitude, the contribution of a single instanton yields[10]

$$\int[DA_\mu]_{n_\pm=1,n_\mp=0}e^{-S(A)}\simeq V\int\frac{d\rho}{\rho^5}D(\rho), \tag{11}$$

where[2]

$$D(\rho)=0.1\left(\frac{8\pi^2}{\bar{g}^2(\rho)}\right)^6\exp\left(-\frac{8\pi^2}{\bar{g}^2(\rho)}\right) \tag{12}$$

denotes the density of instantons of size ρ (in units of ρ^{-4}) and V is the volume of space-time. In the above approximation the saddle-point integration about a multiple-instanton-anti-instanton configuration is

$$\int [DA_\mu]_{n_+,n_-} \simeq \frac{(V)^{n_++n_-}}{n_+!n_-!}\prod_i\int\frac{d\rho_i^\pm}{(\rho_i)^5}D(\rho_i^\pm). \tag{13}$$

In performing the functional integration in Eq. (8) we must also integrate over the quark fields ψ. This involves, in the semiclassical approximation, calculating the determinant of the operator $(i\not\partial - m - \not A)$, where A_μ is the background classical field,

$$\int d\psi\, d\bar\psi\, e^{\int dx\, \bar\psi(i\not\partial-m-\not A)\psi}=\det(i\not\partial-m-\not A)\equiv\Delta(A). \tag{14}$$

In the dilute-gas approximation, where A is a superposition of well-separated instanton fields, the determinant factorizes into a product of the determinants for each individual instanton. Thus,

$$\Delta(A_{n_+,n_-})\simeq\prod_i\Delta(m,\rho_i,\mu), \tag{15}$$

where $\Delta(m,\rho,\mu)$ is the fermion determinant in a background instanton field of size ρ. It is a dimensionless function of the fermion mass, m, of ρ and of the renormalization scale parameter μ. It has been calculated by 't Hooft in the limit $m\rho\to 0$. For large values of $m\rho$ the fermions should decouple completely.[16] We have calculated the large-$m\rho$ limit of $\Delta(m\rho)$. The results can be summarized as follows (for details see the Appendix).

For a single fermion (after renormalization),

$$\Delta(m\rho)=\begin{cases}1.338(\rho m)\,e^{-(2/3)\ln\rho\mu}, & \rho m\ll 1\\ 1-0.16/(m\rho)^2, & \rho m\gg 1.\end{cases} \tag{16}$$

The determinant in the case of N_f fermions of mass m_i, simply factorizes and is thus equal to

$$\prod_{i=1}^{N_f}\Delta(m_i\rho).$$

We are now in a position to evaluate the vacuum expectation value of O. In the dilute-gas approximation as outlined above we will have

$$\langle\text{vac}|O|\text{vac}\rangle=\frac{\displaystyle\sum_{n_+,n_-=0}^{+\infty}\frac{1}{n_+!n_-!}\int\prod\frac{dx_i^\pm d\rho_i^\pm}{(\rho_i^\pm)^5}dR_i^\pm D(\rho_i^\pm)\Delta(m\rho_i^\pm)\langle O\rangle_{A_{n_+,n_-}}}{\displaystyle\sum_{n_+,n_-=0}^{+\infty}\frac{1}{n_+!n_-!}\int\prod\frac{dx_i^\pm d\rho_i^\pm}{(\rho_i^\pm)^5}dR_i^\pm D(\rho_i^\pm)\Delta(m\rho_i^\pm)}, \tag{17}$$

where $\langle O\rangle_{A_{n_+,n_-}}$ is the expectation value of the product of fields represented by O in the background instanton field $(A_\mu)_{n_+,n_-}$. Since this will in general be a function of the instanton coordinates the integration over instanton position $(dx_i^\pm)$ and group orientation $(dR_i$, normalized so that $\int dR_i=1)$ must be performed after the evaluation of $\langle O\rangle_{A_{n_+,n_-}}$.

When the density of instantons is small this can be expanded in a power series in the density, yielding

$$\langle\text{vac}|O|\text{vac}\rangle=\langle O\rangle_0+\sum_\pm\int\frac{dx_\pm d\rho_\pm}{(\rho_\pm)^5}dR_\pm D(\rho_\pm)\Delta(m\rho_\pm)[\langle O\rangle_{A(x_\pm,\rho_\pm,R_\pm)}-\langle O\rangle_0]+O(D^2), \tag{18}$$

where $\langle O\rangle_0$ is the ordinary vacuum expectation value of O. The higher-order terms, which are proportional to higher powers of the density, can easily be evaluated. For example, a contribution to the D^2 terms is

$$\int dx_+dx_-\frac{d\rho_+d\rho_-}{(\rho_+\rho_-)^5}dR_+dR_-D(\rho_+)D(\rho_-)\Delta(m\rho_+)\Delta(m\rho_-)[\langle O\rangle_{A(x_+,\rho_+,R_+,x_-,\rho_-,R_-)}-\langle O\rangle_{A(x_+,\rho_+,R_+)}-\langle O\rangle_{A(x_-,\rho_-,R_-)}+\langle O\rangle_0]. \tag{19}$$

The integration over instanton positions is expected to be convergent. This is because the expectation value of a product of local operators in a background instanton field will approach the ordinary vacuum expectation value when the instanton is centered about a point very far away from the local operators. Consider the second term in the density expansion of $\langle\text{vac}|O|\text{vac}\rangle$. When $x_\pm$ is very far away from the positions of the local fields contained in O expects that

$$\langle O\rangle_{A_\pm(x_i,\rho_i,R_\pm)}-\langle O\rangle_0$$

will vanish as a power of $(x_\pm)^{-1}$. In all of the cases we shall consider below this will be sufficient to render the $\int dx_\pm$ integrals convergent.

Finally we must consider the integration over instanton scale sizes. For small ρ,

$$\bar{g}^2(\rho)\sim\frac{1}{\ln(1/\rho\mu)},$$

and thus the density $D(\rho)$ vanishes rapidly, justifying the dilute-gas approximation. However as ρ increases the density increases leading to a smaller mean instanton–anti-instanton separation and to increasing corrections to the DGA. For large ρ one must include the corrections to the DGA arising from instanton–anti-instanton interactions. In addition other configurations, merons, become important for large ρ. Thus the DGA to the evaluation of $\langle \mathrm{vac}|O|\mathrm{vac}\rangle$ will be reliable only if the ρ integrations are peaked at sufficiently low values of ρ. This can be achieved to some degree by considering O to be a product of local operators at short distances, since one expects that $\langle O\rangle_{A(x,\rho,R)} - \langle O\rangle_0$ will vanish, for $\rho \gg d$, as a power of (d/ρ), where d is the largest scale size associated with O [i.e., if $O = J_\mu(x)J_\nu(y)$, then $d \simeq |x-y|$].

We shall now consider some special examples of operators O, and show that the effects of instantons in the DGA can be summarized by an effective Lagrangian. Consdier first the fermion propagator $O = T\{\bar\psi(x)\psi(y)\}$. We must evaluate this propagator in a background instanton field, A_{n_+,n_-}, as given in Eq. (9). Thus

$$\langle O\rangle_{A_{n_+,n_-}} = \langle x|(i\not\partial - m - \not A_{n_+,n_-})^{-1}|y\rangle \equiv S_{n_+,n_-}(x,y,\Omega_i)\,, \qquad (20)$$

where $\Omega_i = (x_i, R_i, \rho_i)$ labels the instanton coordinates. In the DGA the functional integral will be dominated by well-separated instantons. In that case S_{n_+,n_-} can be calculated in terms of the propagator in a single instanton or anti-instanton field,

$$S_\pm(x,y;\Omega_i) = \langle x|[i\not\partial - m - \not A_\pm(\Omega)]^{-1}|y\rangle\,. \qquad (21)$$

Since A_{n_+,n_-} is a sum of fields localized at well-separated positions S_{n_+,n_-} can be expanded as follows:

$$S_{n_+,n_-} \cong S_0 + \sum_i [S^{(i)} - S_0] + \sum_{i\neq j}[S^{(i)} - S_0]S_0^{-1}[S^{(j)} - S_0] + \cdots + \sum_{i_1\neq i_2 \cdots i_{n_++n_-}} [S^{(i_1)} - S_0]S_0^{-1}\cdots[S^{(i_{n_++n_-})} - S_0]\,, \qquad (22)$$

where S_0 is the free fermion propagator $S_0(x-y) = \langle x|(i\not\partial - m)^{-1}|y\rangle$, $S^{(i)}$ is the propagator in the background field given in Eq. (10), and matrix notation has been employed so that

$$S^{(i)}S_0^{-1}S^{(j)} = \int du\,dv\,\langle x|[i\not\partial - m - \not A_i(\Omega_i)]^{-1}|u\rangle \times\langle u|(i\not\partial - m)|v\rangle \times\langle v|[i\not\partial - m - \not A_j(\Omega_j)]^{-1}|y\rangle\,. \qquad (23)$$

This expression has a simple diagrammatic representation as illustrated in Fig. 1. The terms we have neglected are those where the fermion interacts with say the instanton localized at x_1 both before and after interacting with the instanton localized at x_2. Such a contribution will be negligible if $|x_1 - x_2|$ is very large. Its inclusion in $S^{(n)}$ will lead to corrections which will be of higher order in the instanton density (see Fig. 2).

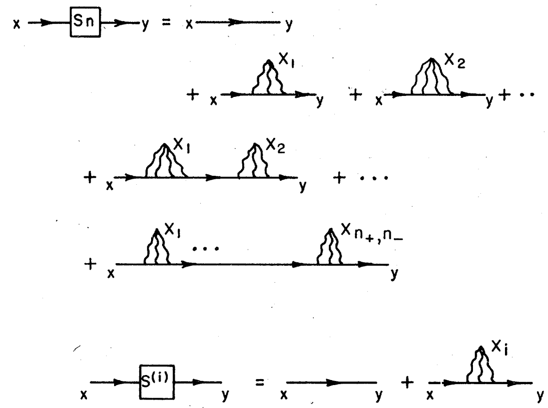

FIG. 1. Leading contributions to the propagators.

We now insert this expansion into Eq. (17). After some algebra one then derives

$$\langle \mathrm{vac}|T[\bar\psi(x)\psi(y)]|\mathrm{vac}\rangle = S(x-y) = \left\langle x\left|\frac{1}{i\not\partial - m - \Sigma}\right|y\right\rangle, \qquad (24)$$

where

$$\langle x|\Sigma|y\rangle = \sum_\pm \int \frac{dx_\pm d\rho_\pm dR_\pm}{(\rho_\pm)^5} D(\rho_\pm)\Delta(\rho_\pm m) \times\langle x|S_0^{-1}[S(x_\pm,\rho_\pm,R_\pm) - S_0]S_0^{-1}|y\rangle\,. \qquad (25)$$

Thus in the DGA the effect of the instantons on the fermion propagator is contained in a self-energy term Σ, which is proportional to the instanton density. The terms that we have neglected in S_{n_+,n_-} would contribute terms of order density squared to Σ. Note that to lowest order in Σ, or equivalently in the density of instantons, we could have derived Eq. (24) directly from Eq. (18).

In a similar fashion we can derive effective fermion interaction terms that summarize the effect of instantons in the DGA when O is a product of many fermion fields. Consider a product of

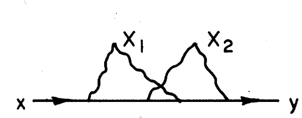

FIG. 2. Nonleading contribution to the propagator.

four fermion fields,

$$ {}_{\alpha\beta}O_{\gamma\delta}(x_1x_2x_3x_4)=\bar\psi_\alpha(x_1)\bar\psi_\beta(x_2)\psi_\gamma(x_3)\psi_\delta(x_4)\,. $$

We then have, in a background A_{n_+,n_-} field,

$$ \langle O\rangle_{A_{n_+,n_-}}=-[S_{n_+,n_-}(x_1,x_3;\Omega_i)]_{\alpha\gamma}[S_{n_+,n_-}(x_2,x_4;\Omega_i)]_{\beta\delta} +[S_{n_+,n_-}(x_1,x_4;\Omega_i)]_{\alpha\delta}[S_{n_+,n_-}(x_2,x_3;\Omega_i)]_{\beta\gamma}\,. \tag{26} $$

If we now replace S_{n_+,n_-} by its approximate expression in the DGA and insert in Eq. (17), we will find that the effective fermion four-point function is that generated by a theory with a fermion propagator given by Eq. (24) plus a four-fermion interaction term. This interaction can most easily be derived by using Eq. (18). Thus,

$$ \begin{aligned} \langle\mathrm{vac}\,|\,{}_{\alpha\beta}O_{\gamma\delta}(x_1x_2,x_3x_4)\quad|\,\mathrm{vac}\rangle = {} & S_0(x_1-x_3)_{\alpha\gamma}S_0(x_2-x_4)_{\beta\delta} \\ & +\sum_\pm\int\frac{dx_\pm d\rho_\pm dR_i}{(\rho_\pm)^5}D(\rho_\pm)\Delta(m\rho_\pm) \\ & \times\{[S_\pm(x_1,x_3;\Omega_\pm)-S_0(x_1-x_3)]_{\alpha\gamma}S_0(x_2-x_4)_{\beta\delta} \\ & +S_0(x_1-x_3)_{\alpha\gamma}[S_\pm(x_2,x_4;\Omega_\pm)-S_0(x_2-x_4)]_{\beta\delta} \\ & +[S_\pm(x_1,x_3;\Omega_\pm)-S_0(x_1-x_3)]_{\alpha\gamma}[S_\pm(x_2,x_4;\Omega_\pm)-S_0(x_2-x_4)]_{\beta\delta}\} \\ & +O(D^2)-(x_3\leftrightarrow x_4,\ \gamma\leftrightarrow\delta)\,, \end{aligned} \tag{27} $$

where we have separated explicitly the terms corresponding to the instanton self-energy corrections to the fermion propagator. This expansion is represented in Fig. 3. We thus deduce that instantons in the DGA generate an effective four-fermion interaction term, given by

$$ \begin{aligned} G_{\alpha\beta,\gamma\delta}(x_1x_2,x_3x_4)= {} & \sum_\pm\int\frac{dx_\pm d\rho_\pm dR_\pm}{(\rho_\pm)^5}D(\rho_\pm)\Delta(m\rho_\pm)[S_\pm(x_1,x_3;\Omega_\pm)-S_0(x_1-x_3)]_{\alpha\gamma} \\ & \times[S_\pm(x_2,x_4;\Omega_\pm)-S_0(x_2-x_4)]_{\beta\delta}-[x_3\leftrightarrow x_4,\ \gamma\leftrightarrow\delta]\,. \end{aligned} \tag{28} $$

Note that this vertex (which incidently is not irreducible) is nonlocal. Thus even though it is an effective four-fermion vertex it will not lead to ultraviolet divergences. Indeed as $x_1\to x_3$ the leading singularities in $S_\pm(x_1,x_3,\Omega_\pm)$ and in $S_0(x_1-x_3)$ will cancel.

If we were to keep all the terms in the expansion of O in the DGA to S_{n_+,n_-}, we would simply find that they are given by the sum of all diagrams generated by the connected vertex G and the fermion propagator S [Eq. (24)]. The higher-order corrections to S_{n_+,n_-} would lead to terms in G proportional to the square of the instanton density. In a similar fashion one can derive effective 6, 8, ..., $2N$, fermion vertices by evaluating in the DGA the vacuum matrix elements of products of 6, 8, ..., $2N$ fermion fields. Thus it is sufficient to calculate the fermion propagator in a background instanton field in order to evaluate the vacuum expectation value of a product of an arbitrary number of quark fields in the DGA. This approximation will be valid as long as the $\int d\rho$ integrations in Eq. (17) are dominated by instantons of size ρ such that $\bar g^2(\rho)/8\pi^2$ is sufficiently small. In principle corrections to the DGA which arise due to the instanton interactions, proportion to $D(\rho)$, as well as higher-order quantum corrections, proportional to $\bar g^2(\rho)/8\pi^2$ can be systematically calculated.

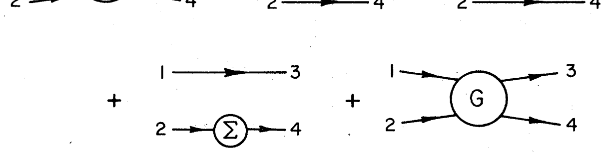

FIG. 3. The four-point function.

III. THE CURRENT TWO-POINT FUNCTION IN THE DGA

In this section we shall calculate the vacuum expectation value of products of hadronic currents in the DGA. In particular we are interested in the two-point function of the electromagnetic current,

$$ \Pi_{\mu\nu}(x-y)=\langle\mathrm{vac}\,|\,T[J_\mu(x)J_\nu(y)]\,|\,\mathrm{vac}\rangle\,, \tag{29} $$

$$ J_\mu=\bar\Psi Q\gamma_\mu\Psi\,. $$

As demonstrated above this can be calculated by summing all Feynman diagrams generated by the fermion propagator and four-fermion interaction as given in Eqs. (24) and (28).

This propagator and vertex can be calculated in terms of the propagator, $S_\pm(x,y,\rho_\pm)$, of a fermion in a single-instanton field, $A_\pm(\rho_\pm)$. This can be written as

$$S_\pm(x,y;\rho_\pm)=\sum_E \frac{\Psi_E(x,\Omega_\pm)\Psi_E^\dagger(y,\Omega_\pm)}{E-m}, \tag{30}$$

where Ψ_E (E) are the eigenfunctions (eigenvalues) of the operator $i\partial\!\!\!/ - A\!\!\!/_\pm(\Omega_\pm)$,

$$\begin{aligned} &iD\!\!\!/\Psi_E(x)=E\Psi_E(x),\\ &\int d^4X\,|\Psi_E(X)|^2=1. \end{aligned} \tag{31}$$

Jackiw and Rebbi[13] have explicitly constructed the solutions to the Dirac equation, Eq. (31), on the five-dimensional sphere, as well as the explicit propagator in the case of $m=0$. A simpler expression for this propagator has been derived by Brown *et al.*[14] using an $O(4)$ formalism. We shall calculate $S_\pm$ in an expansion about $m=0$. This will be useful for discussing the large-momentum behavior of $\Pi_{\mu\nu}$ where terms of order $(m/p)^2$ may be neglected.

Now, when $m=0$ there exists a zero-energy solution to the Dirac equation in the instanton field,

$$(\Psi_0)_{\alpha,i}=\left(\frac{2}{\pi^2}\right)^{1/2}\frac{\rho}{[(x-x_\pm)^2+\rho_\pm^2]^{3/2}} \times\left(i\gamma\cdot\frac{x}{|x|}\gamma_2\frac{1+\gamma_5}{2}\right)_{\alpha,i}, \tag{32}$$

where α (i) is a Dirac (color) index. Separating this term from the $\Psi_{E\neq0}$ eigenfunctions we then expand in powers of m, so that

$$S_\pm(x,y;\Omega_\pm)=-\frac{\Psi_0(x)\Psi_0^\dagger(y)}{m}+\sum_{E\neq0}\frac{\Psi_E(x)\Psi_E^\dagger(y)}{E} +m\sum_{E\neq0}\frac{\Psi_E(x)\Psi_E^\dagger(y)}{E^2}+O(m^2). \tag{33}$$

It turns out that the terms of order m in $S_\pm$ must be kept when calculating the $m\to0$ limit of $\Pi_{\mu\nu}$, since they can combine with the $1/m$ terms to yield a finite result.

Now we can use the explicit expression for

$$S_\pm(x,y,\Omega_\pm)=\sum_{E\neq0}\frac{\Psi_E(x,\Omega)\Psi_E^\dagger(y,\Omega)}{E}, \tag{34}$$

derived from Brown *et al.*, who show that[14]

$$S_\pm(x,y,\Omega_\pm)=\gamma^\mu\vec{D}_\mu\Delta_\pm(x,y,\Omega_\pm)\tfrac{1}{2}(1\pm\gamma_5) +\Delta_\pm(x,y,\Omega_\pm)\overleftarrow{D}_\mu^y\gamma^\mu\tfrac{1}{2}(1\mp\gamma_5), \tag{35}$$

where $D_\mu=\partial/\partial x_\mu - iA_\mu(x-x_i,\rho_i)$ and $\Delta_\pm(x,y,\Omega_\pm)$ is the propagator of a spinless particle (which is a triplet under color). It has the explicit form (as always we work in the singular gauge)

$$\begin{aligned}\Delta_\pm(x,y,\Omega_\pm)=&\frac{1}{4\pi^2(x-y)^2}\left(1+\frac{\rho_\pm^2}{(x-x_\pm)^2}\right)^{-1/2}\\ &\times\left(1+\frac{\rho_\pm^2\sigma_\mp\cdot(x-x_\pm)\sigma_\pm\cdot(y-x_\pm)}{(x-x_\pm)^2(y-x_\pm)^2}\right)\\ &\times\left(1+\frac{\rho_\pm^2}{(y-x_\pm)^2}\right)^{-1/2}\end{aligned} \tag{36}$$

and $(\sigma_\mu)_\pm=(R_{ab}\sigma_b,\mp i)$.

The full propagator, up to terms of order m^2, is then given by

$$S_\pm(x,y;\Omega_\pm)=-\frac{\psi_0(x)\psi_0^\dagger(y)}{m}+S_\pm(x,y,\Omega_\pm) +m\int d^4z\,S_\pm(x,z,\Omega_\pm)S_\pm(z,y,\Omega_\pm). \tag{37}$$

It is instructive to examine the explicit form of $S_\pm$. For an instanton or anti-instanton located at the origin

$$\begin{aligned}S_\pm(x,y;\Omega_\pm)=&S_0(x-y)\left(1+\frac{\rho^2}{x^2}\right)^{-1/2}\left(1+\frac{\rho^2\sigma_\mp x\sigma_\pm y}{x^2y^2}\right)\left(1+\frac{\rho^2}{y^2}\right)^{-1/2}\\ &-\frac{\Delta_0(x-y)}{x^2y^2}\left(1+\frac{\rho^2}{x^2}\right)^{-1/2}\left(\frac{\rho^2}{\rho^2+x^2}\sigma_\mp\cdot x\sigma_\pm\cdot\gamma\sigma_\mp\cdot\Delta\sigma_\pm\cdot y\,\frac{1\pm\gamma_5}{2}\right.\\ &\left.+\frac{\rho^2}{\rho^2+y^2}\sigma_\mp\cdot x\sigma_\pm\cdot\Delta\sigma_\mp\cdot\gamma\sigma_\pm\cdot y\,\frac{1\mp\gamma_5}{2}\right)\left(1+\frac{\rho^2}{y^2}\right)^{-1/2},\end{aligned} \tag{38}$$

where $S_0=-\Delta\!\!\!/ /2\pi^2\Delta^4$, $\Delta_0=1/4\pi^2\Delta^2$, $\Delta=x-y$.

Note that as $\Delta\to0$, $S_\pm$ approaches the free quark propagator S_0. This will guarantee that instantons will not affect the leading short-distance behavior of hadronic currents and that their contribution for large momentum, p, will be suppressed by at least $1/p^2$. Also one easily verifies that $S_\pm-S_0$ vanishes rapidly when the instanton is very far away from x or y, which ensures the convergence of the integral over instanton position when Σ or G is calculated. We also note that $S_\pm$ contains P and T violating terms. These will vanish, however, when we sum over instantons and anti-instantons in a $\theta=0$ vacuum. However if $\theta\neq0$ we must add $S_\pm$ with different phases, $e^{\pm i\theta}$, and the P,T violating terms will survive.

We now proceed to calculate $\Pi_{\mu\nu}$ in the DGA. To lowest order in the density we can use Eq. (18) directly. We must therefore evaluate $\Pi_{\mu\nu}$ in a background instanton field. Clearly,

$$\Pi_{\mu\nu}(x,y,\Omega_\pm)=\langle J_\mu(x)J_\nu(y)\rangle_{A(\Omega_\pm)}=\sum Q_i^{\,2}\Pi^i_{\mu\nu}=-\sum_i \mathrm{Tr}(\gamma_\mu S^i_\pm(x,y,\Omega_\pm)\gamma_\nu S^i_\pm(y,x,\Omega_\pm))Q_i^{\,2}\,, \tag{39}$$

where the sum runs over quark flavors labeled by i, and the trace is over color and Dirac indices. We can calculate $\Pi_{\mu\nu}$ explicitly if we expand S^i in power of the quark mass m_i. In particular we shall calculate $\Pi_{\mu\nu}$ in the limit of vanishing quark masses, in which case the first three terms in the expansion of $S^i_\pm$ will suffice. Owing to the fact that J_μ is a vector current the $1/m^2$ as well as the $1/m$ terms vanish when the traces are performed. This would not have occurred incidentally if we considering scalar currents. In fact it is precisely these terms that give rise to the effective vertex, first discussed by 't Hooft,[10] which has been used to probe for chiral-symmetry breaking in Ref. 2.

Keeping only terms which survive as $m_i\to 0$, we find

$$\Pi^i_{\mu\nu}(x,y;\Omega_\pm)=-\mathrm{Tr}[\gamma_\mu S_\pm(x,y,\Omega_\pm)\gamma_\nu S_\pm(y,x,\Omega_\pm)]+2\,\mathrm{Tr}\left[\gamma_\mu\Psi_0(x)\Psi_0^\dagger(y)\gamma_\nu\int d^4z\,S_\pm(x,z;\Omega_\pm)S_\pm(z,y;\Omega_\pm)\right], \tag{40}$$

the second term arising from the interference of the $O(1/m)$ and $O(m)$ terms in the quark propagator. Using the explicit form of $S_\pm$ the first trace is easily evaluated yielding [we have averaged over instanton group orientations for $G=\mathrm{SU}(3)$]

$$\begin{aligned}-\mathrm{Tr}[\gamma_\mu S_\pm(x,y;\Omega_\pm)\gamma_\nu S_\pm(y,x;\Omega_\pm)]=-\mathrm{Tr}(\gamma_\mu S_0\gamma_\nu S_0)+\frac{3}{2(2\pi^4)}h_xh_y\Big(&S_{\mu\alpha\nu\beta}\frac{\rho_\pm^{\,4}h_xh_y}{\Delta^4}(2\Delta^\alpha\Delta^\beta-g^{\alpha\beta}\Delta^2)\\&+\frac{\rho_\pm^{\,2}}{\Delta^4}(h_y(\Delta^\alpha y^\beta+\Delta^\beta y^\alpha)-h_x(\Delta^\alpha x^\beta+\Delta^\beta x^\alpha))\\&\pm 2\epsilon_{\mu\nu\alpha\beta}\frac{\rho_\pm^{\,2}}{\Delta^4}(h_y\Delta^\alpha y^\beta-h_x\Delta^\beta x^\alpha)\Big)\,,\end{aligned} \tag{41}$$

where $h_x=1/x^2+\rho^2$, $\Delta=x-y$, $\Sigma=x+y$, and $S_{\mu\alpha\nu\beta}=g_{\mu\alpha}g_{\nu\beta}-g_{\mu\nu}g_{\alpha\beta}+g_{\mu\beta}g_{\nu\alpha}$, and we have chosen the instanton to be located at $x_\pm=0$.

The second term can be easily calculated by using the identity (14) $-(\gamma D)^2\frac{1}{2}(1+\gamma_5)=D^2\frac{1}{2}(1+\gamma_5)$, yielding

$$-\frac{3}{2\pi^4}(h_xh_y)^2\frac{\rho^2}{\Delta^2}[(\rho^2+x\cdot y)g_{\mu\nu}+(y_\mu x_\nu-x_\mu y_\nu)\pm\epsilon_{\mu\alpha\nu\beta}x_\alpha y_\beta]\,. \tag{42}$$

Finally we add these contributions and sum over instantons and anti-instantons and subtract $2\langle\Pi_{\mu\nu}\rangle_0$ to obtain

$$\begin{aligned}\delta\Pi_{\mu\nu}&=\Big[\sum_\pm \Pi_{\mu\nu}(x,y,\Omega_\pm)-2\langle\Pi_{\mu\nu}\rangle_0\Big]_{m_i=0}\\&=\sum_i Q_i^{\,2}\frac{3}{4\pi^4}(h_xh_y)^2\rho^2\left(-\frac{2\Sigma^2\Delta_\mu\Delta_\nu}{\Delta^4}+\frac{2\Sigma\cdot\Delta(\Sigma_\mu\Delta_\nu+\Delta_\mu\Sigma_\nu-\Sigma\cdot\Delta g_{\mu\nu})}{\Delta^4}+\frac{2(\Delta^2g_{\mu\nu}-\Delta_\mu\Delta_\nu+\Delta_\mu\Sigma_\nu-\Delta_\nu\Sigma_\mu)}{\Delta^2}\right),\end{aligned} \tag{43}$$

where of course $\langle\Pi_{\mu\nu}\rangle_0=12S^{\mu\nu\alpha\beta}\Delta_\alpha\Delta_\beta/(2\pi^2)^2\Delta^8$, the 3 arising from the color trace. A check on the above calculation is provided by current conservation $\partial^x_\mu\Pi_{\mu\nu}(x,y)=\partial^y_\nu\Pi_{\mu\nu}(x,y)=0$.

Note that this expression vanishes when $\rho=0$, as it must since then the instanton is a pure (singular) gauge. Also note that the light-cone singularity is softer than that present in $(\Pi_{\mu\nu})_0 \underset{\Delta\to 0}{\sim} 1/\Delta^6$. In fact it is suppressed by a factor of Δ^4 as $\Delta\to 0$. It will imply that the instanton corrections will be suppressed by $1/p^4$.[17] Also note that $\delta\Pi^\pm_{\mu\nu}$ behaves as $1/(x_\pm)^6$ for large instanton position $x_\pm$ so that the integration over instanton positions in Eq. (18) will converge. We also observe that it was absolutely necessary to keep the order-m terms in $S_\pm$ in order to obtain the correct zero-mass $\Pi_{\mu\nu}$. If we had dropped these terms the resulting $\Pi_{\mu\nu}$ would not even be conserved. Thus the zero-quark-mass limit must be treated very carefully and one should only set $m=0$ after a complete calculation of gauge-invariant Green's functions.

Our final expression for $\Pi_{\mu\nu}$ in the DGA is then

$$\Pi_{\mu\nu}(x-y)=\sum_i Q_i^{\,2}\left(\Pi^0_{\mu\nu}(x-y)+\int\frac{d^4z\,d\rho}{\rho^5}[D(\rho)\Delta(m_i\rho)\delta\Pi_{\mu\nu}(x,y;z,\rho)+O(m_i^{\,2})]\right). \tag{44}$$

Since $\Pi_{\mu\nu}$ is conserved, it is sufficient to calculate its trace $\Pi(\Delta)=\Pi^\mu_\mu(\Delta)$. We find that

$$\delta\Pi^{\mu}_{\mu}(x,y;z,\rho)=\frac{3}{2\pi^4}(h_{x+z}h_{y+z})^2\frac{\rho^2}{\Delta^4}\{3\Delta^4-2[\Delta(\Sigma+2z)]^2-(\Sigma+2z)^2\Delta^2\}. \tag{45}$$

The integration over instanton position, z, can now be explicitly performed yielding

$$\delta\Pi(\Delta,\rho)=\int d^4z\,\delta\Pi_{\mu\nu}(x,y;z,\rho)=\frac{36}{\pi^2}\frac{\rho^2}{\Delta^2}\rho^2\frac{\partial}{\partial\Delta^2}\left(\frac{1}{\Delta^2}\xi\ln\frac{1+\xi}{1-\xi}\right),\quad \xi^2=\frac{\Delta^2}{\Delta^2+4\rho^2}. \tag{46}$$

In the following section we shall use this expression to discuss the magnitude of the instanton correction to $\Pi_{\mu\nu}$ for large momentum.

IV. e^+e^- ANNIHILATION

In this section we shall discuss some of the theoretical and phenomenological implications of the instanton contributions to the e^+e^- annihilation cross section. First we shall estimate the numerical value of these corrections. We start by calculating the Fourier transform of $\delta\Pi(\Delta,\rho)$:

$$\delta\Pi(P,\rho)=-12\rho^2\left[\frac{1}{(\rho P)^2}-3\int_0^1 dx\,K_2\left(\frac{2\rho P}{(1-x^2)^{1/2}}\right)\right]. \tag{47}$$

We shall be interested in $\delta\Pi(P,\rho)$ for large values of the momentum. In this region we can safely neglect the second term of Eq. (47), which decreases more rapidly then $\exp(-P\rho)$. The error thus made turns out to be less than 10% for the range of momentum we shall consider.

Inserting Eq. (47) into Eq. (44) we calculate the net value of $\Pi(P)$ [as defined in Eq. (1)] including the first two perturbative contributions and the contribution of instantons,

$$\Pi(P^2)\underset{P^2\to\infty}{\approx}\frac{1}{4\pi^2}\sum_i Q_i^{\,2}\left[\ln\left(\frac{P^2}{\mu^2}\right)+\frac{1}{b_0}\ln\left(\frac{g^2}{\bar{g}^2(P^2)}\right)-\frac{16\pi^2}{P^4}\int\frac{d\rho}{\rho^5}D(\rho)\Delta(m_i\rho)\right]. \tag{48}$$

To estimate the instanton contribution we make the following approximations: (a) As discussed in the Appendix the fermion determinant for light quarks suppresses instantons of size $\rho\lesssim\rho_A$, whereas for instantons of size $\rho\gg\rho_A$, $\Delta(m,\rho)\sim 1$. Thus we shall approximate $\Delta(m_i,\rho)$ by $\Theta(\rho-\rho_A)$. (b) For large ρ the DGA surely breaks down. First as the density increases instanton–anti-instanton interactions can no longer be neglected. Also we must include meron configurations, which become important once $x(\rho)=8\pi^2/g^2(\rho)\sim 16$. Thus the contribution of instantons in the DGA approximation must be cut off at some value of $\rho=\rho_c$, and the above integral becomes

$$16\pi^2\int_{\rho_A}^{\rho_c}\frac{d\rho}{\rho^5}\times 0.1\left(\frac{8\pi^2}{g^2(\rho)}\right)^6 e^{-8\pi^2/g^2(\rho)}=16\pi^2\frac{0.1}{11}\int_{x_c}^{x_A}dx\,x^6\exp(-\tfrac{7}{11}x), \tag{49}$$

where we have used the perturbative value for

$$\frac{dx}{d\ln(1/\rho\mu)}=11,$$

neglecting the instanton contributions to the coupling-constant renormalization. To estimate x_c we consider the net fraction of space occupied by instantons and anti-instantons of size less than ρ,

$$f(\rho)=\pi^2\int_0^\rho\frac{d\rho}{\rho}0.1[x(\rho)]^6\exp[-x(\rho)]. \tag{50}$$

When this fraction is of order 1 the DGA surely breaks down. We shall take as an estimate for $x_c=x(\rho_c)$ the value of ρ for which $f(\rho_c)=1$. This yields $x_c\approx 14$, $\rho_c\approx 0.28\mu^{-1}$. Finally since the integrand in Eq. (4a) is a rapidly decreasing function of x, for $x>14$, we can let $x_A\to\infty$. We then obtain for the above integral the value $(8.1\mu)^4$. This value is relatively insensitive to the choice of $x_c=14$, indeed if we had taken $x_c=17$ or 15, we would have obtained the value $(6.6\mu)^4$ or $(7.8\mu)^4$. However, if no cutoff had been imposed, i.e., $x_c=0$, then we obtain $(20.2\mu)^4$.

Let us now compare the instanton contribution Π_I, to the free field theory value Π_0. The ratio of these terms,

$$R_1=\frac{\Pi_1}{\Pi_0}=\left(\frac{8.1\mu}{P}\right)^4\frac{1}{\ln(P^2/\mu^2)}, \tag{51}$$

is given in Table I for various values of P. We have expressed P in units of 4μ, since a reasonable estimate of the hadronic mass scale (M_ρ or M_N) is given by $M_H\sim 4\mu$ (Ref. 2) (i.e., we take $\rho\approx 0.25\mu^{-1}$ corresponding to $x\sim 16$, to be an estimate of the size of a hadron). We note that R_1 is very small until the momentum is of order $2M_H$ and then it increases rapidly. In Table II we compare the instanton contribution Π_I to the second-order perturbation theory contribution Π_1. Here we have made the arbitrary choice for

TABLE I. The ratio of the instanton contribution to the free-field value, $R_1 = \Pi_I/\Pi_0 = (8.1\mu/p)^4/\ln(p^2/\mu^2)$.

R_1	$p/4\mu$	$x(p)$	$\frac{g^2(p)}{8\pi^2}$
7×10^{-6}	20	48	0.021
2.4×10^{-5}	15	45	0.022
1.2×10^{-4}	10	41	0.025
2.4×10^{-3}	5	33	0.03
0.007	4	30.5	0.033
0.025	3	27.3	0.037
0.14	2	22.9	0.044
0.55	1.5	19.7	0.05

the value of $g^2/8\pi^2$ in Eq. (48) to be equal to $e/x_c \sim 0.19$. Again the ratio

$$R_2 = \frac{\Pi_I}{\Pi_1} = \left(\frac{8.1\mu}{P}\right)^4 \left(\tfrac{4}{9}\ln\frac{ex(p)}{x_c}\right)^{-1} \qquad (52)$$

is very small until $P \approx 3M_H$, and then it increases rapidly.

The above estimates are clearly very crude and quite sensitive to the large ρ cutoff. An improved treatment would take into account instanton interactions, the effect of instantons on the coupling-constant renormalization and would treat the mass dependence of the fermion determinant more precisely. The most glaring shortcoming of our calculation is, however, the fact that we have not taken into account merons which presumably are responsible for the dominant vacuum fluctuations on a scale larger than ρ_c. Thus the above calculation, which is reasonable for instantons in the DGA, probably *underestimates* the nonperturbative contributions to $\Pi_{\mu\nu}$.

We have calculated $\Pi_{\mu\nu}$ in the Euclidean region (P^2 spacelike). To compare with experiment it is necessary to continue to the Minkowski region (P^2 timelike), say by expressing $\Pi(p)$ using a dispersion relation in terms of $\sigma_{e^+e^-\to\text{hadrons}}$. Thus the powerlike corrections to $\Pi_{\mu\nu}$ for spacelike momenta would be related to moments of σ. There is however little point in attempting such a comparison at present.

TABLE II. The ratio of the instanton contribution to the second-order perturbation theory contribution, $R_2 = \Pi_I/\Pi_1 = (6.8\mu/p)^4\{\frac{4}{9}\ln[ex(p)/X_c]\}^{-1}$.

R_2	$p/4\mu$	x	$\frac{g^2}{8\pi^2}$
7×10^{-5}	20	48	0.021
2.4×10^{-4}	15	45	0.022
1.2×10^{-3}	10	41	0.025
2.4×10^{-2}	5	33	0.03
6×10^{-2}	4	30.5	0.033
0.19	3	27.3	0.037
0.43	2.5	25.3	0.04
1.1	2	22.9	0.044
4.1	1.5	19.7	0.05

We can however draw some conclusions from our calculation which we believe will not be altered by an improved treatment.

1. The magnitude of the nonperturbative instanton contributions to $\Pi_{\mu\nu}(P)$ is a rapidly varying function of P [$\sim(CM_H/P)^4$, where $C\approx 1$]. These corrections are negligible for $P \gtrsim CM_H$, but increase rapidly for $P \lesssim CM_H$. Since the effective coupling $g^2(P)/8\pi^2$ remains small down to $P\sim CM_H$ one expects that there will be a sharp transition from the region of asymptotic freedom ($P \gtrsim CM_H$), where there are small logarithmic corrections to free field behavior to the region ($P \lesssim CM_H$) where there are large nonperturbative effects. This could explain precocious scaling.

2. The nonperturbative effects become comparable, and then rapidly overwhelm, the perturbative corrections for values of the momentum which correspond to very small couplings. In our calculations we find that this occurs when $g^2/8\pi^2 \sim \frac{1}{20}$. Thus, one cannot trust perturbation theory once the effective coupling, $g^2/8\pi^2$, is of order $\frac{1}{20}$.

Finally we note that the large-momentum behavior of $\Pi_{\mu\nu}(P)$ could be used to calculate meson masses. Migdal[18] has proposed a scheme where one attempts to match the large-Euclidean-momentum behavior of the vacuum expectation values of products of hadronic operators O_J = (say $\bar\psi\gamma_{\mu_1}\not{D}_{\mu_2}\cdots\not{D}_{\mu_J}\psi$) with a sum of meson poles. He has developed efficient techniques for deriving the resulting meson spectrum. This method appears to fail if one only takes into account the perturbative corrections to free field behavior.[15] It would therefore be very interesting to include the nonperturbative corrections in Migdal's scheme. Note that $\langle 0|T[O_J(x)O_J(y)]|0\rangle$ can be calculated in the DGA using exactly the same procedure discussed in this paper.

V. CONCLUSIONS

In this paper we have analyzed the effect of vacuum tunneling (instantons) on the short-distance behavior of hadronic currents within the dilute-gas approximation. In particular we evaluated these corrections for e^+e^- annihilation. We have seen that these nonperturbative effects are substantial and overwhelm the standard perturbative corrections to asymptotic freedom at values of the momentum which correspond to rather small couplings.

There are, however, many problems which remain to be resolved. The dilute-gas approximation is insufficient for a truly quantitative treatment of instanton effects. One must improve upon this approximation taking into account instanton interactions and the contribution of meron configurations. Given such an improved treatment it would then be worthwhile to confront these corrections with experiment, as well as to use them to get a handle on the hadronic spectrum.

In addition one would like to extend this analysis to deal with deep-inelastic scattering, where one directly probes the spacelike region and where the experimental information is so rich. The analysis of the instanton corrections to deep-inelastic scattering, however, is much more involved than that of e^+e^- annihilation.

In the standard analysis of deep-inelastic scattering one utilizes in addition to the renormalization group plus perturbation theory the existence of an operator-produce expansion. This expansion, however, appears to break down once tunneling is taken into account. This is perhaps to be expected if one recalls the intuitive physical motivation for a local operator-product expansion. Namely the product $A(x)B(y)$ of local operators for values of $|x-y|$ much less than the characteristic length scale of a local theory should be indistinguishable from a local operator. Thus, it is quite plausible, and can be proved to be true to all orders in perturbation theory, that an asymptotic expansion exists as $|x-y|\to 0$,

$$A(x)B(y) \underset{|x-y|\approx 0}{\approx} \sum_n C_n(x-y)O_n\left(\frac{x+y}{2}\right). \tag{53}$$

The Wilson coefficients C_n are c-number functions whose $x-y$ dependence can be determined by use of the renormalization group and O_n are a complete set of local operators.

In the presence of tunneling the above intuitive picture breaks down. Given a finite density of instantons and a finite $|x-y|$ there is always a nonvanishing probability that a tunneling event (instanton) occurs in between the space-time events x and y. Therefore, there is a clean distinction between the effect of a perturbation on the system by means of a local operator $A(x,t_1)$ followed by a perturbation by means of $B(x,t_2)$ and a perturbation by means of a local operator

$$O\left(x, \frac{t_1+t_2}{2}\right)$$

no matter how small t_1-t_2 is, as long as one can tunnel between two degenerate vacuums in the time interval t_1-t_2. To be sure as $t_1-t_2\to 0$ this requires instantons of sizes $\rho \lesssim t_1-t_2$ and thus the vanishing density of these instantons as $\rho\to 0$ ensures that such events will be rare. However, we have reason to expect that such tunneling events will invalidate a local operator-produce expansion.

Indeed if we analyze deep-inelastic scattering in the DGA, using the effective *nonlocal* Lagrangian which summarizes the net contribution of instantons, we find in addition to nonperturbative corrections to the Wilson coefficients of the standard operator-product expansion new contributions that appear not to be describable in terms of local operators. To be sure we can determine the q^2 dependence of the instanton corrections $(1/q^2)$, however, it is not clear that the coefficient of such corrections can be estimated in the absence of explicit hadronic wave functions. This problem is currently under investigation.

Notes added in proof. After submission of this paper we received a paper by L. Balieu, J. Ellis, M. K. Gaillard, and W. J. Zakrzewski, CERN Report No. TH-2482, 1978 (unpublished) dealing with the same subject. They evaluate $\delta\Pi_{\mu\nu}$ due to a single instanton, obtaining a result identical to ours. (We would like to thank them for bringing to our attention certain typographical errors in our original manuscript.) They integrate over the dilute gas instanton density, with small bare quark masses, without taking into account dynamical mass generation. They then attempt to deduce prediction for the behavior of $\delta\Pi_{\mu\nu}$, and thereby δR, in the timelike Minkowski region. Since the first term in Eq. (47) has no cut, and the cut of the second term is dominated by $p\approx 0$, they conclude that $\mathrm{Im}\delta\Pi_{\mu\nu}$ is calculable as $Q^2\to\infty$, falling rapidly to zero like $Q^{-11-N_f/3}$ (up to logarithmic corrections, where N_f is the number of quark flavors). We disagree completely with this analysis for the following reasons:

1. The DGA cannot be used to discuss the analytic properties of Green's functions. In this approximation one does not obtain confinement and the elimination of the zero-mass states. Thus, for massless quarks, $\delta\Pi_{\mu\nu}\sim 1/p^2+\cdots$ and contains a pole at $p^2=0$. An improved treatment will surely remove this pole from the physical spectrum replacing it with a cut at $p^2=4M_\pi^2$. However, this requires control over the infrared dynamics of the theory, which is beyond the domain of the DGA.

2. Even within the spirit of using the DGA, there are corrections to Eq. (47) arising from perturbative gluon corrections of order $(\ln p^2)^n/p^2$ which have cuts at $p^2=0$. Furthermore the appearance of $1/p^2$ in Eq. (47) is due to the fact that one has expanded about zero mass in the quark propagators. If one keeps the mass finite in these propagators then the pole at $p^2=0$ in $\delta\Pi_{\mu\nu}$ is replaced by a square-root branch point at $p^2=4M^2$. This is irrelevant for large spacelike p^2, however, it

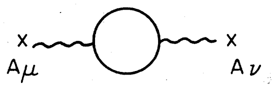

FIG. 4. The second-order-in-A contribution to the fermion determinant.

clearly affects the discontinuity of $\delta\Pi_{\mu\nu}$ for timelike p^2. Another way of saying this is that mass corrections to $\delta\Pi_{\mu\nu}$, of order $(m^2)^N/(p^2)^{N+1}$, can only be neglected as $p^2\to\infty$ in the spacelike region.

Thus, even within the spirit of the DGA the correct conclusion would be that

$$\frac{\Delta R}{R} \underset{p^2\to\infty}{\sim} \frac{c}{p^4},$$

where c is a constant which cannot be calculated without making assumptions regarding the instanton density.

ACKNOWLEDGMENTS

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APPENDIX: THE MASS DEPENDENCE OF THE FERMION DETERMINANT

In calculating the density of instantons in the DGA one must evaluate the determinant of the operator $(i\partial\!\!\!/ + A\!\!\!/ - m)$ for a background instanton field A_μ

$$\mathrm{Det}(i\partial\!\!\!/ + A\!\!\!/ - m) = \exp[\mathrm{Tr}\ln(i\partial\!\!\!/ + A\!\!\!/ - m)]. \qquad (A1)$$

Here we shall discuss the mass dependence of the determinant.

The determinant whose logarithm is simply the sum of all connected one-loop fermion diagrams in the background field, is divergent. The divergence is canceled by the appropriate coupling-constant counterterm. The renormalized determinant is then a dimensionless function of m/μ and $\rho\mu$ where μ is the renormalization scale parameter and ρ the instanton size.

This determinant has been evaluated explicitly in the limit of zero mass by 't Hooft.[10] Using Pauli-Villars regulators he derived

$$\Delta(m/\mu,\rho\mu) \underset{m\to 0}{\simeq} 1.338\,\rho m \exp(-\tfrac{2}{3}\ln\rho\mu). \qquad (A2)$$

The factor of $m\rho$ arises due to the existence of the zero-energy mode for vanishing fermion mass in a background gauge field of topological quantum number one. The factor $-\frac{2}{3}\ln\rho\mu$ is exactly the correct magnitude to combine with the gauge field determinant to yield, after renormalization the value $8\pi^2/g^2(\rho\mu)$ for the instanton action where $g^2(\rho\mu)$ is the effective coupling including the effects of renormalization due to fermion loops. Both of these factors are thus derivable from general principles. The remaining part of the determinant (1.338) requires explicit calculation.

On the other hand in the limit of large masses, $m\to\infty$, we expect that the fermions should decouple completely. Indeed the decoupling theorems[16] imply that as $m\to\infty$ the total effect of the fermions (to order $1/m^2$) in the theory can be summarized by a coupling-constant renormalization. We shall show below that, after making the appropriate coupling-constant renormalization,

$$\Delta(m/\mu,\rho\mu) \underset{m\to\infty}{\sim} 1-\frac{0.16}{(m\rho)^2}. \qquad (A3)$$

To illustrate the above consider the contribution to $\ln\Delta$ of the diagram in Fig. 4 which is

$$\int\frac{d^4P}{(2\pi)^4}A_\mu(P)\Pi_{\mu\nu}(P)A_\nu(-P),$$

where $A_\mu(P)$ is the Fourier transform of the instanton field and $\Pi_{\mu\nu}(P)$ is the standard vacuum polarization tensor

$$\Pi_{\mu\nu}(P)=\frac{1}{2\pi^2}(P_\mu P_\nu - g_{\mu\nu}P^2) \times\left[\tfrac{1}{6}\ln\frac{4\pi\mu^2}{\Lambda^2}-\int_0^1 d\alpha\,\alpha(1-\alpha) \times\ln\left(\frac{m^2+P^2\alpha(1-\alpha)}{4\pi\mu^2}\right)\right]. \qquad (A4)$$

The logarithmic divergence is removed by the coupling-constant renormalization. In other words, the gauge field action

$$\frac{1}{g_0^{\,2}}\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2(x) = \frac{1}{g_0^{\,2}}\int A_\mu(P)(P_\mu P_\nu - g_{\mu\nu}P^2) \times A_\nu(-P)\frac{d^4P}{(2\pi)^4}+O(A^3)$$

contains a piece precisely equal to the above pole term.

Thus after renormalization this diagram contributes to $\ln\Delta$ the term

$$\ln\Delta=\int\frac{d^4P}{(2\pi)^4}A_\mu(P)A_\nu(-P)(g_{\mu\nu}P^2-P_\mu P_\nu) \times\frac{1}{2\pi^2}\int_0^1 d\alpha\,\alpha(\alpha-1)\ln\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2} +\cdots. \qquad (A5)$$

When $m\to 0$ the integral can be evaluated and will yield $a\ln\rho\mu + b$. The $a\ln\rho\mu$ term, together with similar terms from the diagrams in Fig. 5, yields

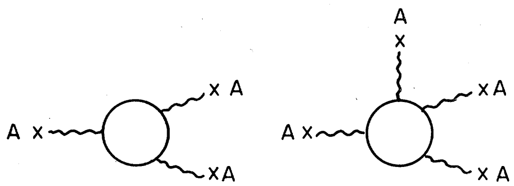

FIG. 5. Other divergent contributions.

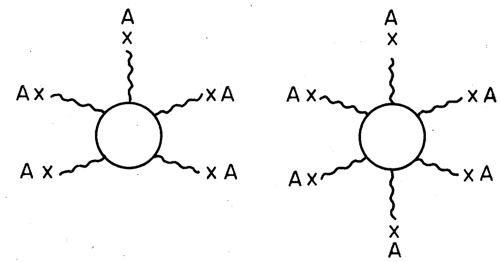

FIG. 6. Convergent contributions to the fermion determinant.

the $-\frac{2}{3}\ln\rho\mu$ in Eq. (A2). These are the only diagrams that diverge as $\rho\to 0$ [note that $A_\mu(P)\sim(1/P^3)_{P\to\infty}\exp(-\rho P)$] and indeed the only diagrams that depend on $\rho\mu$.

For large mass, however,

$$\Pi^{\rm ren}_{\mu\nu}(P) \underset{m\to\infty}{\sim} (g_{\mu\nu}P^2 - P_\mu P_\nu) \times\left[\frac{1}{12\pi^2}\ln\frac{m^2}{4\pi\mu^2}+O\left(\frac{P^2}{m^2}\right)\right]\ldots .$$

Thus, in the limit of large mass this terms, as well as similar terms from the divergent graphs in Fig. 5, will yield

$$\ln\Delta\sim\frac{1}{12\pi^2}\int {\rm Tr}F_{\mu\nu}{}^2(x)\,dx\,\ln\frac{m}{\mu}+O\left(\frac{1}{m^2\rho^2}\right). \tag{A6}$$

The $1/m^2\rho^2$ terms arise from the $O(P^2/m^2)$ terms in the divergent graphs as well as from all the convergent graphs in Fig. 6. It is clear from dimensional analysis that the coefficient of $\ln(m/\mu)$ must be the gauge field action, and thus this term could be absorbed in a finite renormalization of the coupling. We shall indeed perform this finite renormalization by adding a term,

$$\frac{1}{12\pi^2}\ln\left(1+\frac{m^2}{4\pi\mu^2}\right),$$

to $8\pi^2/g^2$ that does not affect the zero-mass theory.

In that case, $\ln\Delta\sim_{m\to\infty}O(1/m^2\rho^2)$, and we shall now proceed to evaluate the coefficient of the $1/m^2$ term. This is easily done by recognizing that the coefficient of $1/m^2$ must be the integral of a local gauge-invariant operator, constructed out of the background field, of dimension six. In other words, when we pull out factors of $1/m$ from the appropriate Feynman diagrams the remaining m-independent contribution arises from large internal momentum compared to the momentum carried by the external field. Thus,

$$\ln\Delta=\left(a\ln\frac{\Lambda^2}{\mu^2}+b\ln\frac{m}{\mu}\right)\int d^4x\,{\rm Tr}F_{\mu\nu}{}^2 + \sum_{i=0}^{\infty}\left(\frac{1}{m^2}\right)\sum_j^i \quad d^4x\,O_i^{(j)}[A_\mu(x)], \tag{A7}$$

where $O_i^{(j)}$ are gauge-invariant local operators of dimension $4+2i$.

The coefficient of $1/m^2$ is expressible in terms of dimension-six operators. It would seem that there are three such operators, ${\rm Tr}\{D_\mu F_{\alpha\beta}D_\alpha F_{\mu\beta}\}$, ${\rm Tr}\{D_\alpha F_{\mu\nu}D_\alpha F_{\mu\nu}\}$, and ${\rm Tr}\{F_{\alpha\beta}F_{\beta\gamma}F_{\gamma\alpha}\}$. However, when integrated over space only one of these is independent if one recalls that $D^\mu F_{\mu\nu}=0$ (since the background instanton field is a solution of the equations of motion) and that $[D_\mu,D_\nu]=F_{\mu\nu}$. Thus the $1/m^2$ term in $\ln\Delta$ is given by $(c/m^2)\times\int d^4x\,{\rm Tr}(D_\mu F_{\alpha\beta}D_\alpha F_{\mu\beta})$. The value of the constant c is then easy to calculate in terms of the $1/m^2$ term generated by the Feynman diagram in Fig. 4,

$$-\int A_\mu(P)(g_{\mu\nu}P^2-P_\mu P_\nu)\,\frac{P^2}{30m^2}\times A_\nu(-P)\frac{d^4P}{(2\pi)^4}+O(A^3)$$
$$=-\frac{c}{m^2}\int A_\mu(P)(g_{\mu\nu}P^2-P_\mu P_\nu)\times A_\nu(-P)P^2\frac{d^4P}{(2\pi)^4}+O(A^3). \tag{A8}$$

We conclude that $c=1/60\pi^2$. The $1/m^2$ term in $\ln\Delta$ is given by

$$\frac{1}{60\pi^2m^2}\int{\rm Tr}\{D_\mu F_{\alpha\beta}D_\alpha F_{\mu\beta}\},$$

which for an instanton field is equal to $-4/25(m\rho)^2$. Therefore, after renormalization

$$\Delta\underset{m\rho\to\infty}{\sim}\exp\left(-\frac{4}{25}\frac{1}{(m\rho)^2}\right)\cong 1-\frac{4}{25}\frac{1}{(m\rho)^2}. \tag{A9}$$

We conclude that the net affect of a fermion of mass m is to substantially supress the contribution of instantons of size $\rho\ll 1/m$. On the other hand, for instantons of size $\rho\approx 1/m$ the fermion contribution to the density of instantons can be ignored.

Until now we have treated m as a constant parameter. This is true of the quark-mass parameter, i.e., the explicit $m\bar\psi\psi$ term in the Lagrangian. However, there can also be a momentum-dependent dynamically generated mass. Indeed we believe that the light quarks (whose mass parameters are very small ~10 MeV) derive their mass from the dynamical chiral-symmetry-breaking mechanism discussed in Pef. 2. How do we interpret Eqs. (A2) and (A9) when m is momentum dependent, i.e., when the quark propagator, in the true chirally noninvariant vacuum, is $1/[\not P+m(P)]$? Since the background instanton field falls off rapidly for $P \gg 1/\rho$ the dominant contributions to $\ln \Delta$ are from quarks of momentum $\sim 1/\rho$. Thus crudely speaking, m in Eqs. (A2) and (A9) should be replaced by $m(1/\rho)$.

According to the picture described in Ref. 2 we expect that the light quark dynamical masses are strongly momentum dependent, vanishing rapidly for large momentum and conversely becoming very large as the momentum is reduced. The precise form of $m(1/\rho)$ requires control over the chirally asymmetric vacuum, which is not yet available in the case of two or more light quarks. However a reasonable approximation, given the above picture, is to say that

$$\Delta\left(\frac{m(\rho^{-1})}{\mu}, \rho\mu\right) \simeq 0, \quad \text{for } \rho < \rho_A$$

where $m(\rho^{-1}) \ll 1$, and

$$\Delta\left(\frac{m(\rho^{-1})}{\mu}, \rho\mu\right) \simeq 1, \quad \text{for } \rho > \rho_A$$

where $m(\rho^{-1}) \approx 1$. Thus in this approximation the effect of the light quark is simply to suppress instantosn of size less than ρ_A, and for instantons of size greater than ρ_A the effect of the quarks on the density of instantons may be neglected. We may then replace $D(\rho\mu)\,\Delta((m/\mu)\,\rho\mu)$ by $D(\rho\mu)\theta(\rho-\rho_A)$ for the purpose of estimating, in the DGA, the density of instantons.

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