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Real-photon spectral weight functions, imaginary part of vacuum polarization, and electromagnetic vertices

Charles Chahine*

Department of Physics, Brandeis University, Waltham, Massachusetts 02154

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We introduce the concept of a real-photon spectral weight function for any cross section involving charged particles as a simple approximation taking into account the soft part of photon emission to all orders in perturbation theory. The spectral weight function replaces the energy-momentum conservation δ function in the elastic cross section. The spectral weight function is computed in closed form in space time and in the peaking approximation in momentum space. We apply the spectral weight function description to the imaginary part of vacuum polarization Im Π and to electron-proton scattering. We derive a spectral representation for Im Π and compare its content with the known fourth-order result, showing in particular the identity of the soft and peaking approximations in lowest order. The virtual-photon radiative corrections are discussed in part, with emphasis on the threshold behavior of the vertex functions. A relativistic generalization of the electric nonrelativistic vertex function is given, whose asymptotic behavior is appropriate to use in conjunction with the spectral weight function.

I. INTRODUCTION

In recent years, much effort has been devoted to building coherent spaces on which Feynman amplitudes[1] are free from infrared divergences. Recently,[2] a definite example, the scattering of an electron by a weak external potential, has been worked out using a modified Lehmann-Symanzik-Zimmermann (LSZ) formalism. The result obtained, when expanded in perturbation theory, agrees with Schwinger's[3] result. We must note, however, that in most of these approaches real and virtual photons are not treated on the same footing: Real soft photons (defined with respect to a cutoff) are included to all orders, whereas virtual-photon radiative corrections are included up to some order. The consistency of the procedure when nonperturbative effects are expected to be important remains unclear. Viewed in the context of quantum field theory, the early work of Bloch and Nordsieck[4] demonstrates nonperturbatively the infrared cancellation between soft-photon emission and virtual radiative corrections. The prevalent field-theory treatment of infrared divergences to all orders rests on the work of Yennie, Frautschi, and Suura[5,6] (YFS) and Jauch and Rohrlich.[7] As is well known, the prominent result is the exponentiation in the cross section of the lowest-order infrared divergence when the emission of an arbitrary number of soft photons is included, and also in the amplitude when radiative corrections are included to all orders. Once this result is established, the cancellation of the infrared divergence in the "inclusive" cross section (defined as the cross section which includes the emission of an arbitrary number of soft photons) is the same as in lowest-order perturbation theory. One is left with an infrared-finite cross section with nonperturbative effects coming from both real and virtual photons.

The so-called exponentiation of infrared divergence is rigorous only in the limit where all photon momenta go to zero. To extract physical information from this favorable circumstance, one separates hard and soft contributions (defined with respect to a noncovariant cutoff), sums up the soft parts, and computes the hard contributions up to some order in perturbation theory. The cutoff is then related to the energy resolution of the experimental apparatus. Taking into account globally the contribution of real soft photons in this context has been possible until now only in potential scattering[5,6] where only energy resolution is effective, although the analysis of more complicated situations has been attempted.[8]

The first purpose of the present article is to present a simple and attractive picture in which the whole effect of soft-photon emission is described by a real-photon spectral weight function. This function replaces the energy-momentum conservation δ function in the elastic cross-section formula and contains as a factor $\lambda^{\alpha A}$, where λ is the photon mass regulator, α is the fine-structure constant, and $A \equiv 2\overline{A}$ is a conventional function which depends on the momenta of the charged particles. Since we do not use a cutoff on photon energy, we must say what we mean by soft photon. In fact, we define the *soft part* of a cross section as the contribution in which all matrix elements are approximated by their soft limit (all photon momenta go to zero). We note that the soft part receives contributions from photon momenta which are large as well as soft. The advantage of this definition is to avoid the introduction of a cutoff. The hard part of a cross section will be the difference between the exact and the soft contributions. In this paper, we shall not attempt (except in fourth order) to derive systematically the hard parts, which can be done following the Grammer-Yennie[6] analysis. The role of the spectral weight function is to sum up in closed from the soft contribution to any cross section.

The simplification brought by soft photons is the fact that they are emitted independently with the provision that they satisfy energy-momentum conservation. They are characterized by an individual relative emission probability density which is $-\alpha(2\pi^2)^{-1}|j|^2\delta_+(k^2-\lambda^2)$ ($|j|^2=j^\mu j_\mu<0$) for an emitted photon of four-momentum k, where j^μ is the classical spin-zero current

$$j^\mu=\frac{p'^\mu}{k\cdot p'}-\frac{p^\mu}{k\cdot p}\,. \tag{1.1}$$

p and p' are the momenta of the charged particle before and after scattering. The important feature of the soft-photon probability density is the well-known peaking of the emitted photon along p or p' directions and the dk/k spectrum which leads to the infrared singularities.

The real-photon spectral weight function, which takes the place of the energy-momentum conservation δ function, will be called $E_\lambda(p_i;K)$, where p_i are the momenta of the charged particles and K is the effective photon momentum. K is, of course, the argument of the original δ function. We shall be able to compute in closed form $E_\lambda(p_i;x)$, the Fourier transform with respect to K of $E_\lambda(p_i;K)$. Since $E_\lambda(p_i;x)$ involves an exponential of Spence functions with complicated arguments, it is unlikely that one can compute the exact spectral weight function $E_\lambda(p_i;K)$. However, the frame-dependent *photon energy* spectral weight function $E_\lambda(p_i;K_0)$, defined as the three-momentum integral of $E_\lambda(p_i;K)$, will be computed in closed form. The use of $E_\lambda(p_i;K_0)$, is equivalent to the standard approach in potential scattering.

The function $E_\lambda(p_i;x)$ obeys the scaling equation $E_\lambda(p_i;\rho x)=\rho^{-\alpha A}E_\lambda(p_i;x)$ with $\rho>0$ and, consequently, $E_\lambda(p_i;\rho K)=\rho^{\alpha A-4}E_\lambda(p_i;K)$. The origin of this nonperturbative property (E_λ sums up the infrared "lns" to a power) is the known dependence of E_λ on λ and the lack of a cutoff. Moreover, $E_\lambda(p,p;K)=1$, corresponding to the fact that an unaccelerated charged particle cannot radiate. We thus are led to introduce an approximation to the spectral weight function which retains the important properties we just described and incorporates the expected peaking of the emitted photons along the directions of the charged particles. This will be called the peaking approximation, the corresponding spectral weight function being denoted $E_{1\lambda}(p_i;K)$. Going from the soft (exact) to the peaking approximation amounts to the replacement

$$\frac{-\alpha}{2\pi^2}|j|^2\delta_+(k^2-\lambda^2)\to\alpha I_1(p,p';k)$$
$$=\alpha\bar{A}\int_{\lambda/2m}^{\infty}\frac{d\sigma}{\sigma}\left[\delta^4(k-\sigma l)+\delta^4(k-\sigma l')\right],$$

which yields (1.2)

$$\begin{aligned}E_\lambda(p,p';K)\to E_{1\lambda}(p,p';K)&=\left(\frac{\lambda^2e^{2\gamma}}{4m^2}\right)^{-\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})\int_0^\infty d\sigma\,d\sigma'(\sigma\sigma')^{\alpha\bar{A}-1}\times\delta^4(k-\sigma l-\sigma' l)\\&=\left(\frac{\lambda^2e^{2\gamma}l\cdot l'}{2m^2}\right)^{-\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})\,\frac{\delta^2(K_T)\theta(K^2)\theta(K_0)}{(K^2)^{1-\alpha\bar{A}}}\,,\end{aligned} \tag{1.3}$$

where the second form is valid in the Breit frame. Here l and l' are light-cone momenta, $l^2=l'^2=0$ with $l_0,l_0'>0$ which are linear combinations of p and p', γ and Γ are Euler's constant and function, respectively, and m is the electron mass. The normalization of the scalar product $l\cdot l'$ is related to the part of the infrared cross section which is assumed to exponentiate. We have determined this normalization in such a way that the photon energy spectral functions $E_\lambda(p,p';K_0)$ and $E_{1\lambda}(p,p';K_0)$ *coincide in the* Breit (c.m.) for p-p' spacelike (timelike).

The second point we shall discuss is the relevance of the peaking approximation to the actual physical problems involving radiative corrections. In this article, we shall exhibit the general formulas for two processes, the imaginary part of vacuum polarization ImΠ and the electron-proton scattering cross section in the one-photon exchange approximation. We will discuss very carefully ImΠ to get a precise idea of the usefulness and limitations of the peaking approximation, leaving the detailed discussion of radiative corrections in $e+p$ scattering to a forthcoming article.[9] Using the spectral weight function E_1, we derive a nonperturbative spectral form ImΠ_1, which is a functional of the electromagnetic vertices F_1 and F_2. The function ImΠ_1 contains the whole contribution of the (e^+e^-) intermediate state to all orders in perturbation theory with the infrared correlated arbitrary number of photons emitted according to the peaking approximation. We proceed to check this formula by expanding up to fourth order using the known expressions of the vertices up to second order. We compute also the hard part of ImΠ to order e^4. The remarkable fact is that with the choice of the normalization of $l\cdot l'$ made as explained above, the soft and peaking approximations give identical results up to order e^4. Adding the soft and hard parts, we reproduce the Källén-

Sabry[10,11] formula. We conclude from this computation that the hard part is very small near threshold and behaves as $\ln s$ at high energy, the soft or peaking contribution behaving as $\ln^2 s$ while the total contribution is known[12,13] to be constant.

In the third place, we discuss the nonperturbative formula for ImΠ with emphasis on the threshold behavior where the formula is expected to become exact. We point out that the use of the usual YFS $\exp(\alpha B)$ factor[5,6] which contains the correct λ dependence and the exponentiated leading $\ln^2 q^2$ terms found by many authors[14] is incompatible with the spectral representation of ImΠ and has to be modified somewhat. The reason is that $\exp(\alpha B)$ exponentiates the lowest-order threshold singularity $\alpha/4v$, v being the relative volocity. This difficulty is solved by emphasizing the nonrelativistic (NR) nature of the vertex function F_0 near threshold which can be obtained to all orders by solving the Schrödinger equation for a truncated Coulomb potential. The function F_0 has poles corresponding to the NR positronium bound states. When expanded in perturbation in the domain $v \ll 1$ and $\alpha/v \ll 1$, it coincides with the known terms of the threshold expansion of the electric vertex function up to fourth order.[15] Our analysis suggests the following generalization of the NR result which possesses the expected virtues

$$F_e = F_1 + \frac{\alpha}{2\pi} F_2 = e^{\alpha \bar{A}_1 \gamma}\Gamma(1+\alpha \bar{A}_1) e^{\alpha(B+1/2\pi)}, \quad (1.4)$$

where $\bar{A}_1$ is the analytic function whose real part is $\bar{A}$. We note that apart from the factor $\exp(\alpha/2\pi)$, whose origin is the normalization at zero momentum transfer, Eq. (1.4) begins to differ from $\exp(\alpha B)$ only at order α^2 and leaves the high-energy behavior essentially unmodified. A similar form is proposed for F_2. The limitations of the peaking approximation are also discussed in relation to the real part of vacuum polarization.

This article is organized as follows. In Sec. II, we introduce the general description of soft-photon emission by a spectral weight function beginning with ImΠ, discussing similarly $e+p$ scattering and generalizing the result for any cross section. Section III is first devoted to the detailed computation of the spectral weight function $E_\lambda(p_i;x)$. We then pause to derive the photon-energy spectral weight function $E_\lambda(p_i;K_0)$, thus making contact with YFS results. We then proceed to introduce the peaking approximation, derive the general spectral weight function $E_{1\lambda}(p_i;K)$, and fix the $l\cdot l'$ normalization. In Sec. IV, we first derive the nonperturbative formula for ImΠ in the peaking approximation. This formula is expanded up to fourth order [this is equivalent to using $\alpha I_1(k)$ of Eq. (1.2)]; the hard contribution is identified and computed. We then derive the NR form of the vertex F_e, the exact threshold value of ImΠ, and the relativistic generalizations for F_e and F_2. Finally, a formal expression for $\Pi_1(0)$ is derived which shows the limitation of the peaking approximation. The $e+p$ radiative corrections, which are started in this paper, are discussed in the context of the peaking approximation in a forthcoming article.[9]

II. SOFT-PHOTON EMISSION EFFECT ON A CROSS SECTION

We shall demonstrate how a cross section is modified when an arbitrary number of real soft photons is taken into account. The basic result we shall establish is that the whole effect of taking into account the soft photons is to "broaden" the energy-momentum conservation $\delta^{(4)}$ function which appears in the cross section. This broadened function we shall call spectral weight function. As a first typical example, we establish how the soft photons modify the electron-positron pair intermediate-state contribution to the imaginary part of vacuum polarization (ImΠ) in QED. Owing to unitarity, ImΠ behaves as an inclusive cross section for timelike momentum q. We discuss next soft-photon emission in electron-proton scattering (including the deep-inelastic region) in the one-photon exchange approximation with a spacelike momentum transfer q. Our discussion incorporates the Schwinger[3] problem as a limiting case in which the electromagnetic potential created by the "heavy" nucleon is considered as static. We finally give the spectral weight function in the general case of scattering which involves any number of charged particles.

A. Imaginary part of vacuum polarization including soft photons

We denote by $\Pi_{\mu\nu}(q)$ the proper vacuum polarization tensor in QED. From Lorentz and gauge invariance, its general form is

$$\Pi_{\mu\nu} = (-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi(q^2). \quad (2.1)$$

The imaginary part of $\Pi(q^2)$ is given by the unitarity relation

$$\text{Im}\Pi(q^2) = \frac{1}{3q^2}\sum_{p^{(n)}=q} |\langle 0|J_\mu(0)|n\rangle|^2, \quad (2.2)$$

where $\langle 0|J_\mu(0)|n\rangle$ is the proper matrix element of the current between the vacuum and the state $|n\rangle$ of four-momentum $p^{(n)}$. If we want to use Eq. (2.2) nonperturbatively, then we must include with every intermediate state $|n\rangle$ containing e^+e^- pairs an arbitrary number of soft real photons. This is so because the matrix elements are infrared divergent due to radiative corrections and the infrared cancellations occur between real and virtual photons. We shall concentrate here on the contributions to Eq. (2.2) of the states $|n\rangle$, which contain, in addition to one e^+e^- pair, an arbitrary number of soft photons. Later, it will be clear how the present approach could be generalized to take into account the contribution of intermediate states with more pairs and an arbitrary number of soft photons. By writing down explicitly the space summations and integrations implicit in Eq. (2.2), one has

$$\mathrm{Im}\Pi(q^2)=\frac{\pi}{3q^2}\sum_{n=0}\int\frac{d^4p_+d^4p'}{(2\pi)^3n!}\,\delta(p_+{}^2-m^2)\theta(+p_0)\delta(p'^2-m^2)\theta(p_0')\prod_{i=1}^{n}\frac{d^4k_i\delta(k_i{}^2-\lambda^2)\theta(k_{i0})}{(2\pi)^3}$$
$$\times\sum_{s,s'}\sum_{\text{polarizations}}|\langle 0|J_\mu(0)|p_+,p';k_1\cdots k_n\rangle|^2\delta^4(q-p'-p_+-\textstyle\sum k_i)\,, \qquad (2.3)$$

where p', s' and p_+, s are respectively, the momenta and spins of the electron and positron, k_i, ϵ_i are the momenta and polarizations of the photons, and m is the electron mass. We have given the photon a mass λ since, although $\Pi(q^2)$ is infrared finite, the separate terms of the right-hand side of Eq. (2.3) are infrared divergent. In fourth-order perturbation theory, the infrared finiteness of $\Pi(q^2)$ has been shown by Jost and Luttinger,[16] who computed the relevant Feynman graphs and by Källén and Sabry,[10] who used Eq. (2.2). The cancellation of infrared divergences in the right-hand side of Eq. (2.3) to all orders in perturbation theory entails the general proof given by Jauch and Rohrlich[7] and by Yennie, Frautschi, and Suura[5,6] (YFS).

The basic result which permits the treatment of the infrared divergence to all orders in perturbation theory is that one knows the leading singularities of the matrix elements for the emission of an arbitrary number of soft photons in terms of the matrix element of the same process without photon emission. Explicitly, one has

$$\langle 0|J_\mu(0)|p_+,p';k_1\cdots k_n\rangle \underset{k_i\to 0}{\sim} \langle 0|J_\mu(0)|p_+,p'\rangle e^n \times\epsilon_1\cdot j(k_1)\cdots\epsilon_n\cdot j(k_n)\,, \qquad (2.4)$$

where e is the electron charge and $j^\mu(k)$ is the classical current which is given by

$$j^\mu(k)=\frac{p'^\mu}{k\cdot p'}-\frac{p_+^\mu}{k\cdot p_+}\,. \qquad (2.5)$$

Equation (2.4) expresses the fact that soft photons are dynamically independent.

Let us define the "soft part" of $\mathrm{Im}\Pi$ as the contribution to Eq. (2.3) which uses (2.4) for all on-mass-shell values of the k_i. Since we do not introduce any cutoff on photon energies, the "soft part" of $\mathrm{Im}\Pi$ clearly receives contributions from nonsoft photons. The remainder of $\mathrm{Im}\Pi$, which we call the hard part, is infrared finite and can be computed term by term in perturbation theory. Up to fourth order in e, the soft and hard parts will be discussed in Sec. IV, where the former is shown to dominate at high energy. The splitting of $\mathrm{Im}\Pi$ into hard and soft parts, which we just introduced, is convenient but not unique. Conventionally, for example,[3,5] hard and soft parts are defined using a cutoff on photon energies, the cutoff being related to the experimental resolution of the detection apparatus. In fact, as long as one discusses the "inclusive cross section" $\mathrm{Im}\Pi$, no cutoff is necessary.

Let us now demonstrate that the soft part of $\mathrm{Im}\Pi$ exponentiates and involves a spectral weight function. The sum over photon polarization is done using the relation

$$\sum_{\text{polarizations}}(\epsilon\cdot j)^2=-j^\mu j_\mu\,. \qquad (2.6)$$

In principle, since the photon is considered as massive, the polarization sum is in fact $-(j)^2+(k\cdot j)/\lambda^2$ and $k\cdot j$ is not identically zero since the current is no longer conserved. The modified current of Eq. (2.5) is

$$j^\mu=p'^\mu(k\cdot p'+\lambda^2/2)^{-1}-p_+^\mu(k\cdot p_++\lambda^2/2)^{-1}\,.$$

However, it is known[11] that $k\cdot j$ is proportional to λ^2 and the longitudinal polarization does not contribute in the limit $\lambda\to 0$. One can also neglect λ^2 compared to $k\cdot p$ or $k\cdot p'$, the error being not larger than λ/m. Using (2.3), (2.4), and (2.6), the soft part of $\mathrm{Im}\Pi$ is

$$\mathrm{Im}\Pi_{\text{soft}}(q^2)=\frac{1}{24\pi^2q^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2\pi)^{3n}}\int d^4p_+d^4p'\delta_+(p_+{}^2-m^2)\delta_+(p'^2-m^2)\sum_{s,s'}|\langle 0|J_\mu(0)|p_+,p'\rangle|^2\prod_{i=1}^{n}d^4k_i\delta_+(k_i{}^2-\lambda^2)$$
$$\times|ej^\mu(k_i)|^2\delta^4\Big(q-p_+-p'-\sum k_i\Big)\,. \qquad (2.7)$$

The series (2.7) can be summed using the x representation of the δ function, which makes the photons look as if they were kinematically independent:

$$\delta^4\left(q-p_+-p'-\sum k_i\right)=\int \frac{d^4x}{(2\pi)^4}\, e^{i(q-p_+-p')\cdot x}\prod_i e^{-ik_i x}\,, \tag{2.8}$$

and we obtain

$$\mathrm{Im}\Pi_{\mathrm{soft}}(q^2)=\frac{1}{24\pi^2 q^2}\int d^4p_+ d^4p'\delta_+(p_+{}^2-m^2)\delta_+(p'^2-m^2)E_\lambda(p_+,p';q-p_+-p')\sum_{s,s'}|\langle 0|J_\mu(0)|p_+,p'\rangle|^2\,. \tag{2.9}$$

We shall call $E_\lambda(p_+,p';K)$ the real-photon spectral weight function in momentum space where $K=q-p_+-p'$ represents the missing four-momentum due to photon emission. This function is the Fourier transform of $\tilde{E}_\lambda(p_+,p';x)$,

$$E_\lambda(p_+,p';K)=\int\frac{d^4x}{(2\pi)^4}\,e^{iKx+\alpha\tilde{I}_\lambda(p_+,p';x)}$$
$$\equiv\int\frac{d^4x}{(2\pi)^4}\,e^{iKx}\tilde{E}_\lambda(p_+,p';x)\,, \tag{2.10}$$

where α is the fine-structure constant and $\tilde{I}_\lambda(p_+,p';x)$ is the infrared-divergent function

$$\tilde{I}_\lambda(p_+,p';x)=-\frac{1}{2\pi^2}\int d^4k\,\delta_+(k^2-\lambda^2)e^{-ik\cdot x}|j^\mu(k)|^2\,, \tag{2.11}$$

which is the Fourier transform of the relative probability for one-soft-photon emission. It is important to note that Eq. (2.9) looks exactly like the elastic unitarity equation [the term $n=0$ in Eq. (2.7)] except for the fact that the energy-momentum conservation factor $\delta^4(q-p_+-p')$ has been replaced by the spectral weight function $E_\lambda(p_+,p';q-p_+-p')$, the argument of δ and the (third) argument of E_λ being the same. One can think of the spectral weight function E_λ as a broadened δ function due to real-photon emission.

Let us now perform the spin summations in Eq. (2.9). The charge and magnetic-moment vertex functions of the electron (to all orders in perturbation theory) are introduced as usual by

$$\langle 0|J_\mu(0)|p_+,p'\rangle=\bar{u}(p')\left[F_1(a)\gamma_\mu+\frac{i\alpha}{4\pi m}\sigma_{\mu\nu}\times(p'+p_+)^\nu F_2(a)\right]v(p_+)\,, \tag{2.12}$$

where a is the squared electron-positron energy in their c.m. frame,

$$a=(p'+p_+)^2\geq 4m^2\,. \tag{2.13}$$

The vertices are normalized according to

$$F_1(0)=1 \text{ to all orders},\quad F_2(0)=1+O(\alpha)\,. \tag{2.14}$$

The trace resulting from the spin sum is readily computed and Eq. (2.9) becomes

$$\mathrm{Im}\Pi'_{\mathrm{soft}}(q^2)=\frac{1}{6\pi^2q^2}\int d^4p_+d^4p'\,\delta_+(p'^2-m^2)\delta_+(p_+{}^2-m^2)\times E_\lambda(p_+,p';q-p_+-p')X_\lambda(a)\,, \tag{2.15}$$

where

$$X_\lambda(a)=|F_1(a)|^2(a+2m^2)+\frac{3\alpha a}{2\pi}\mathrm{Re}(F_1F_2^*)+\left(\frac{\alpha}{2\pi}\right)^2\frac{a(a+8m^2)}{8m^2}|F_2|^2\,. \tag{2.16}$$

Of course, the λ dependence of E_λ and X_λ, which originates from real photons and virtual radiative corrections, respectively, cancels. The full content of Eq. (2.15) can be exploited if one has a nonperturbative knowledge of the F_i vertices. This will be the subject of Sec. IV.

B. Electron-proton scattering including soft-photon emission

As another example of where a spectral weight function appears, let us discuss electron-proton scattering including radiative corrections and real-photon emission. We shall limit ourselves to the approximation in which only one photon is exchanged between the hadronic and leptonic vertices. Furthermore, we shall neglect photon emission and radiative corrections at the hadronic vertex. We use conventional notation[25] and kinematics except that our spinors are normalized to $\bar{u}u=2m$.

It is straightforward to write down the hadron inclusive cross section with emission of r photons from the lepton vertex

$$d\sigma_r = \frac{1}{r!}\,\frac{2M\alpha^2}{[(P\cdot p)^2 - M^2m^2]^{1/2}}\int\frac{d^3p'}{2E'}\prod_{i=1}^{r}\frac{d^4k_i\delta_+(k_i^2-\lambda^2)}{(2\pi)^3}\,\frac{d^4q}{(q^2)^2}\,\delta^4\Big(p-q-p'-\sum k_i\Big)$$
$$\times W^{\mu\nu}(P,q)\langle p,s|J_\mu(0)|p',s',k_i,\epsilon_i\rangle\langle p',s',k_i,\epsilon_i|J_\nu(0)|p,s\rangle\,, \tag{2.17}$$

where

$$W^{\mu\nu}(P,q) = \frac{1}{2M}\sum_{n,\,\text{spin average}}\langle P,S|J_\mu(0)|n\rangle\langle n|J_\nu(0)|P,S\rangle(2\pi)^3\delta^4(P+q-P_n)\,. \tag{2.18}$$

Here (p,s) and (p',s') are the momenta and spins of the incident and scattered electron, respectively, and q is the momentum of the exchanged photon. Similarly, P and P_n are the momenta of the proton and the hadronic state $|n\rangle$, S is the proton spin, and M is its mass. As before, k_i, ϵ_i are the momenta and polarizations of the emitted photons.

Let us compute the soft part of the cross section in the sense defined previously, the equation similar to (2.4) being

$$\langle p',s',k_1,\epsilon_1\cdots k_r,\epsilon_r|J_\mu(0)|p,s\rangle \underset{k_i\to 0}{\sim} \langle p',s'|J_\mu(0)|p,s\rangle e^r\epsilon_1\cdot j(k_1)\cdots\epsilon_r\cdot j(k_r)\,, \tag{2.19}$$

and where,

$$j^\mu = \frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\,. \tag{2.20}$$

Using again the x representation of $\delta^4(p-q-p'-\sum k_i)$, summing over electron spins, photon-number, momenta, and polarizations, one gets

$$d\sigma_{\text{soft}} = \sum_{r=0}^{\infty} d\sigma_r = \frac{2M\alpha^2}{[(P\cdot p)^2 - m^2M^2]^{1/2}}\int\frac{d^3p'}{2E'}\,\frac{d^4q}{(q^2)^2}\,W^{\mu\nu}(P,q)X_{(\lambda)\mu\nu}(p,p')E_\lambda(p,p';p-q-p')\,, \tag{2.21}$$

where

$$X_{(\lambda)\mu\nu} = \tfrac{1}{2}\sum_{s,s'}\langle p,s|J_\mu(0)|p',s'\rangle\langle p',s'|J_\nu(0)|p,s\rangle\,. \tag{2.22}$$

E_λ is the spectral weight function which is defined by Eqs. (2.10), and (2.11) takes care of real-photon emission. The infrared divergence of the tensor $X_{(\lambda)\mu\nu}$ (λ is the photon mass and not a tensor index) coming from virtual soft photons is canceled by that of E_λ. This tensor is easily computed in terms of the F_i vertices for spacelike momentum $p'-p$,

$$a = (p'-p)^2 < 0\,. \tag{2.23}$$

The calculation is simplified using the Gordon decomposition to write

$$\langle p',s'|J_\mu(0)|p,s\rangle = \bar u(p')\Big[F_e\gamma_\mu - \frac{\alpha}{4\pi m}(p+p')_\mu F_2\Big]u(p)\,, \tag{2.24}$$

where

$$F_e = F_1 + (\alpha/2\pi)F_2\,. \tag{2.25}$$

Computing the trace over γ matrices, Eq. (2.22) gives

$$X_{(\lambda)\mu\nu} = F_1^2(ag_{\mu\nu} + 2p_\mu p'_\nu + 2p_\nu p'_\mu) + \frac{\alpha}{\pi}F_1F_2[ag_{\mu\nu} - (p-p')_\mu(p-p')_\nu] + \Big(\frac{\alpha}{2\pi}\Big)^2F_2^2$$
$$\times\Big\{a\Big[g_{\mu\nu} - \frac{(p+p')_\mu(p+p')_\nu}{4m^2}\Big] - (p-p')_\mu(p-p')_\nu\Big\}\,. \tag{2.26}$$

A more explicit form of Eq. (2.21) is obtained by introducing the conventional W_1 and W_2 structure functions,

$$W^{\mu\nu}(P,q) = \frac{1}{M^2}\Big(P^\mu - \frac{\nu}{q^2}q^\mu\Big)\Big(P^\nu - \frac{\nu}{q^2}q^\nu\Big)W_2(q^2,\nu) - \Big(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\Big)W_1(q^2,\nu)\,,\quad \nu = P\cdot q\,. \tag{2.27}$$

The contraction between $X_{(\lambda)\mu\nu}$ and $W^{\mu\nu}$ using (2.26) and (2.27) is straightforward. However, the soft approximation violates somewhat current conservation, which implies

$$q^\mu X_{(\lambda)\mu\nu} = q^\nu X_{(\lambda)\mu\nu} = 0 . \tag{2.28}$$

The violation involves terms proportional to $K = p - q - p'$, the effective photon momentum. In the spirit of the soft approximation, we shall ignore these terms, that is, we shall enforce Eq. (2.28) in the $X^{\mu\nu}_{(\lambda)} W_{\mu\nu}$ contraction. Moreover, we can include part of the hard-photon contribution by assuming that the electron electromagnetic vertices are taken as functions of q^2 instead of $a = (p' - p)^2$, since the included photons are emitted from external legs. One gets

$$X^{\mu\nu}_{(\lambda)} W_{\mu\nu} = F_1{}^2[(a + 4EE')W_2 - 2W_1(a + 2m^2)] + \frac{\alpha}{\pi} F_1 F_2 \{[a - (E - E')^2]W_2 - 3aW_1\} + \left(\frac{\alpha}{2\pi}\right)^2 F_2{}^2 \left\{\left[a - \frac{a}{4m^2}(E + E')^2 - (E - E')^2\right]W_2 - a\,\frac{(a + 8m^2)}{4m^2} W_1\right\}, \tag{2.29}$$

where we have used the lab system of reference to write

$$P \cdot p = ME, \quad P \cdot p' = ME'. \tag{2.30}$$

In this system, the variable a at high energy is

$$a = -4EE' \sin^2(\tfrac{1}{2}\theta), \tag{2.31}$$

where θ is the electron scattering angle. One can check Eq. (2.29) by noting that for $F_2 = 0$, this equation reduces to the familiar form

$$4EE' \cos^2(\tfrac{1}{2}\theta)[W_2 + 2W_1 \tan^2(\tfrac{1}{2}\theta)]F_1{}^2 \tag{2.32}$$

and F_1 is 1 if radiative corrections are neglected. We shall discuss Eq. (2.21) in more detail in a forthcoming paper.[9]

C. Contribution of real soft photons in the general case

The generalization of the above results is now straightforward. Consider an arbitrary process and let q be the sum of (ingoing) momenta of the observed neutral particles (including hard photons if any) in the initial and final states. Let p_i be the physical momenta of ith charged particle, boson or fermion, and z_i its charge (in units of $e > 0$). We associate also with each particle a variable $\theta_i = \pm 1$ according to whether the particle is outgoing ($\theta_i = 1$) or incoming ($\theta_i = -1$). The energy-momentum conservation which appears in the cross section reads $\delta^4(q - \sum_i p_i \theta_i \epsilon_i)$, where $\epsilon_i = z_i / |z_i|$ is +1 for a particle and −1 for an antiparticle. As before, we define the soft contribution to the cross section as the sum of the elastic cross section and the cross section for emission of an arbitrary number of soft photons. Using the same technique as above, it is clear that the soft contribution to the cross section is obtained by replacing the δ function by a spectral weight function with the same argument. This argument is of course the effective momentum of the soft photons. The real photons' spectral weight function in configuration space is

$$\tilde{E}_\lambda(p_i; x) = \exp[\alpha \tilde{I}(p_i; x)], \tag{2.33}$$

with

$$\tilde{I}(p_i; x) = -\frac{1}{2\pi^2} \int d^4k\, \delta_+(k^2 - \lambda^2) |j^\mu(k)|^2 e^{-ik\cdot x}, \tag{2.34}$$

where $j^\mu(k)$ is a generalized classical current defined by

$$j^\mu(k) = \sum_i \frac{\theta_i z_i p_i^\mu}{p_i \cdot k}. \tag{2.35}$$

Using charge conservation,

$$\sum z_i \theta_i = 0, \tag{2.36}$$

it is easily seen that Eq. (2.34) can be written[5] as a sum of contributions from pairs of charged particles

$$\tilde{I}(p_i; x) = -\sum_{i<j} z_i z_j \theta_i \theta_j \tilde{I}_{ij}(p_i, p_j; x), \tag{2.37}$$

with

$$\tilde{I}_{ij}(p_i, p_j; x) = \frac{-1}{2\pi^2} \int d^4k\, \delta_+(k^2 - \lambda^2) e^{-ik\cdot x} \times \left| \frac{p_i^\mu}{k \cdot p_i} - \frac{p_j^\mu}{k \cdot p_j} \right|^2. \tag{2.38}$$

The corresponding spectral weight function in momentum space is then $E_\lambda(p_i; q - \sum p_i \theta_i \epsilon_i)$, where $E_\lambda(p_i; K)$ is the Fourier transform of (2.33)

$$E_\lambda(p_i; K) = \int \frac{d^4x}{(2\pi)^4} e^{iK\cdot x} \tilde{E}_\lambda(p_i; x). \tag{2.39}$$

Although an explicit expression for $\tilde{E}_\lambda(p_i; x)$ will be given in the next section, it does not seem possible to obtain a useful expression for its Fourier transform $E_\lambda(p_i; K)$ which is needed in practical applications. We shall, however, exhibit a simple, approximate form for $E_\lambda(p_i; K)$ and discuss its physical meaning and limitations.

III. REAL-PHOTON SPECTRAL WEIGHT FUNCTIONS

This section is first devoted to the computation of the spectral weight function $\tilde{E}_\lambda(p_i;x)$ first introduced in Eqs. (2.10) and (2.11) and generalized in Eqs. (2.33) to (2.38). Since we do not introduce any cutoff on photon energies, it turns out that it is the x variable which compensates for the dimension of the photon mass λ. Consequently, the known power-law dependence of E on λ manifests itself as a scaling property of E in the x variable. Since it is unlikely that one can compute the momentum spectral weight function in closed form, we pause to derive the photon energy spectral weight function $E_\lambda(p_i;K_0)$ which can be computed in closed form. This function is frame dependent and has its simplest form in the Breit (c.m.) frame for spacelike (timelike) momentum transfer. The form of $E_\lambda(p_i;K_0)$ is quite similar to the YFS result in potential scattering. We then introduce a Lorentz-invariant approximation $E_{1\lambda}(p_i;K)$ to the exact spectral weight function which we call the peaking approximation. It obeys in particular the scaling property and has a clear physical interpretation: The emitted photons are collinear with the light-cone momenta l or l' which are linear combinations of p and p', the momenta of the charged particles before and after scattering. The "scale" of the function $E_{1\lambda}(p_i;K)$, which is set by the scalar product $l\cdot l'$, is fixed by identifying the photon energy spectral functions $E_\lambda(p_i;K_0)$ and $E_{1\lambda}(p_i;K_0)$ in the Breit (c.m.) frame.

A. Spectral weight function in space-time

It is clear from Eqs. (2.33) and (2.38) that a general spectral weight function in space-time is the product of spectral functions over all distinct pairs (i,j) of charged particles with momenta (p_i,p_j). The logarithm of a spectral weight function pair involves the integral

$$I'(p_i,p_j;x)=\frac{p_i\cdot p_j}{\pi^2}\int d^4k\,\frac{e^{-ikx}\delta_+(k^2-\lambda^2)}{(k\cdot p_i)(k\cdot p_j)}\,, \tag{3.1}$$

where I' is $\tilde{I}$ defined in Eq. (2.38) with the terms $p_i=p_j$ suppressed. Using Feynman's formula to combine the denominators and introducing $u=k\cdot x$ one gets

$$I'=\frac{p_i\cdot p_j}{\pi^2}\int_0^1 d\alpha\int du\,e^{-iu}\times\int d^4k\,\frac{\delta(u-k\cdot x)\delta_+(k^2-\lambda^2)}{(k\cdot P_\alpha)^2}\,, \tag{3.2}$$

where

$$P_\alpha=\alpha p_i+(1-\alpha)p_j\,. \tag{3.3}$$

With the charged particles being on the mass shell, $p_i^2=m_i^2$ and $E_i\geqslant m_i$, the four-vector P_α is timelike,

$$P_\alpha^2=m_i^2\alpha^2+m_j^2(1-\alpha)^2+2\alpha(1-\alpha)p_i\cdot p_j \geqslant[\alpha m_i+(1-\alpha)m_j]^2\geqslant 0 \tag{3.4}$$

(we assume $m_i\geqslant m_j$). We can evaluate the k integral in a reference frame where $\vec{\mathrm{P}}_\alpha=0$. Setting $E_\alpha=P_{0\alpha}=(P_\alpha^2)^{1/2}$ and $r=|\vec{x}|$, we obtain, after a short calculation,

$$I'=\frac{p_i\cdot p_j}{\pi}\int_0^1\frac{d\alpha}{P_\alpha^2}\,[\theta(x^2)I_\alpha^+ +\theta(-x^2)I_\alpha^-]\,, \tag{3.5a}$$

where

$$I_\alpha^-=\frac{x^2}{r}\int\frac{du\,e^{-iu}}{ux_0-r(u^2-\lambda^2x^2)^{1/2}}\,,$$
$$I_\alpha^+=2\int du\,e^{-iu}\,\frac{\theta(ux_0)\theta(u^2-\lambda^2x^2)(u^2-\lambda^2x^2)^{1/2}}{u^2+r^2\lambda^2}\,. \tag{3.5b}$$

Only the real part of these integrals is infrared divergent. A convenient way to separate the λ dependence is to introduce a cutoff S which will disappear in the final result. Setting $\lambda=0$ whenever possible, one obtains

$$I_\alpha^-=2\int_0^\infty\frac{du}{u}[\cos u-\theta(S-u)] +\frac{x^2}{r}\int_{-S}^{+S}\frac{du}{ux_0-r(u^2-\lambda^2x^2)^{1/2}} +\frac{ix^2}{r}\int_{-\infty}^{+\infty}\frac{\sin u\,du}{r|u|-x_0u}\,, \tag{3.6a}$$

$$I_\alpha^+=2\int_0^\infty\frac{du}{u}[\cos u-\theta(S-u)] +2\int_{-S}^{+S}\frac{du\,\theta(ux_0)\theta(u^2-\lambda^2x^2)(u^2-\lambda^2x^2)^{1/2}}{u^2+r^2\lambda^2} -2i\int_{-\infty}^{+\infty}\frac{du\,\theta(ux_0)\sin u}{|u|}\,. \tag{3.6b}$$

The first integrals in (3.6a) and (3.6b) are related to Euler's constant $\gamma=0.58$ by

$$\int_0^\infty\frac{du}{u}[\cos u-\theta(S-u)]=-(\gamma+\ln S)\,. \tag{3.7}$$

In the second integrals of Eqs. (3.6), it is convenient to change variables $u\to v=\lambda u$. The third integrals are trivial.

One obtains after some rearrangement

$$I_\alpha^- = -2(\gamma + \ln S) + 2\left[\int_0^{s/\lambda} \frac{dv}{(v^2 - x^2)^{1/2}} - x_0^2 \int_0^\infty \frac{dv}{(v^2 + r^2)(v^2 - x^2)^{1/2}}\right] - i\pi \frac{x_0}{r}$$
$$= -\left[\ln \frac{\lambda^2(-x^2)e^{2\gamma}}{4} + \frac{x_0}{r} \ln \frac{r + x_0}{r - x_0} + \frac{i\pi x_0}{r}\right]. \quad (3.8a)$$

In the same way, we obtain from (3.6b)

$$I_\alpha^+ = -\left[\ln \frac{\lambda^2 x^2 e^{2\gamma}}{4} + \frac{x_0}{r} \ln \frac{x_0 + r}{x_0 - r} + i\pi\epsilon(x_0)\right]. \quad (3.8b)$$

Using the covariant expressions for x_0 and r,

$$x_0 = (x\cdot P_\alpha)/(P_\alpha^2)^{1/2}, \quad r = \{[(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]/P_\alpha^2\}^{1/2}, \quad (3.9)$$

Eq. (3.5) becomes

$$I'(p_i, p_j; x) = -\frac{p_i \cdot p_j}{\pi} \int_0^1 \frac{d\alpha}{P_\alpha^2}\left\{\ln\left[\frac{\lambda^2(-x^2)e^{2\gamma}}{4}\right] + \frac{x\cdot P_\alpha}{[(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}} \ln \frac{x\cdot P_\alpha + [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}{x\cdot P_\alpha - [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}\right\} \quad (3.10)$$

where x_0 has a negative imaginary part ($x_0 \to x_0 - i\epsilon$), so the logarithms read

$$\ln(-x^2) = \ln|x^2| + i\pi\epsilon(x_0)\theta(x^2) \quad (3.11a)$$

and

$$\ln \frac{x\cdot P_\alpha + [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}{x\cdot P_\alpha - [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}} = \ln\left|\frac{x\cdot P_\alpha + [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}{x\cdot P_\alpha - [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}\right| + i\pi\theta(-x^2). \quad (3.11b)$$

The α integration of the first term of Eq. (3.10) is elementary. The second term leads to a large number of Spence functions with complicated arguments.[3,18] Subtracting from (3.10) the terms with $p_i = p_j$, the explicit form of the function $\tilde{I}$ of Eq. (2.38) is

$$\tilde{I}_{ij}(p_i, p_j; x) = -\bar{A}\left(\frac{p_i\cdot p_j}{m_i m_j}\right) \ln\left(\frac{-\lambda^2 x^2 e^{2\gamma}}{4}\right) - \frac{p_i\cdot p_j}{\pi}\int_0^1 \frac{d\alpha}{P_\alpha^2}\frac{x\cdot P_\alpha}{[(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}} \ln \frac{x\cdot P_\alpha + [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}{x\cdot P_\alpha - [(x\cdot P_\alpha)^2 - x^2 P_\alpha^2]^{1/2}}$$
$$+\left\{\frac{x\cdot p_i}{2\pi[(x\cdot p_i)^2 - m_i^2 x^2]^{1/2}} \ln \frac{x\cdot p_i + [(x\cdot p_i)^2 - m_i^2 x^2]^{1/2}}{x\cdot p_i - [(x\cdot p_i)^2 - m_i^2 x^2]^{1/2}} + (i \to j)\right\}, \quad (3.12)$$

where P_α is defined in Eq. (3.3). For later convenience, we shall use both the function $\bar{A}$ and the standard function[5] A, which are defined by

$$A(X_{ij}) \equiv 2\bar{A} = \frac{2}{\pi}\int_0^1 d\alpha\left(\frac{p_i\cdot p_j}{P_\alpha^2} - 1\right)$$
$$= -\frac{\vec{k}^2}{4\pi^2}\int d\Omega\left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)^2, \quad (3.13)$$

where $X_{ij} = p_i\cdot p_j/m_i m_j$. The α integration is trivial and one gets

$$A(X) = \frac{2}{\pi}\left[\frac{X}{2(X^2-1)^{1/2}} \ln\left|\frac{X + (X^2-1)^{1/2}}{X - (X^2-1)^{1/2}}\right| - 1\right]. \quad (3.14)$$

Note that A is real and non-negative since j^μ is orthogonal to k^μ and therefore spacelike. The high-energy behavior of the function A is

$$A \underset{X\gg 1}{\sim} \frac{2}{\pi}\left(\ln \frac{2p_i\cdot p_j}{m_i m_j} - 1\right). \quad (3.15)$$

For later reference, let us write explicitly the form of the function A in the case $m_i = m_j = m$. In terms of the variable $a = (p_+ + p')^2$ for ImΠ [cf. (2.13)] and $a = (p' - p)^2$ for $e + p$ scattering [cf. (2.23)], we have

$$A = \frac{2}{\pi}\left(\frac{1+u^2}{2u} \ln\left|\frac{1+u}{1-u}\right| - 1\right) \quad (3.16)$$

and

$$u = \left(\frac{a - 4m^2}{a}\right)^{1/2}. \quad (3.17)$$

Equation (3.12), combined with (2.33)–(2.38), gives the spectral weight function in space-time for any process.

The essential problem we have to face now is to compute the Fourier transform of $\exp[\alpha\tilde{I}(p, p'; x)]$. It is unlikely that such a program can be done without resorting to some approximation. Fortunately, we can be guided in the choice of our approximation by a scaling property of the spectral weight function which is a direct consequence of the infrared behavior of the emitted photons. A glance at

Eq. (3.12) shows that

$$E_\lambda(p_i;\rho x)=\rho^{-\alpha A}E_\lambda(p_i;x), \tag{3.18}$$

where $A=2\bar{A}$ is, for a general process, defined by [compare Eq. (2.37)]

$$A=2\bar{A}=-\sum_{i<j} z_i z_j \theta_i\theta_j A(X_{ij}) \tag{3.19}$$

and $A(X)$ is given in (3.13). The corresponding scaling property of the spectral weight function in momentum space is

$$E_\lambda(p_i;\rho K)=\rho^{\alpha A-4}E_\lambda(p_i;K). \tag{3.20}$$

The origin of the exponent -4 is easily understood since $E_\lambda(p_i;K)\to\delta^4(K)$ when $\alpha\to 0$. Let us note also two simple and important properties of the spectral weight function associated with a pair of charged particles with momenta p and p', which result from the classical current definition:

$$E_\lambda(p,p';K)=E_\lambda(p',p;K) \tag{3.21}$$

and

$$E_\lambda(p,p;K)=1. \tag{3.22}$$

The last property expresses the fact that a charged particle which is not accelerated cannot radiate.

Before exhibiting a nice and useful approximation to the general spectral function $E_\lambda(p_i;K)$, we shall pause to discuss a simple case where the spectral function can be computed exactly. This will permit us to make contact with the standard YFS work.[5,6]

B. Energy spectral weight function

There exists a large class of physical problems for which only energy resolution is effective. As a typical example, let us pursue the discussion of electron-proton scattering. Introducing the effective photon four-momentum $K=p-q-p'$, Eq. (2.21) reads

$$d\sigma_{\text{soft}}=\frac{2\alpha^2}{p_{\text{Lab}}}\int\frac{d^3p'}{2E'}\frac{d^4K}{[(p-p'-K)^2]^2}\times W^{\mu\nu}(P,p-p'-K)\times X_{(\lambda)\mu\nu}(p,p')E_\lambda(p,p';K), \tag{3.23}$$

where $W_{\mu\nu}X^{\mu\nu}_{(\lambda)}$ is explicitly stated in (2.29). Of course, one expects that only small values of K contribute to this integral. We are seeking a physical situation where only K_0 plays a significant role, so we can neglect $\vec{K}$ in $W_{\mu\nu}$ and $(q^2)^{-2}$. It is easily seen that this is the case of (almost) elastic scattering in the Breit (B) reference frame where $\vec{p}+\vec{p}'=0$, $E=E'$. In effect, taking $\vec{p}$ along the positive z axis and neglecting the electron mass, one has the following kinematical relations:

$$-\frac{q^2}{2}\sim\nu=P\cdot q=-P_0K_0-P_z(2E_B-K_z) \tag{3.24}$$

and

$$2E_B{}^2=p\cdot p',\quad 2P_0E_B=P\cdot(p+p'),\quad 2P_zE_B=P\cdot(p'-p), \tag{3.25}$$

where we have neglected the transverse component of $\vec{K}$. According to (3.25), $|P_z|\ll P_0$ for elastic scattering and K_z can be neglected in (3.24), since $K^2>0$. Now we can evaluate the $\vec{K}$ integral in (3.23), thus introducing a photon-energy spectral weight function:

$$E_\lambda(p,p';K_0)=\int d^3K\,E_\lambda(p,p';K)=\int\frac{dt}{2\pi}e^{K_0t}\tilde{E}_\lambda(p,p';x_0=t,\vec{x}=0). \tag{3.26}$$

Of course, the function $E_\lambda(K_0)$, computed in the lab frame, would appear directly in Eq. (3.23) if one considers the problem at hand as potential scattering.[5] From (3.12), we see that for $\vec{x}=0$, the t dependence of $\tilde{E}_\lambda$ is a power law and reads

$$\tilde{E}_\lambda(p,p';x_0=t,\vec{x}=0)=\left(\frac{\lambda e^\gamma}{2}i(t-i\epsilon)\right)^{-\alpha A}e^{\alpha F(r)}, \tag{3.27}$$

where

$$F(r)=-\frac{(1+u^2)}{\pi u}\int_0^1\frac{d\beta}{(1-\beta^2u^2)\beta}\ln\frac{1+\beta u}{1-\beta u}+\frac{1}{\pi u}\ln\frac{1+u}{1-u} \tag{3.28}$$

and

$$u=\frac{p_B}{E_B}=\left(\frac{p\cdot p'-m^2}{p\cdot p'+m^2}\right)^{1/2}=\left(\frac{a-4m^2}{a}\right)^{1/2}. \tag{3.29}$$

Although F has been computed in the Breit frame, it is a Lorentz-invariant function. Its explicit form involves a Spence function ϕ or a dilogarithm Euler function Li_2:

$$F(r)=-\frac{(1+r^2)}{\pi(1-r^2)}\left[\tfrac{1}{2}\ln^2 r+2\,\mathrm{Li}_2(1-r)\right]-\frac{1}{\pi}\frac{(1+r)}{(1-r)}\ln r=\bar{A}(r)\ln r-\frac{(1+3r)}{2\pi(1-r)}\ln r-\frac{2(1+r^2)}{\pi(1-r^2)}\mathrm{Li}_2(1-r). \tag{3.30}$$

Here,[15]

$$\mathrm{Li}_2(y)=-\int_0^y\frac{dt}{t}\ln(1-t)=-\phi(-y)+\frac{\pi^2}{12} \tag{3.31}$$

and r is a kinematical invariant, frequently used in what follows:

$$r=\frac{1-u}{1+u}=X-(X^2-1)^{1/2},\quad X=p\cdot p'/m^2\,. \quad (3.32)$$

The second form of $F(r)$ in Eq. (3.30) uses (3.14) and (3.32). At high energy, $X\gg 1$, $r\sim m^2/2p\cdot p'$, and from Eq. (3.30) one gets

$$F(r)\underset{r\to 0}{\sim}-\frac{1}{2\pi}\ln^2 r-\frac{1}{\pi}\ln r-\frac{\pi}{6}+O(r\ln r)\,. \quad (3.33)$$

Looking back to Eqs. (3.26) and (3.27), we see that the time integral can be computed by deforming the contour in the complex t plane or, equivalently, by using an integral representation of Euler's Γ function written in the form

$$[i(t-i\epsilon)]^{-\nu}=\Gamma^{-1}(\nu)\int_0^\infty d\sigma\,\sigma^{\nu-1}e^{-i\sigma t}\quad (\mathrm{Re}\,\nu>0) \quad (3.34)$$

to integrate over t first, with the result

$$E_\lambda(p,p';K_0)=\left(\frac{\lambda e^\gamma}{2}\right)^{-\alpha A}\Gamma^{-1}(\alpha A)\frac{\theta(K_0)}{K_0^{\,1-\alpha A}}e^{\alpha F(r)}. \quad (3.35)$$

This equation is very similar to the YFS[5] result in potential scattering. In the latter case, the photon-energy spectral weight function is computed in the lab frame, whereas Eq. (3.35) is valid in the Breit frame. The difference, however, is not very significant at high energy since it appears in the angular part of F. We note also that Eq. (3.35) is an exponentiated form of Schwinger's[3,11] lowest-order computation of soft-photon emission. Note that the radiative tail is characterized by the spectrum $\Gamma^{-1}(\alpha A)K_0^{\,-1+\alpha A}$ which is integrable at $K_0=0$.

C. The spectral weight function in the peaking approximation

We discuss now the general spectral weight function in momentum space. We shall seek an approximation for $E_\lambda(p_i;K)$ which satisfies (3.21) and (3.22) and the scaling equation (3.20). Since the former equation is nonperturbative, we shall insist that the approximate spectral weight function satisfying this equation, let us call it $E_{1\lambda}$, can be computed in closed form. In a previous paper[19] and also in the first version of this manuscript, we chose for $\tilde{I}(p_i;x)$ the form [compare Eqs. (3.12) and (3.27)]

$$\tilde{I}\to\tilde{I}_1(p_i;x)=-\bar{A}\ln(-\lambda^2x^2e^{2\gamma}/4)\,.$$

The resulting spectral weight function in momentum space is[19]

$$E_1(p_i;K)=\frac{2(\lambda^2e^{2\gamma})^{-\alpha\bar{A}}}{\pi(K^2)^{2-\alpha\bar{A}}}\Gamma^{-1}(\alpha\bar{A})\Gamma^{-1}(\alpha\bar{A}-1)\times\theta(K^2)\theta(K_0)\,.$$

The three-momentum integral of this function, defined by *analytic continuation* in $\alpha\bar{A}$, leads, of course, to the first factors in (3.35). This approximation has been criticized, with good reason, on two grounds. First, if E_1 is to describe the photon four-momentum spectrum (in experiments where three-momentum resolution is also precise), it must be an integrable function of $\vec{K}$ without resorting to analytic continuation. Second, the Fourier transform of $\tilde{I}$, which represents the relative probability of one soft-photon emission will be

$$I_1(p_i;k)=(A/\pi)\theta(k_0)\delta'(k^2-\lambda^2)\,,$$

which is not positive.[20] Furthermore, the function E_1 obtained previously is negative for small αA. Thus the preceding choice for E_1 cannot be defended. Fortunately, we have found a very practical approximation which does not suffer from these objections.

To introduce the peaking approximation, we associate with every pair of charged particles appearing in Eq. (2.38) with momenta p and p' two positive-frequency light-cone momenta l and l' defined by

$$l=\frac{2[r^{1/2}mm'N(r)]^{1/2}}{1-r^2}\left(\frac{p}{m}-\frac{rp'}{m'}\right),$$
$$l'=\frac{2[r^{1/2}mm'N(r)]^{1/2}}{1-r^2}\left(\frac{p'}{m'}-\frac{rp}{m}\right). \quad (3.36)$$

The kinematical variable r, chosen such that $l^2=l'^2=0$, is then given by Eq. (3.32). We have kept the solution $0\le r\le 1$ so that $l_0, l_0'>0$.

At high energy, $r\to 0$ like

$$r\sim\frac{1}{2X}=\frac{mm'}{2p\cdot p'} \quad (3.37)$$

and the momenta l and l' become nearly parallel to p and p', respectively. The normalization factor $N(r)$, which is supposed to exponentiate, is discussed later. It is introduced in Eq. (3.36) in such a way that $N(r)\sim$const, at high energy, at least for the two examples discussed in Sec. II. From (3.36) we get

$$l\cdot l'=2mm'r^{-1/2}N(r)\,. \quad (3.38)$$

By definition, the peaking approximation to Eq. (3.12) is

$$\tilde{I}_1(p,p';x)=-\bar{A}(X)\ln\left[\frac{-\lambda^2e^{2\gamma}}{4mm'}(l\cdot x)(l'\cdot x)\right], \quad (3.39)$$

where l and l' are defined in Eq. (3.36) and x_0 has a negative imaginary part. When only one pair of charged particles is involved (as in ImΠ or $e+p$ scattering of Sec. II), the space-time spectral weight function, in the peaking approximation, is

$$\tilde{E}_{1\lambda}(p,p';x)=\exp(\alpha\tilde{I}_1)$$
$$=\left[\frac{-\lambda^2 e^{2\gamma}}{4mm'}(l\cdot x)(l'\cdot x)\right]^{-\alpha\bar{A}}. \qquad (3.40)$$

The four-dimensional Fourier transform of (3.40) is easily obtained using twice the integral representation of Eq. (3.34). One immediately gets

$$E_{1\lambda}(p,p';K)=\left(\frac{\lambda^2 e^{2\gamma}}{4mm'}\right)^{-\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})$$
$$\times\int_0^\infty d\sigma\int_0^\infty d\sigma'(\sigma\sigma')^{\alpha\bar{A}-1}\delta^4(K-\sigma l-\sigma' l'). \qquad (3.41)$$

This equation is a simple and important result of this paper. Of course, $E_{1\lambda}$ fulfills (3.21), (3.22), and the scaling equation (3.20). Note also that $E_{1\lambda}$ is proportional to $[N(r)]^{-\alpha\bar{A}}$ as may be seen by the change of variables $\sigma\to\sigma[N(r)]^{1/2}$, $\sigma'\to\sigma'[N(r)]^{1/2}$. Physically, Eq. (3.35) means that the effective photon momentum K is in the plane formed by l and l', that is, $E_{1\lambda}$ is proportional to $\delta^2(K_T)$, where K_T is the transverse component of K. The meaning of the peaking approximation is clarified by computing $I_1(p,p';k)$, the Fourier transform of (3.39) which represents (when multiplied by α) the relative probability for one-photon emission with momentum k. This probability reads

$$\alpha I_1(p,p';k)=\alpha\bar{A}\int_{\lambda/2m}^\infty\frac{d\sigma}{\sigma}[\delta^4(k-\sigma l)+\delta^4(k-\sigma l')]. \qquad (3.42)$$

Thus, in the peaking approximation, only photons collinear with l or l' are considered. To prove Eq. (3.42), it is enough to compute the integral

$$\int d^4k\,e^{-ik\cdot x}\int_{\lambda/2m}^\infty\frac{d\sigma}{\sigma}\delta^4(k-\sigma l)=\int_{\lambda/2m}^\infty\frac{d\sigma}{\sigma}e^{-i\sigma l\cdot x} \qquad (3.43)$$

in the limit of small $\lambda/2m$. Setting $\lambda/2m=0$ in the imaginary part, one gets, using (3.7),

$$\int_{\lambda/2m}^\infty\frac{d\sigma}{\sigma}e^{-i\sigma l\cdot x}=-\gamma-\ln\frac{\lambda}{2m}|l\cdot x|-\frac{i\pi}{2}\epsilon(l\cdot x)$$
$$-\int_0^{\lambda|l\cdot x|/2m}\frac{dt}{t}(\cos t-1),$$
$$=-\ln\frac{\lambda e^{\gamma}i}{2m}(l\cdot x-i\epsilon), \qquad (3.44)$$

where, in the last form, we have neglected the integral which vanishes with λ. Equation (3.42) is thus established.

Generalization of the above results to any cross section is now a simple routine. The space-time spectral weight function, in the peaking approximation, is [compare Eqs. (2.37), (2.38), and (3.40)]

$$\tilde{E}_{1\lambda}(p_i;x)=\prod_{i<j}\left[-\frac{\lambda^2 e^{2\gamma}}{4m_i m_j}(l_i\cdot x)(l_j\cdot x)\right]^{-\alpha\bar{A}'_{ij}}, \qquad (3.45)$$

where, to simplify writing, we have set

$$\bar{A}'_{ij}=-z_i z_j\theta_i\theta_j\bar{A}(X_{ij}). \qquad (3.46)$$

The momentum-space version of (3.45) is

$$E_{1\lambda}(p_i;K)=\left[\prod_{i<j}\left(\frac{\lambda^2 e^{2\gamma}}{4m_i m_j}\right)^{-\alpha\bar{A}'_{ij}}\Gamma^{-2}(\alpha\bar{A}'_{ij})\right]$$
$$\times\int_0^\infty\prod_{i<j}d\sigma_i d\sigma_j(\sigma_i\sigma_j)^{\alpha\bar{A}'_{ij}-1}$$
$$\times\delta^4\left(K-\sum_i\sigma_i l_i\right). \qquad (3.47)$$

Finally, the relative probability for one-photon emission in the peaking approximation is

$$\alpha I_1(p_i;k)=-\sum_{i<j}z_i z_j\theta_i\theta_j\bar{A}(X_{ij})$$
$$\times\int_0^\infty\frac{d\sigma}{\sigma}[\delta^4(k-\sigma l_i)+\delta^4(k-\sigma l_j)]. \qquad (3.48)$$

In this paper, we shall not discuss further these general forms but concentrate on the pair spectral weight function given in (3.41). A more explicit form of this equation is obtained by doing the σ and σ' integrations. One gets the simplest result in the Breit frame where $\vec{p}+\vec{p}'=0$ and thus $\vec{l}+\vec{l}'=0$. In this frame we have

$$l_0^2=l\cdot l'/2=m^2 r^{-1/2}N(r) \qquad (3.49)$$

according to (3.38). After computing the Jacobian, Eq. (3.41) reads

$$E_{1\lambda}(p,p';K)=2\Gamma^{-2}(\alpha\bar{A})[\lambda^2 e^{2\gamma}r^{-1/2}N(r)]^{-\alpha\bar{A}}$$
$$\times\frac{\delta^2(K_T)\theta(K^2)\theta(K_0)}{(K^2)^{1-\alpha\bar{A}}}. \qquad (3.50)$$

This is, of course, the generalization of the photon-energy spectral weight function, Eq. (3.35). The essential feature of Eq. (3.50) is the appearance of an *integrable* singularity at $K_0=\vec{K}$. The scaling equation (3.20) has been satisfied covariantly, thanks to $\delta^2(K_T)$, which takes care of two powers of inverse K momentum.

A further confidence in the peaking approximation is obtained by computing the corresponding energy spectral weight function, defined as in Eq. (3.26). Owing to $\delta^2(K_T)$, there remains only an integral over K_z which is proportional to $\Gamma^2(\alpha\bar{A})\Gamma^{-1}(2\alpha\bar{A})$. The result is

$$E_{1\lambda}(p,p';K_0)=\int d^3K\,E_{1\lambda}(p,p';K)$$
$$=\Gamma^{-1}(\alpha A)\left\{\frac{[N(r)]^{1/2}\lambda e^{\gamma}}{2r^{1/4}}\right\}^{-\alpha A}\frac{\theta(K_0)}{K_0^{\,1-\alpha A}}, \tag{3.51}$$

where, as before, $A=2\overline{A}$. Comparison between (3.51) and (3.35) shows that the exact and peaking photon-energy spectral functions are identical in the Breit frame if $N(r)$ is chosen such that

$$\overline{A}[\tfrac{1}{2}\ln r-\ln N(r)]=F(r). \tag{3.52}$$

By taking into account Eq. (3.30), we get

$$A(r)\ln N(r)=\frac{2(1+r^2)}{\pi(1-r^2)}\,\mathrm{Li}_2(1-r)+\frac{1+3r}{2\pi(1-r)}\ln r. \tag{3.53}$$

In particular, $N(r)\sim\sqrt{e}$ at high energy. Using the normalization of Eq. (3.52), Eq. (3.50) becomes

$$E_{1\lambda}(p,p';K)=2\Gamma^{-2}(\alpha\overline{A})(\lambda e^{\gamma})^{-\alpha A}e^{\alpha F(r)}$$
$$\times\frac{\delta^2(K_T)\theta(K^2)\theta(K_0)}{(K^2)^{1-\alpha\overline{A}}}, \tag{3.54}$$

which is our explicit form of the spectral weight function in the Breit frame.

We shall see in the next section that with the normalization of Eq. (3.52), the spectral weight functions $E_\lambda(K)$ and $E_{1\lambda}(K)$ coincide in perturbation theory up to order α. This means in particular that, in lowest order at least, the relative probability of one-soft-photon emission

$$-\alpha(2\pi^2)^{-1}|j|^2\delta_+(k^2-\lambda^2)$$

can be replaced by $\alpha I_1(k)$, given in Eq. (3.42), which happens to be much simpler to handle in practical computations.

IV. THE QED VACUUM-POLARIZATION FORMULA AND THE ELECTROMAGNETIC VERTICES

In this section, we shall first use the spectral weight function in the peaking approximation $E_{1\lambda}(p,p';K)$ to derive spectral representations for the imaginary part of vacuum polarization in QED. In particular, the spectral form with respect to a, the squared c.m. electron-positron energy, has a characteristic soft-photon tail.[8] Since our formula is a functional of the electromagnetic vertices, a quantitative discussion entails a nonperturbative knowledge of these functions.

As a test of the peaking approximation, we compare the formula we obtain for ImΠ with the known fourth-order result. We find that the soft and peaking approximations coincide up to this order. The hard contribution to ImΠ, which is included neither in the soft nor in the peaking approximation, is computed and shown to be very small near threshold and behaves as $\ln s$, whereas the soft contribution behaves as $\ln^2 s$ at high energy.

The nonperturbative spectral representation of ImΠ leads us to discuss the threshold behavior of the electromagnetic (em) vertices. The exact form of the electric vertex function F_e near threshold is derived from the solution of the Schrödinger equation for a truncated Coulomb potential. It has poles corresponding to the nonrelativistic positronium bound-states energies. The resulting threshold behavior of ImΠ is then easily computed and agrees with another derivation.[21]

A natural extension of the nonrelativistic vertex function F_e is given with the correct threshold and infrared dependence. The proposed F_e has also the correct high-energy behavior found by many authors[14] and differs (the difference appears only in fourth order) from the Grammer-Yennie[6] form by the presence of the positronium bound-state poles.

The nonperturbative form of F_e is used as an ingredient in the discussion of the radiative corrections in $e+p$ scattering discussed in the next paper.[9]

A. Non-perturbative, spectral representation of ImΠ

Knowledge of the spectral weight function in closed form allows us to derive a simple nonperturbative representation of ImΠ in the peaking approximation. Putting (3.41) into (2.15), we get

$$\mathrm{Im}\Pi_1(s)=\frac{1}{6\pi^2 s}\int d^4p'd^4p_+\delta_+(p'^2-m^2)\delta_+(p_+^{\,2}-m^2)\left(\frac{\lambda^2}{4m^2}\right)^{-\alpha\overline{A}}$$
$$\times\Gamma^{-2}(\alpha\overline{A})X_\lambda(a)\int_0^\infty d\sigma d\sigma'(\sigma\sigma')^{\alpha\overline{A}-1}\delta^4(q-p_+-p'-\sigma l_+-\sigma' l'), \tag{4.1}$$

where a and $X_\lambda(a)$ are given in Eqs. (2.13) and (2.16), respectively, whereas l and l' are given by [compare Eq. (3.36)]

$$l_+=\frac{2[r^{1/2}N(r)]^{1/2}}{1-r^2}(p_+-rp'),$$
$$l'=\frac{2[r^{1/2}N(r)]^{1/2}}{1-r^2}(p'-rp_+), \tag{4.2}$$

the relation between r and a being here [compare Eq. (3.32)]

$$a=m^2(1+r)^2/r. \tag{4.3}$$

We shall absorb part of the normalization factors [cf. (3.38)] by changing variables

$$\sigma\to[r^{1/2}N(r)]^{1/2}\sigma,\quad \sigma'\to[r^{1/2}/N(r)]^{1/2}\sigma'.$$

Upon introducing the noninfrared combination of the vertices

$$X(a)=[\lambda^2 e^{2\gamma} r^{-1/2} N(r)]^{-\alpha\bar{A}} X_\lambda(a)$$
$$=(\lambda e^{\gamma})^{-\alpha A} X_\lambda(a) e^{\alpha F(r)}, \tag{4.4}$$

Eq. (4.1) becomes

$$\mathrm{Im}\Pi_1(s)=\frac{1}{6\pi^2 s}\int da \int_0^\infty d\sigma\, d\sigma'\,(\sigma\sigma')^{\alpha\bar{A}-1}(4m^2)^{\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})X(a)$$
$$\times\int d^4p' d^4p_+ \delta_+(p'^2-m^2)\delta_+(p_+{}^2-m^2)\delta((p_+ +p')^2-a)\delta^4(q-Cp_+-C'p'), \tag{4.5}$$

where

$$C=1+\frac{2\sqrt{r}\,(\sigma-r\sigma')}{1-r^2},\quad C'=1+\frac{2\sqrt{r}\,(\sigma'-r\sigma)}{1-r^2}. \tag{4.6}$$

Let us compute the phase-space integral in Eq. (4.5) in the frame where $\vec{q}=0$ and $q_0=\sqrt{s}$. Then

$$\sqrt{s}-CE_+-C'E'=0,\quad Cp_z+C'p_z'=0, \tag{4.7}$$

and Eq. (4.5) becomes, after an elementary calculation,

$$\mathrm{Im}\Pi_1(s)=\frac{1}{6\pi s\sqrt{s}}\int da(4m^2)^{\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})X(a)\int_0^\infty d\sigma\, d\sigma'(\sigma\sigma')^{\alpha\bar{A}-1}\frac{|p_z'|}{C^2C'}\delta((p_+ +p')^2-a). \tag{4.8}$$

Note that p_z' and $(p_+ +p')^2$ are to be expressed in terms of C, C' using Eq. (4.7). The argument of the δ function is also computed using this equation and found to be

$$(p_+ +p')^2-a=[s-m^2(C-C')^2-aCC']/CC'$$
$$=[s-a-2m\sqrt{a}\,(\sigma+\sigma')-4m^2\sigma\sigma']/CC', \tag{4.9}$$

where we have used (4.6) to obtain the last form of Eq. (4.9). It is very convenient to introduce the variable

$$b=4m^2\sigma\sigma', \tag{4.10}$$

which is easily seen to be the squared mass of the effective photon momentum ($b=K^2$). Equation (4.8) now reads

$$\mathrm{Im}\Pi_1(s)=\frac{1}{6\pi s\sqrt{s}}\int da\, db\, b^{\alpha\bar{A}-1}\Gamma^{-2}(\alpha\bar{A})X(a)\int_0^\infty d\sigma\, d\sigma'\,\delta(b/4m^2-\sigma\sigma')|p_z'/C|\delta(s-a-b-2m\sqrt{a}\,(\sigma+\sigma')). \tag{4.11}$$

Integration over σ and σ' amounts to computing a simple Jacobian which is found to be

$$2m\sqrt{a}|\sigma-\sigma'|=[\Delta(s,a,b)]^{1/2}, \tag{4.12}$$

where

$$\Delta(s,a,b)=s^2+a^2+b^2-2as-2bs-2ab. \tag{4.13}$$

From Eqs. (4.6) and (4.7) we found

$$p_z'/C=[a(a-4m^2)]^{1/2}/2\sqrt{s}. \tag{4.14}$$

The double spectral representation of $\mathrm{Im}\Pi$ finally is

$$\mathrm{Im}\Pi_1(s)=\frac{1}{6\pi s^2}\int_{4m^2}^{s} da[a(a-4m^2)]^{1/2}X(a)\Gamma^{-2}(\alpha\bar{A})\int_0^{(\sqrt{s}-\sqrt{a})^2}\frac{db\, b^{\alpha\bar{A}-1}}{[\Delta(s,a,b)]^{1/2}}. \tag{4.15}$$

(A factor of 2 is included in this result corresponding to the domains $\sigma>\sigma'$ and $\sigma<\sigma'$.)

We note that the b integral is convergent without analytic continuation in $\alpha\bar{A}$. The power of b, $\alpha\bar{A}-1$, is the same as in Eq. (3.50) since $b=K^2$. Equation (4.15) gives in particular the "spectrum" of $\mathrm{Im}\Pi$ in terms of the electron-positron pair mass $\sqrt{a}$ and the "missing mass" $\sqrt{b}$.

We shall now integrate over the photon spectrum in Eq. (4.15). By the change of variable

$$x=b/(\sqrt{s}-\sqrt{a})^2,$$

the b integral is[22]

$$\Gamma^{-2}(\alpha\bar{A})\int_0^{(\sqrt{s}-\sqrt{a})^2}\frac{db\,b^{\alpha\bar{A}-1}}{[\Delta(s,a,b)]^{1/2}}=\frac{\Gamma^{-2}(\alpha\bar{A})(\sqrt{s}-\sqrt{a})^{\alpha A}}{s-a}\int_0^1\frac{dx\,x^{\alpha\bar{A}-1}}{(1-x)^{1/2}(1-zx)^{1/2}}$$
$$=\frac{\Gamma^{-1}(\alpha A)[2(\sqrt{s}-\sqrt{a})]^{\alpha A}}{2(s-a)}\,{}_2F_1\left(\tfrac{1}{2},\frac{\alpha A}{2};\frac{\alpha A+1}{2};z\right), \tag{4.16}$$

where

$$z=\left(\frac{\sqrt{s}-\sqrt{a}}{\sqrt{s}+\sqrt{a}}\right)^2 \tag{4.17}$$

and ${}_2F_1\,(a,b;c;z)$ is the hypergeometric function which is not expected to play a significant role, since for $s=a$, ${}_2F_1=1$.

From Eqs. (4.15) and (4.16) we get

$$\mathrm{Im}\Pi_1(s)=\frac{1}{12\pi s^2}\int_{4m^2}^s\frac{da[a(a-4m^2)]^{1/2}}{(s-a)^{1-\alpha A(a)}}\left(\frac{\sqrt{s}+\sqrt{a}}{2}\right)^{-\alpha A(a)}X(a)\Gamma^{-1}(\alpha A(a))\,{}_2F_1\left(\tfrac{1}{2},\frac{\alpha A}{2};\frac{\alpha A+1}{2};\left(\frac{\sqrt{s}-\sqrt{a}}{\sqrt{s}+\sqrt{a}}\right)^2\right). \tag{4.18}$$

Here, $A(a)$ is given in Eq. (3.16) and $X(a)$ in Eqs. (4.4), (3.30), and (2.16). Of course, the a integral is convergent even in the infrared region where $s\sim a$. The essential feature of this equation is the broadening of the $\delta(s-a)$ factor which we would have had for the elastic unitarity contribution to $\Gamma^{-}(\alpha A)(s-a)^{\alpha A-1}$ which is characteristic of the soft-photon tail. As we shall see, the hypergeometric function will not contribute up to order e^4.

At this point, it may be instructive to derive an approximate form of Eq. (4.18) using the photon-energy spectral weight function, Eq. (3.35). Taking the effective photon momentum $K=q-p_+-p'$ as an integration variable, Eq. (2.15) becomes

$$\mathrm{Im}\Pi_{\text{soft}}(s)=\frac{1}{6\pi^2 s}\int da\,\frac{d^3p'}{2E'}\,d^4K\,\delta(a-(q-K)^2)\delta_+((q-p'-K)^2-m^2)X_\lambda(a)E_\lambda(p',q-p'-K;K). \tag{4.19}$$

In the frame where $\vec{q}=0$, $q_0=\sqrt{s}$, we shall make the following approximations:

(i) Neglect K in the second argument of E_λ.
(ii) Neglect $\vec{K}$ in the arguments of the two δ functions.

We can then integrate over $\vec{K}$ which brings the photon-energy spectral weight function given in Eq. (3.35). The approximate form of Eq. (4.19) is then

$$\mathrm{Im}\Pi_{\text{soft}}\simeq\frac{1}{6\pi^2 s}\int da\,\frac{d^3p'}{2E'}\,dK_0\,\delta(a-(\sqrt{s}-K_0)^2)\delta((\sqrt{s}-E'-K_0)^2-E'^2)X(a)\Gamma^{-1}(\alpha A)\theta(K_0)/K_0^{\,1-\alpha A}, \tag{4.20}$$

where $X(a)$ is defined in Eq. (4.4). Using the δ functions, we integrate over K_0 and E' with the result

$$\mathrm{Im}\Pi_{\text{soft}}=\frac{1}{12\pi s}\int_{4m^2}^s\frac{da[(a-4m^2)/a]^{1/2}}{(s-a)^{1-\alpha A}}\,\Gamma^{-1}(\alpha A)X(a)\left(\frac{\sqrt{s}+\sqrt{a}}{2}\right)^{-\alpha A}\frac{(\sqrt{s}+\sqrt{a})}{2\sqrt{a}}. \tag{4.21}$$

Of course, Eqs. (4.21) and (4.18) are very similar. In particular, the regular parts of the integrands are the same for $s=a$. As will be clear shortly, these equations are equivalent up to fourth order in perturbation theory. Before discussing nonperturbative applications, it is necessary to know the importance of the neglected terms in going from the exact to the soft and then to the peaking approximations of $\mathrm{Im}\Pi$. We shall now analyze the respective contributions up to fourth order in perturbation theory.

B. Perturbation expansion of ImΠ to fourth order

Perturbation expansion of Eq. (4.18) involves a Laurent expansion of the distribution $\Gamma^{-1}(\alpha A)(s-a)^{\alpha A(a)-1}$ in the power of the exponent. This is slightly more complicated than the known expansion[23] of $\Gamma^{-1}(\lambda)x^\lambda$ in powers of λ due to the dependence of the exponent on the variable a. The approach we follow is a direct generalization to take into account this dependence. The expansion is expected to converge for $\alpha A\lesssim 1$. We write Eq. (4.18) in the form

$$\mathrm{Im}\Pi_1(s)=\frac{\alpha A(s)}{12\pi s^2}\int_{4m^2}^{s}\frac{da\,\psi(s,a)}{(s-a)^{1-\alpha A(s)}}\,, \qquad (4.22)$$

where

$$\psi(s,a)=[a(a-4m^2)]^{1/2}X(a)(s-a)^{-\alpha A(a)}$$
$$\times\Gamma^{-1}(\alpha A(a))\,{}_2F_1/\alpha A(s)\,. \qquad (4.23)$$

To expand Eq. (4.22) in a perturbation series, we must first integrate by parts once or, more simply, introduce a small cutoff ϵ which we shall fix in a convenient way. In the following, the limit $\epsilon\to 0$ is understood. We can write Eq. (4.22) in the form

$$\mathrm{Im}\Pi_1(s)=\frac{1}{12\pi s^2}\left[\alpha A(s)\int_{4m^2}^{s-\epsilon}\frac{da\psi(s,a)}{(s-a)^{1-\alpha A(s)}}+\psi(s,s)\epsilon^{\alpha A(s)}\right], \qquad (4.24)$$

where, to obtain the second term of this equation, we have approximated $\psi(s,a)$ by $\psi(s,s)$ in the interval $[s-\epsilon,s]$. Let us write

$$\psi(s,a)=\psi^{(0)}(s,a)+\alpha\psi^{(2)}(s,a)+\cdots\,, \qquad (4.25)$$

where higher orders in α are not needed for $\mathrm{Im}\Pi$ up to fourth order. Equation (4.22) becomes

$$\mathrm{Im}\Pi_1(s)=\frac{1}{12\pi s^2}\left[\psi^{(0)}(s,s)+\alpha A(s)\int_{4m^2}^{s-\epsilon}\frac{da\psi^{(0)}(s,a)}{s-a}+\alpha\psi^{(2)}(s,s)+\alpha A(s)\psi^{(0)}(s,s)\ln\epsilon+O((\alpha A)^2)\right]. \qquad (4.26)$$

We note that, up to fourth order, $\psi^{(2)}$ appears only for $s=a$, which means that Eqs. (4.18) and (4.21) are equivalent up to this order.

From Eqs. (4.4) and (2.16), the expansion of $X(s)$ is

$$X(s)=(2m^2+s)+\alpha(2m^2+s)\times[-A(s)\ln\lambda e^{\gamma}+F(s)+2\,\mathrm{Re}F_1^{(2)}(s)+3sF_2^{(2)}(s)/2\pi(2m^2+s)]\,, \qquad (4.27)$$

where $F(s)$ is the function given in Eq. (3.30) and $F_1^{(2)}$ and $F_2^{(2)}$ are the lowest-order radiative corrections to the (proper) vertices,

$$F_1=1+\alpha F_1^{(2)}+\alpha^2F_1^{(4)}+\cdots\,, \qquad (4.28)$$

$$F_2=F_2^{(2)}+\alpha F_2^{(4)}+\cdots\,, \qquad (4.29)$$

with $F_1^{(2)}(0)=F_2^{(2)}(0)=1$ according to the normalization (2.14). The explicit expressions of $F_1^{(2)}$ and $F_2^{(2)}$ are given here for convenience:

$$\pi F_1^{(2)}(s)=-\ln\frac{\lambda}{m}\left(1-\frac{1+v^2}{2v}\ln\left|\frac{1+v}{1-v}\right|\right)-1+\frac{1+2v^2}{4v}\ln\left|\frac{1+v}{1-v}\right|$$
$$+\frac{1+v^2}{2v}\left[-\phi\left(-\frac{1-v}{1+v}\right)-\tfrac{1}{4}\ln^2\frac{1+v}{1-v}+\frac{\pi^2}{4}\theta(1-v)-\ln\left|\frac{1+v}{1-v}\right|\ln\frac{2v}{1+v}\right]$$
$$+i\pi\theta(s-4m^2)\left[\frac{1+v^2}{4v}\ln\frac{4m^2v^2}{\lambda^2(1-v^2)}-\frac{1+2v^2}{4v}\right]$$
$$=\pi B(s)-\tfrac{1}{2}\left[1-\frac{1+v^2}{2v}\ln\left|\frac{1+v}{1-v}\right|+i\pi\theta(s-4m^2)\frac{1+v^2}{2v}\right], \qquad (4.30)$$

$$F_2^{(2)}(s)=-\frac{(1-v^2)}{2v}\ln\left|\frac{1+v}{1-v}\right|+\frac{i\pi}{2v}(1-v^2)\theta(s-4m^2)\,. \qquad (4.31)$$

In these equations and in the following, the variable v is defined by [compare Eq. (3.17)]

$$v=\left(\frac{s-4m^2}{s}\right)^{1/2}. \qquad (4.32)$$

$B(s)$ is the standard YFS function which we shall use later. Note that when $0<s<4m^2$, the logarithms in Eqs. (4.30) and (4.31) have to be replaced by arctan functions and $\theta(1-v)$ is to be considered as zero. The Spence function ϕ is defined in Eq. (3.31). It is easy to verify that, using Eq. (3.16) for A and (4.30) and (4.31), $X(s)$ as given in Eq. (4.27) is free from infrared divergence. We note also that F_2 begins to be infrared divergent[15] only at order e^4. A straightforward expansion of Eq. (4.23) using (4.27) leads to

$$\psi^{(0)}(s,a)=[a(a-4m^2)]^{1/2}(2m^2+a)A(a)/A(s) \qquad (4.33)$$

and

$$\psi^{(2)}(s,s)=(2m^2+s)[s(s-4m^2)]^{1/2}[-A(s)\ln\lambda\sqrt{s}+F(s)+2\,\mathrm{Re}F_1^{(2)}(s)+3sF_2^{(2)}(s)/2\pi(2m^2+s)]\,. \qquad (4.34)$$

Choosing $\epsilon=\lambda\sqrt{s}$ in Eq. (4.26), this equation becomes

$$\mathrm{Im}\Pi_1(s)=\frac{v(3-v^2)}{24\pi}+\alpha\mathrm{Im}\Pi_1^{(2a)}+\alpha\mathrm{Im}\Pi^{(2b)}+O((\alpha A)^2)\,, \tag{4.35}$$

where we define

$$\mathrm{Im}\Pi_1^{(2a)}(s)=\frac{1}{12\pi s^2}\left\{\int_{4m^2}^{s-\lambda\sqrt{s}}\frac{da[a(a-4m^2)]^{1/2}(2m^2+a)A(a)}{s-a}+F(s)(2m^2+s)[s(s-4m^2)]^{1/2}\right\} \tag{4.36}$$

and

$$\begin{aligned}\mathrm{Im}\Pi^{(2b)}(s)&=\frac{v}{12\pi}\left[(3-v^2)\mathrm{Re}F_1^{(2)}(s)+\frac{3}{2\pi}\mathrm{Re}F_2^{(2)}(s)\right]\\&=\frac{1}{48\pi^2}\left\{-4v(3-v^2)\ln\frac{\lambda}{m}\left(1-\frac{1+v^2}{2v}\ln\frac{1+v}{1-v}\right)-4v(3-v^2)+2v^2(4-v^2)\ln\frac{1+v}{1-v}+2(3-v^2)(1+v^2)\right.\\&\qquad\left.\times\left[\frac{\pi^2}{4}-\phi\left(-\frac{1-v}{1+v}\right)-\tfrac{1}{4}\ln^2\frac{1+v}{1-v}-\ln\frac{1+v}{1-v}\ln\frac{2v}{1+v}\right]\right\}.\end{aligned} \tag{4.37}$$

The physical meaning of Eq. (4.35) is clear. The first term represents the contribution to ImΠ in lowest order. (Note that a factor α has been factored out in defining ImΠ as the imaginary part of vacuum polarization.) The second term, $\alpha\mathrm{Im}\Pi_1^{(2a)}$, represents the contribution of the intermediate state (e^+,e^-,γ) in lowest order of the peaking approximation. We have checked that Eq. (4.36) is also obtained using $\alpha I_1(p_+,p';k)$ given in Eq. (3.42) as the relative probability for one-photon emission, that is,

$$\mathrm{Im}\Pi_1^{(2a)}(s)=\frac{1}{6\pi^2 s}\int\frac{d^3p'}{2E'}\frac{d^3p_+}{2E_+}d^4k[(p_++p')^2+2m^2]I_1(p_+,p';k)\delta^4(q-p_+-p'-k)\,. \tag{4.38}$$

Finally, the third term, $\alpha\mathrm{Im}\Pi^{(2b)}$, represents the virtual-photon radiative corrections to the e^+e^- intermediate state. We note that Eq. (4.37) is not quite the same as Eq. (23) of Ref. 10, since these authors compute the improper vacuum polarization. Their equation is recovered by using the form factors instead of the vertex functions in Eq. (4.37).

The first term of Eq. (4.36) is computed in Appendix A. The (e^+,e^-,γ) intermediate-state contribution to ImΠ in the peaking approximation is found to be

$$\begin{aligned}\mathrm{Im}\Pi_1^{(2a)}(s)=\frac{1}{48\pi^2}\Bigg\{&2(1+v^2)(3-v^2)\left[\tfrac{1}{4}\ln^2\frac{1+v}{1-v}-\ln\frac{1+v}{1-v}\ln\frac{4v}{(1+v)^2}-2\phi\left(\frac{1-v}{1+v}\right)-3\phi\left(-\frac{1-v}{1+v}\right)-\frac{3\pi^2}{4}\right]\\&-4v(3-v^2)\left(\frac{1+v^2}{2v}\ln\frac{1+v}{1-v}-1\right)\ln\frac{\lambda}{m}-\tfrac{1}{2}\ln\frac{1+v}{1-v}[3v^4+6v^2-9+12v(3-v^2)]\\&+4v(3-v^2)\ln\frac{(1+v)^3}{8v^2}+v(27-5v^2)\Bigg\}\,,\end{aligned} \tag{4.39}$$

where v is defined in Eq. (4.32). Before discussing the high- and low-energy limits of the different contributions to ImΠ, we shall compute the hard contribution which is not included either in the soft or in the peaking approximations.

The hard contribution comes from the difference between $\langle 0|J_\mu(0)|p_+,p',k\rangle$ computed to lowest order and its soft-photon limit as defined in Eq. (2.4). It is expected to be small near threshold since the emitted photon is necessarily soft due to phase-space limitations. To lowest order in perturbation theory, the "bremsstrahlung amplitude" is given by

$$\langle 0|J_\mu(0)|p_+,p',k\rangle=e\bar{u}(p')\left[\frac{\not\epsilon(\not p'+\not k+m)\gamma^\mu}{(p'+k)^2-m^2}+\frac{\gamma^\mu(-\not p_+-\not k+m)\not\epsilon}{(p_++k)^2-m^2}\right]v(p_+)\,, \tag{4.40}$$

where $\not p=\gamma^\mu p_\mu$. With the help of the Dirac equation and the γ matrices algebra, Eq. (4.40) is brought to the form

$$\langle 0|J_\mu(0)|p_+,p',k\rangle=e\bar{u}(p')A_{\mu\rho}v(p_+)\epsilon^\rho\,, \tag{4.41}$$

where

$$A^{\mu\rho}=\gamma^\mu j^\rho+\frac{[\gamma^\rho,\not k]\gamma^\mu}{4k\cdot p'}+\frac{\gamma^\mu[\gamma^\rho,\not k]}{4k\cdot p_+}\,. \tag{4.42}$$

The first term of Eq. (4.42) is sometimes called the electric part of the current, the remainder being its magnetic part. The electric part squared gives, of course, the soft contribution of $\mathrm{Im}\Pi$. The magnetic part squared and the interference term will give the hard part of $\mathrm{Im}\Pi$. The sum over spins and polarizations of the square of Eq. (4.40) involves a trace over γ matrices, which gives

$$\sum_{\text{spins}}\sum_{\text{polarizations}}|\langle 0|J_\mu(0)|p_+,p',k\rangle|^2=8[-(p_+\cdot p'+2m^2)|j|^2+R]\,, \tag{4.43}$$

with

$$R=\left(\frac{k\cdot p_+}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p_+}\right)\left(1-\frac{m^2}{k\cdot p'}-\frac{m^2}{k\cdot p_+}\right)+2p_+\cdot p'\left(\frac{1}{k\cdot p'}+\frac{1}{k\cdot p_+}\right). \tag{4.44}$$

The remainder R gives the hard-photon contribution $\alpha\,\mathrm{Im}\Pi^{(2)}_{\text{hard}}$ to $\mathrm{Im}\Pi$. Comparing this with the normalization of Eq. (2.3) we get

$$\mathrm{Im}\Pi^{(2)}_{\text{hard}}=\frac{1}{6\pi^4 s}\int\frac{d^3p'}{2E'}\,\frac{d^3p_+}{2E_+}\,\frac{d^3k}{2w}\,\delta^4(q-p_+-p'-k)R\,. \tag{4.45}$$

There is no need for a photon mass here since the integral is infrared finite. This phase-space integral poses no problem and can even be expressed in terms of elementary functions. Introducing the integration variable $a=(p_++p')^2$, we get

$$\begin{aligned}\mathrm{Im}\Pi^{(2)}_{\text{hard}}&=\frac{1}{12\pi^2 s^2}\int_{4m^2}^{s}da\left[(s+a-4m^2)\ln\frac{1+(a-4m^2/a)^{1/2}}{1-(a-4m^2/a)^{1/2}}-(s+a)\left(\frac{a-4m^2}{a}\right)^{1/2}\right]\\&=\frac{1}{48\pi^2}\left\{[6-4(1-v^2)+\tfrac{7}{4}(1-v^2)^2]\ln\frac{1+v}{1-v}-11v+\tfrac{7}{2}v(1-v^2)\right\},\end{aligned} \tag{4.46}$$

where the second form has been obtained using the same method as in Appendix A. By adding Eq. (4.46) to Eq. (4.39), we obtain

$$\begin{aligned}\mathrm{Im}\Pi^{(2a)}\equiv\mathrm{Im}\Pi^{(2a)}_1+\mathrm{Im}\Pi^{(2)}_{\text{hard}}=\frac{1}{48\pi^2}\Bigg\{&2(1+v^2)(3-v^2)\left[\tfrac{1}{4}\ln^2\frac{1+v}{1-v}-\ln\frac{1+v}{1-v}\ln\frac{4v}{(1+v)^2}-2\phi\left(\frac{1-v}{1+v}\right)\right.\\&\left.-3\phi\left(-\frac{1-v}{1+v}\right)-\frac{3\pi^2}{4}\right]\\&-4v(3-v^2)\left(\frac{1+v^2}{2v}\ln\frac{1+v}{1-v}-1\right)\ln\frac{\lambda}{m}+\ln\frac{1+v}{1-v}[\tfrac{1}{4}(v^4-10v^2+33)-6v(3-v^2)]\\&+\frac{v}{2}(39-17v^2)+4v(3-v^2)\ln\frac{(1+v)^3}{8v^2}\Bigg\}.\end{aligned} \tag{4.47}$$

The most important fact about this result is the identity of Eq. (4.47) with Eq. (48) of Ref. 10, which gives the full fourth order contribution of the (e^+,e^-,γ) intermediate state. Remembering that $\mathrm{Im}\Pi_1$ is the peaking contribution to $\mathrm{Im}\Pi$ and the definition of $\mathrm{Im}\Pi^{(2)}_{\text{hard}}$, Eqs. (4.43) and (4.45), we conclude that the soft and peaking approximations of $\mathrm{Im}\Pi$ coincide at order e^4. In other words, we get the same result if the relative probability of a soft-photon emission, which is $-\alpha(2\pi^2)^{-1}|j|^2\delta_+(k^2-\lambda^2)$, is replaced by $\alpha I_1(k)$, given in Eq. (3.42). We have indeed checked that the difference

$$\begin{aligned}\alpha\,\mathrm{Im}\Pi^{(2a)}_{\text{soft}}-\alpha\,\mathrm{Im}\Pi^{(2a)}_1=\frac{1}{6\pi^2 s}\int\frac{d^3p'}{2E'}\,\frac{d^3p_+}{2E_+}\,&d^4k[(p_++p')^2+2m^2]\delta^4(q-p_+-p'-k)\\&\times[-\alpha(2\pi^2)^{-1}|j|^2\delta_+(k^2-\lambda^2)-\alpha I_1(p_+,p';k)]\end{aligned} \tag{4.48}$$

vanishes. [The limit $\lambda\to 0$, implicit in Eq. (4.48), is not uniform and must be taken carefully after the relevant integrations are done. The difficult part of the calculation is in fact the soft part, dealt with in Ref. 10.]

Let us discuss the result thus far obtained.

(i) From Eq. (4.46), the threshold and high-energy behavior of $\mathrm{Im}\Pi^{(2)}_{\text{hard}}$ are given by

$$\mathrm{Im}\Pi^{(2)}_{\text{hard}}\underset{v\to 0}{\sim}\frac{v^5}{9\pi^2}+O(v^7)\,, \tag{4.49}$$

$$\mathrm{Im}\Pi^{(2)}_{\mathrm{hard}} \underset{s\to\infty}{\sim} \frac{1}{8\pi^2}\left[\ln\frac{s}{m^2}+O(1)\right]. \tag{4.50}$$

(ii) From Eq. (4.47), apart from the term proportional to $\ln(\lambda/m)$, the total contribution of the (e^+, e^-, γ) intermediate state behaves as

$$\mathrm{Im}\Pi^{(2a)} \underset{v\to 0}{\sim} -\frac{2}{3\pi^2}\left[v\ln v+O(v)\right], \tag{4.51}$$

$$\mathrm{Im}\Pi^{(2a)} \underset{s\to\infty}{\sim} \frac{1}{24\pi^2}\left[\ln^2\frac{s}{m^2}-3\ln\frac{s}{m^2}+O(1)\right]. \tag{4.52}$$

(iii) From Eq. (4.37), the e^+e^- contribution behaves like

$$\mathrm{Im}\Pi^{(2b)} \underset{v\to 0}{\sim} \frac{1}{8\pi^2}\left[\frac{\pi^2}{2}-4v\ln v+O(v)\right], \tag{4.53}$$

$$\mathrm{Im}\Pi^{(2b)} \underset{s\to\infty}{\sim} \frac{1}{24\pi^2}\left[-\ln^2\frac{s}{m^2}+3\ln\frac{s}{m^2}+O(1)\right]. \tag{4.54}$$

(iv) By adding Eqs. (4.37) and (4.47), we obtain the total contribution to the imaginary part of vacuum polarization to order e^4,

$$\mathrm{Im}\Pi^{(2)}=\frac{1}{48\pi^2}\left\{-2(1+v^2)(3-v^2)\left[2\phi\left(\frac{1-v}{1+v}\right)+4\phi\left(-\frac{1-v}{1+v}\right)+\frac{\pi^2}{2}+\ln\frac{1+v}{1-v}\ln\frac{8v^2}{(1+v)^3}\right]\right.$$
$$\left.-4v(3-v^2)\ln\frac{8v^2}{(1+v)^3}+\tfrac{1}{4}\left[-7v^4+22v^2+33-24v(3-v^2)\right]\ln\frac{1+v}{1-v}+\frac{v}{2}(15-9v^2)\right\}. \tag{4.55}$$

Adding the lowest-order contribution, the first term of Eq. (4.35), the imaginary part of vacuum polarization, up to fourth order, behaves like

$$\mathrm{Im}\Pi \underset{v\to 0}{\sim} \frac{v}{24\pi}\left(1+\frac{3\pi\alpha}{2v}+\cdots\right) \tag{4.56}$$

and

$$\mathrm{Im}\Pi \underset{s\to\infty}{\sim} \frac{1}{12\pi}\left(1+\frac{3\alpha}{4\pi}+\cdots\right). \tag{4.57}$$

The last equation coincides with the known value of $\mathrm{Im}\Pi$ deduced, for example, from the renormalization group.[12,13] The threshold behavior of $\mathrm{Im}\Pi$ is dominated by the e^+e^- contribution, Eq. (4.53). This point is discussed further in the next subsection.

By comparing Eqs. (4.49)–(4.57), we can learn many interesting facts. Although $\mathrm{Im}\Pi_{\mathrm{hard}}$ was expected to be small near threshold, it is surprising to see how small it is compared to $\mathrm{Im}\Pi$. We are also happy to find that the high-energy behavior of $\mathrm{Im}\Pi^{(2)}_{\mathrm{hard}}$ varies as $\ln s$ whereas the peaking or soft approximation varies as $\ln^2 s$. However, if we compare $\mathrm{Im}\Pi^{(2)}_{\mathrm{hard}}$ to the total $\mathrm{Im}\Pi$, which varies as a constant, the situation may seem less optimistic. However, the nonperturbative formula (4.18) depends on $X(a)$, that is, on the vertices which must be also computed nonperturbatively. It may be possible to choose the soft-part (or peaking) approximation of the F_i, which also exponentiates, in such a way as to guarantee a constant $\mathrm{Im}\Pi$ up to fourth order, when the hard parts of real and virtual radiative corrections are neglected.

C. Virtual-photon radiative corrections and threshold behavior of ImΠ

An extensive study of virtual-photon radiative corrections, analogous to real-photon emission, which we have just presented, is beyond the scope of this paper and will be discussed in a future publication.[25] Here we shall concentrate first on those aspects of radiative corrections relevant to the threshold behavior of the vertices and then propose a relativistic generalization. As we have seen, the whole e^+, e^- intermediate-state contribution appears through $X(a)$ given in Eq. (4.4). From the YFS analysis[5,6] of the radiative corrections to all order in perturbation theory, the vertices F_1 and F_2 are shown to be of the form

$$F_i(a)=e^{\alpha B(a)}\mathfrak{F}_i(a),\quad i=1,2 \tag{4.58}$$

where the $\mathfrak{F}_i$ are free from infrared divergence and B is the standard YFS infrared-divergent function

$$B((p'+p_+)^2)=\frac{i}{(2\pi)^3}\int\frac{d^4k}{k^2-\lambda^2+i\epsilon}\times\left(\frac{2p'-k}{k^2-2p'\cdot k+i\epsilon}+\frac{2p_++k}{k^2+2k\cdot p_++i\epsilon}\right)^2. \tag{4.59}$$

The explicit expression of the B function was given in Eq. (4.30). In particular, the threshold behavior of B is

$$B \underset{v\to 0}{\sim} \frac{\pi}{4v} - \frac{1}{2\pi} + i\theta(s-4m^2)\left(\frac{1}{2v}\ln\frac{2mv}{\lambda}\right), \quad (4.60)$$

which implies that if (4.58) is used in Eq. (4.18), $\mathfrak{F}_i(a)$ being computed perturbatively, the integral in the latter equation will badly diverge at threshold. In fact, there is no physical reason for the exponentiation of the threshold behavior of Eq. (4.60). In perturbation theory, the imaginary parts of F_1 and F_2 have been computed analytically[15] up to order α^2. The threshold singularities, with normalization of Eq. (2.14), are

$$\mathrm{Im}F_1 \underset{v\to 0}{\sim} \frac{\alpha}{4v}\left(\ln\frac{4m^2v^2}{\lambda^2} - 1\right) + \frac{\pi\alpha^2}{16v^2}\ln\frac{4m^2v^2}{\lambda^2} + \cdots, \quad (4.61a)$$

$$\mathrm{Im}F_2 \underset{v\to 0}{\sim} \frac{\pi}{2v} + \frac{\alpha\pi^2}{8v^2} + \cdots. \quad (4.61b)$$

It is clear that the threshold behavior of the fourth-order terms does not allow a standard dispersion relation to compute the real part as discussed by the authors of Ref. 15. A similar increase of the singularities with the order, near threshold, is expected also for the real parts of the F_i.

The origin of the threshold singularities is the long range of the Coulomb potential and more particularly, the existence of the positronium bound states just below threshold. In fact, perturbation theory breaks down when $\alpha/v \gg 1$, and none of Eqs. (4.58), (4.60), or (4.61) are reliable in this domain. Fortunately, nonrelativistic (NR) quantum mechanics applies in the domain $v \ll 1$ and gives the nonperturbative behavior of the combination

$$F_e \equiv F_1 + (\alpha/2\pi)F_2. \quad (4.62)$$

The fact that F_e is the relevant vertex function in the NR limit can be shown by expressing $X(a)$ in terms of F_e and F_2,

$$X(a) = (\lambda e^{\gamma})^{-\alpha A} e^{\alpha F(r)}\left[|F_e|^2(a+2m^2) + \frac{\alpha}{2\pi}(a-4m^2)\mathrm{Re}(F_e F_2^*) + \left(\frac{\alpha}{4\pi m}\right)^2\frac{(a-4m^2)^2}{2}|F_2|^2\right], \quad (4.63)$$

and noting that only the term proportional to $|F_e|^2$ contributes for $v \ll 1$. The vertex function for two spinless, oppositely charged particles with relative orbital momentum $l=0$ is computed in Appendix B and reads

$$F_0 = \Gamma(1-i\nu)\exp\left(\frac{\pi\nu}{2} + i\nu\ln 2\kappa R\right), \quad (4.64)$$

where

$$\nu \equiv \frac{\mu\alpha}{\kappa} = \frac{\alpha}{v_{\mathrm{rel}}} = \frac{\alpha}{2v} \simeq \frac{\alpha m}{(a-4m^2)^{1/2}} \quad (4.65)$$

and R is a large cutoff radius of the Coulomb potential which will be related to the photon mass. Let us note that F_0 contains the divergent (in the limit $R\to\infty$) Coulomb phase factor as well as the poles corresponding to the nonrelativistic positronium bound states.

We shall demonstrate that the perturbation expansion of F_0 coincides with the NR limit of F_e known only from perturbation theory. We expand Eq. (4.64) in powers of α. This will be valid in the domain where $v \ll 1$ and also $\nu = \alpha/2v \ll 1$, for example, for $v \sim 10^{-1}$.

Using the known expansion of the Γ function

$$\Gamma(1+z) = 1 - \gamma z + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right)z^2 + O(z^3), \quad (4.66)$$

where γ is again Euler's constant, one gets

$$F_{0\ \mathrm{perturbation}} \sim 1 + \frac{\pi\alpha}{4v} - \frac{\alpha^2}{4v^2}\left[\tfrac{1}{2}\ln^2(2mvRe^{\gamma}) - \frac{\pi^2}{24}\right] + i\left(\frac{\alpha}{2v} + \frac{\pi\alpha^2}{8v^2}\right)\ln(2mvRe^{\gamma}) + O(\nu^3). \quad (4.67)$$

Using the definition (4.62) together with the known perturbative results of F_1 and F_2 taken in the NR limits [Eqs. (4.61), (4.29), and (4.30)] we obtain

$$F_{e\ \mathrm{perturbation}} \underset{v\ll 1}{\sim} 1 + \frac{\pi\alpha}{4v} + i\left(\frac{\alpha}{2v} + \frac{\pi\alpha^2}{8v^2}\right)\ln\frac{2mv}{\lambda} + O\left(\frac{\alpha^2}{v^2}\right). \quad (4.68)$$

We have not attempted to compute the α^2/v^2 term, if any, in the imaginary part of Eq. (4.68) which can in principle be obtained from the analytical expressions given in Ref. 15. However, the cancellation of the α/v terms in the imaginary part of F_e between Eqs. (4.61a) and (4.61b) confirms that F_e is the right combination to compare with F_0. A glance at the last two equations shows that the known terms in Eq. (4.68) coincide with the corresponding ones in Eq. (4.67) provided the following identification is made:

$$Re^{\gamma} = \lambda^{-1}. \quad (4.69)$$

This shows that F_0 is the exact NR limit of F_e,

$$F_e^{\mathrm{NR}} = F_0 = \Gamma\left(1 - \frac{i\alpha}{2v}\right) \times \exp\left(\frac{\pi\alpha}{4v} + \frac{i\alpha}{2v}\ln\frac{2mv}{\lambda e^{\gamma}}\right). \quad (4.70)$$

This result can be used to discuss the nonperturbative threshold behavior of F_e where perturbation theory breaks down. Very close to threshold, $\alpha/v \gg 1$, we can use the Stirling formula to obtain

$$F_e \underset{\alpha/v \gg 1}{\sim} \sqrt{2\pi}\left(\frac{\alpha}{2v}\right)^{1/2} \exp\left[-\frac{i\alpha}{2v}\ln\left(\frac{\alpha m}{\lambda e^{\gamma}}\right)\right], \quad (4.71)$$

a result clearly nonanalytic in α.

We can discuss the threshold behavior of $\mathrm{Im}\Pi$. We note from Eq. (4.70) that, since only the phase of F_e is infrared divergent, the e^+e^- contribution to $\mathrm{Im}\Pi$ is *infrared convergent by itself* in the NR domain. The reason is that the exponent of λ, αA, is very small in the NR domain

$$\alpha A \underset{v\to 0}{\sim} \frac{8\alpha v^2}{3\pi}. \quad (4.72)$$

Thus we conclude that when intermediate states with photons begin to contribute, the NR limit, Eq. (4.70), ceases to be valid. We can, however, use this equation to obtain the exact threshold behavior of $\mathrm{Im}\Pi$. Since, in the NR domain, αA is to be considered as negligible, $\Gamma^{-1}(\alpha A)(s-a)^{\alpha A-1}$ in Eq. (4.18) is equivalent to $\delta(s-a)$. In other words, we get the exact threshold behavior by multiplying the lowest order by $|F_e|^2$. This gives [compare the first term of Eq. (4.35)]

$$\mathrm{Im}\Pi(s) \underset{v \ll 1}{\sim} \frac{v}{8}|F_e|^2 = \frac{\alpha}{8(1-e^{-\pi\alpha/v})}, \quad (4.73)$$

where, to obtain the second form, we have made use of Eq. (4.70) and of the relation[26]

$$\Gamma(1+i\nu)\Gamma(1-i\nu) = \frac{2\pi\nu e^{\pi\nu}}{e^{2\pi\nu}-1}. \quad (4.74)$$

A similar discussion of $\mathrm{Im}\Pi$ has been given by Barbieri *et al.*[15] and by Schwinger,[11] where $|\psi(0)|^2 = |F_e|^2$, $\psi(0)$ being the wave function of relative motion computed at the origin in configuration space. Note that Eq. (4.73) predicts the nonperturbative result

$$\mathrm{Im}\Pi(4m^2) = \alpha/8, \quad (4.75)$$

which is probably modified by contributions of intermediate states below the $s=4m^2$ threshold, three-photon intermediate state, for example, and so on.

Once the threshold problem is solved, the next step is to obtain a relativistic extension of the vertices which keeps the exact threshold behavior and contains the correct infrared factor. We shall suggest a simple relativistic generalization of Eq. (4.70). Comparing the power of λ in Eq. (4.70) and the coefficient of $\ln(\lambda/m)$ in Eq. (4.30), one must have

$$\frac{i\alpha}{2v} \xrightarrow[\text{relativistic}]{} -\alpha\bar{A}_1 \equiv \frac{\alpha}{\pi}\left(1-\frac{1+v^2}{2v}\ln\frac{v+1}{v-1}\right). \quad (4.76)$$

The real part of the analytic function $\bar{A}_1$ is recognized as the function $\bar{A}=A/2$ which is negligible in the nonrelativistic domain [cf., Eq. (4.72)] while its imaginary part, $(1+v^2)/2v$, goes to $1/2v$ as $v\to 0$. From Eq. (4.70), we note that the argument of the exponential is the nonrelativistic limit of $\alpha(B+1/2\pi)$ as it is seen from Eq. (4.60). The relativistic generalization of Eq. (4.70), which we call $F_{e\,\text{soft}}$, is thus

$$F_{e\,\text{soft}} = e^{\alpha\bar{A}_1\gamma}\Gamma(1+\alpha A_1)e^{\alpha(B+1/2\pi)} \quad (4.77)$$

and $F_e(0)=1+\alpha/2\pi+O(\alpha^2)$ as required by the normalization of Eq. (2.14). This form of the electric vertex F_e has the correct infrared factor and, of course, the exact threshold behavior. The poles of the Γ function in Eq. (4.77) gives the positronium bound-state energies with relativistic corrections.[27,28] Furthermore, the ultraviolet behavior of F_e agrees with the exponentiation of the leading "ln" found by many authors,[14]

$$F_1 \underset{a\to-\infty}{\sim} \exp\left[-\frac{\alpha}{4\pi}\ln^2\left(-\frac{a}{m^2}\right)\right], \quad (4.78)$$

if F_2 is asymptotically negligible with respect to F_1.

A similar study of the vertex function F_2 involves the electron-positron spins and will not be undertaken here. As a guess, we propose

$$F_{2\,\text{soft}} = F_2^{(2)}e^{\alpha\gamma\bar{A}_1}\Gamma(1+\alpha\bar{A}_1)e^{\alpha(B+1/2\pi)}, \quad (4.79)$$

where $F_2^{(2)}$ is given in Eq. (4.31).

Let us conclude this discussion by noting that from Eqs. (4.77), (4.30), and (4.31), the vertex F_e can be written as

$$F_e = e^{\alpha\gamma\bar{A}_1}\Gamma(1+\alpha\bar{A}_1)e^{\alpha(B+1/2\pi)} \times\left[1+\frac{\alpha}{\pi}\left(\frac{v}{2}\ln\frac{v+1}{v-1}-1\right)+O(\alpha^2)\right], \quad (4.80)$$

where the "hard" correction is seen to be regular at the threshold $v=0$.

We shall end this paper by showing the limitations of the nonperturbative formula for $\mathrm{Im}\Pi$. Instead of computing its (super) high-energy behavior, we shall write down a formal expression for the real part $\Pi(0)$. [A similar computation is possible for all the derivatives $\Pi^{(p)}(0)$.] Using the representation (4.15), we get, after interchanging the order of integrations,

$$\begin{aligned}\Pi_1(0) &= \frac{1}{6\pi^2}\int_{4m^2}^{\infty} da[a(a-4m^2)]^{1/2}X(a)\Gamma^{-2}(\alpha\bar{A}) \times\int_0^{\infty} db\, b^{\alpha\bar{A}-1} \times\int_{(\sqrt{b}+\sqrt{a})^2}^{\infty}\frac{ds'}{s'^3[\Delta(s',a,b)]^{1/2}}\\ &= \frac{1}{6\pi^2}\int_{4m^2}^{\infty} da[a(a-4m^2)]^{1/2}X(a)K(a).\end{aligned} \quad (4.81)$$

where $K(a)$ results from the elementary s' integration,

$$K(a)=\frac{\Gamma^{-2}(\alpha\bar{A})}{2}\int_0^\infty \frac{db\, b^{\alpha\bar{A}-1}}{(a-b)^5}\times[(a^2+b^2+4ab)\ln(a/b)-3(a^2-b^2)]\,. \qquad (4.82)$$

Note that the integrand is regular at $a=b$, as one may easily check. The function $K(a)$ is computable in closed form. We have used the trick to compute the separate terms for $a'=-a>0$ and to continue analytically the result thus obtained. We get the very simple result

$$K(a)=\frac{a^{\alpha\bar{A}-3}}{8}\Gamma^2(3-\alpha\bar{A}) \qquad (4.83)$$

and

$$\Pi_1(0)=\frac{1}{48\pi^2}\int_{4m^2}^\infty da\,\frac{[a(a-4m^2)]^{1/2}}{a^{3-\alpha\bar{A}(a)}}X(a)\Gamma^2(3-\alpha\bar{A}) \qquad (4.84)$$

and $X(a)$ is given in (4.63). The salient feature of the equations is the appearance of the double poles at $\alpha\bar{A}=3,4,\ldots,$ whose origin can be traced to the divergence of the b integral in Eq. (4.82) for superhigh energy. Thus our formula for ImΠ ceases to be valid at energies such that $\alpha\bar{A}\sim 1$, where hard-photon contributions and/or pairs of charged particles will come into play.

The formalism we have presented will be the starting point for the computation of radiative corrections in $e+p$ scattering dealt with in a forthcoming article.[9]

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APPENDIX A: CONTRIBUTION OF $(e^+e^-\gamma)$ IN THE PEAKING APPROXIMATION, Im$\Pi^{(2a)}$

Let us compute the first term of Eq. (4.36)

$$\mathrm{Im}\Pi_1'(s)=\frac{1}{12\pi s^2}\times\int_{4m^2}^{s-\lambda\sqrt{s}}\frac{da[a(a-4m^2)]^{1/2}(a+2m^2)A(a)}{s-a}, \qquad (A1)$$

where A is given in (3.16). The change of variable [compare Eq. (3.32)]

$$a=\frac{m^2(1+r)^2}{r} \qquad (A2)$$

brings Eq. (A1) to the form

$$\mathrm{Im}\Pi_1'(s)=-\frac{m^4}{6\pi^2 s^2}\times\int_{x_\lambda}^1\frac{dr}{r^3}\,\frac{(1+4r+r^2)[(1-r^2)^2+(1-r^4)\ln r]}{(r-x)(x^{-1}-r)}, \qquad (A3)$$

where

$$s=\frac{m^2(1+x)^2}{x},\quad x=\frac{1-(s-4m^2/s)^{1/2}}{1+(s-4m^2/s)^{1/2}}, \qquad (A4)$$

and

$$x_\lambda=x\left[1+\frac{\lambda}{(s-4m^2)^{1/2}}\right]. \qquad (A5)$$

We can simplify Eq. (A3) somewhat by changing $r\to 1/r$ in some terms. We get

$$\mathrm{Im}\Pi_1'=\frac{m^4}{6\pi^2 s^2}\int_{x_\lambda}^{1/x_\lambda}\frac{dr(1+4r+r^2)(r\ln r-r+1/r)}{(r-x)(x^{-1}-r)}. \qquad (A6)$$

Only the following terms lead to Spence functions and are computed using the appendix of Ref. 15:

$$\int_{x_\lambda}^{1/x_\lambda}\frac{dr\ln r}{r-1/x}=\int_{x_\lambda}^{1/x_\lambda}\frac{dr\ln r}{r-x}=\mathrm{Li}_2(1-x^2)-\ln x\ln\frac{\lambda\sqrt{x}}{m(1-x)(1-x^2)}=2\phi(x)+2\phi(-x)+\frac{\pi^2}{2}-\tfrac{1}{2}\ln^2 x-\ln x\ln(1+x)-\ln\frac{\lambda}{m}\ln x. \qquad (A7)$$

After a tedious computation of the elementary terms of Eq. (A6), we get

$$\mathrm{Im}\Pi_1'=\frac{1}{6\pi^2(1+x)^4}\Big\{2(1+x^2)(1+4x+x^2)\left[\tfrac{1}{4}\ln^2x+\tfrac{1}{2}\ln x\ln(1+x)-\phi(x)-\phi(-x)-\frac{\pi^2}{4}\right]-(1-x^2)(1+4x+x^2)\left(-\frac{(1+x^2)}{1-x^2}\ln x-1\right)\ln\frac{\lambda}{m}+\ln x\left[\frac{1+4x^2+x^4}{2}+\tfrac{3}{2}(1-x^2)(1+4x+x^2)\right]-(1-x^2)(1+4x+x^2)\ln(1-x^2)(1-x)+(1-x^2)[\tfrac{3}{4}(1+x^2)+2(1+x^2+4x)]\Big\}. \qquad (A8)$$

By adding the second term of Eq. (4.36), where $F(s)$ is given in Eq. (3.30) with $r \to x$, we get Eq. (4.39).

APPENDIX B: NONRELATIVISTIC VERTEX FUNCTION FOR SPINLESS PARTICLES

Let us assume that the vertex function for two spinless particles with opposite charges results from the exchange of Coulomb photons described by the cutoff Coulomb potential

$$V(r) = -\frac{\alpha}{r}\,\theta(R-r)\,, \tag{B1}$$

where R is a large cutoff which we will relate to the photon mass. Although the solution of this problem is well known to many readers, we shall present here the main steps of the derivation for completeness. The radial Schrödinger equation for relative orbital momentum l is

$$y_l'' + \left[\kappa^2 - 2\mu V(r) - \frac{l(l+1)}{r^2}\right] y_l(r) = 0\,, \quad y_l = r\psi_l\,, \tag{B2}$$

where $\kappa = \mu v_{\rm rel}$ is the relative momentum, $\mu = m/2$ is the relative mass, and $v_{\rm rel}$ is the relative velocity,

$$v_{\rm rel} \sim 2v \sim \frac{(a-4m^2)^{1/2}}{m}\,, \tag{B3}$$

a being the invariant mass squared of the electron-positron pair. The solution of Eq. (B2), regular at the origin, continuous as well as its first derivative at $r = R$ is

$$\begin{aligned} y_l &= y_{l,\,\rm Coulomb} \quad \text{for } r \le R\,, \\ y_l &= \sin(\kappa r - \tfrac{1}{2} l\pi + \delta_l) \quad \text{for } r \ge R\,, \end{aligned} \tag{B4}$$

where $y_{l,\,\rm Coulomb}$ is the regular solution of Eq. (B2) for the Coulomb potential $-\alpha/r$, and δ_l is the total phase shift

$$\delta_l \equiv \sigma_l + \nu \ln(2\kappa R) = \operatorname{Arg}\Gamma(l+1-i\nu) + \nu\ln(2\kappa R)\,, \tag{B5}$$

where

$$\nu = \frac{\mu\alpha}{\kappa} \sim \frac{\alpha m}{(a-4m^2)^{1/2}}\,. \tag{B6}$$

The Jost function $D_l(K^2) \equiv f_l(-K)$ is defined by

$$D_l(K^2) = e^{-i\delta_l} \lim_{r\to 0} y_{\rm free}(r)/y(r)\,. \tag{B7}$$

Using the known[26] behavior of $y_{\rm Coulomb}$ near the origin, one gets

$$D_l = e^{-i\delta_l}(c_l)_{\nu=0}/c_l\,, \tag{B8}$$

where the normalization constants c_l are given by

$$c_l = \frac{2^l}{(2l+1)!}\,|\Gamma(l+1-i\nu)|\,e^{\pi\nu}\,. \tag{B9}$$

Equation (B7) now reads

$$D_l(\kappa^2) = l!\,e^{-i\nu\ln 2\kappa r - \pi\nu/2}\,\Gamma^{-1}(l+1-i\nu)\,. \tag{B10}$$

From the analyticity properties of D_l and the vertex functions F_l, it is known that $F_l = D_l^{-1}$, and, in particular,

$$F_0 = D_0^{-1} = e^{i\nu\ln 2\kappa r}\,e^{\pi\nu/2}\,\Gamma(1-i\nu)\,. \tag{B11}$$

Generalization of this result to include spins is possible. We shall not attempt to do it here. We will simply identify F_0 as the nonrelativistic limit of the combination F_e, Eq. (4.62).

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*On leave of absence from Laboratoire de Physique Corpusculaire, Collège de France, Paris 5e.

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