PHYSICAL REVIEW D VOLUME 18, NUMBER 12 15 DECEMBER 1978

Confinement through tensor gauge fields

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(Received 17 March 1978)

Using the O(3,2)-symmetric de Sitter solution of Einstein's equation describing a stongly interacting tensor field we show that hadronic bags confining quarks can be represented as de Sitter "microuniverses" with radii given by $1/R^2 = \lambda\kappa^2/6$. Here κ^2 and λ are the strong-coupling and the "cosmological" constants which appear in the Einstein equation used. Surprisingly the energy spectrum for the two-body hadronic states is essentially the same as that for a harmonic-oscillator potential, though the wave functions are completely different. The Einstein equation can be extended to include color for the tensor fields.

I. INTRODUCTION

Quarks may or may not be exactly confined. It appears certain, however, that if they do exist, whether fractionally or integrally charged, they are at least partially confined in the sense of exhibiting the Archimedes effect: They are light inside a hadron and heavy outside. (Exact confinement would correspond to infinite mass for quarks outside the hadron.)

Much effort has been devoted to the elucidation of a confinement mechanism employing colored vector gluons and vector color dynamics (or VCD) and some success has been reported. Recently, there have been attempts to do this with scalar[1] fields (SCD). Here we wish to report on our efforts to explain confinement by means of tensor fields[2] (TCD). In this approach the basic forces supposedly responsible for the confinement of quarks are carried by a set of strongly interacting colored tensor fields which satisfy equations of motion similar to the Einstein equations for the gravitational field.

There is a simple reason for believing that Einstein-type equations may be relevant to the confinement question: One of the simplest and most symmetrical solutions of the Einstein equations in the absence of source terms, but with a cosmological parameter, is the de Sitter solution. This solution is invariant with respect to a group of orthogonal transformations, either O(3, 2) or O(4, 1) depending on the sign of the cosmological parameter. Geometrically the solution describes a "closed" universe with constant curvature. *The size of this universe as well as the gravitational pressure and density are determined by the cosmological parameter.* The motion of a quantized scalar test particle in the de Sitter universe can be found by solving the appropriate Klein-Gordon equation. In the case of O(3, 2) symmetry *it is found that the energy levels of this system are discrete and integrally spaced in units of the inverse radius of the universe*. [The O(4, 1) alternative is not so easily interpreted and we shall not consider it here.] Now the idea is to scale the de Sitter universe down (by appropriately choosing the parameters in the Einstein equations for a strongly interacting tensor field) to the size of a hadron and look upon it as a solitonlike solution to the equations of motion. Such a microuniverse is embedded in ordinary, flat spacetime and represents a kind of bag in which colored objects are contained.

Of course, in any realistic approximation to the strong-interaction dynamics one would have to take account of the presence of quarks and their action as *sources* for the strong tensor field. The de Sitter microuniverse referred to above was a matter-source-free solution of the self-interacting nonlinear tensor-field equations. There is of course a constant "density" of the tensor-field matter represented by the cosmological term. However, if, in addition, quark matter can be realistically concentrated at a point, then another solution becomes available: The Schwarzschild–de Sitter solution. In this case the solution has a $1/r$ singularity representing the source concentration, but it approaches the de Sitter form as the distance from the singularity is increased. The strength of the $1/r$ singularity is measured by a mass-like parameter μ representing a kind of "mechanical" mass to be associated with quarks. If μ is small compared with the inverse de Sitter radius (so that there are no event horizons, see the remarks at the end of Sec. V), one may then reasonably consider the $\mu = 0$ case as a zeroth-order approximation and perturbatively compute the deviations in energy levels as a power series in μ. In the long run, of course, one would hope to be able to avoid this $1/r$ singularity by treating the quark degrees of freedom dynamically. For a charged or a rotating source quark there are the additional $1/r^2$ singularities [see Sec. III, Eq. (3.15 and Sec. IV, Eq. (4.18)].

In the crude approximation adopted here the hadron is represented as a de Sitter–Schwarzschild microuniverse, with a $1/r$ singularity at the center representing a "source" quark (or a diquark). It is a kind of hadronic bag. Into this world one can think of putting an additional quark—treated as a test particle in that its contribution to the source is ignored—in order to complete the hadronic structure. For the case of scalar test and source quarks, the discrete energy spectrum obtained by solving the Klein-Gordon equation will be interpreted as a prototype mass spectrum for the excited hadron of integer spin. The spectrum is essentially the same as that of the harmonic oscillator, with discrete eigenstates, no continuum, and thus no dissociation.

One may endow the test quark with spin and solve the energy eigenvalue problem for a Dirac equation in the de Sitter–Schwarzschild background. This system would represent half-integer-spin hadrons. One may be more ambitious and instead of a de Sitter–Schwarzschild solution use a de Sitter–Kerr background—the Kerr terms representing a spinning source. This would provide spin-orbit and spin-spin coupling terms in the potential.

So much for the isolated hadron: A de Sitter-type microuniverse consisting of a source quark (or diquark) and a test quark. We must now attempt to describe its exterior and, in particular, the interaction between two such microuniverses. In other words, we must embed the microuniverses in physical spacetime and define, for example, their gravitational masses. This necessitates considering the gravitational field $g_{\mu\nu}$ in addition to the strong tensor field which we call $f_{\mu\nu}$. Such a complex has been formulated in the past in the f-g theory.[3] We shall find that the equations of this kind of model help to justify the assumed de Sitter form of the $f_{\mu\nu}$ microuniverse *in that exact solutions to the coupled f-g equations can be obtained.* Gravity occupies a special position among the external forces which can probe a microuniverse, since it is described by a tensor field which can mix with the strong tensor $f_{\mu\nu}$. Other probing forces, such as electromagnetism or weak interactions, may be made to act in the standard fashion in accord with the usual gauge principles, but we have not gone into this question. In addition one may also have strong gluons which may also be confined, through the agency of the tensor field $f_{\mu\nu}$.

The main problem for a realistic description of hadrons, which must be solved, concerns the incorporation of color. We need to make a distinction between quarks and antiquarks and also introduce the observed color selectivity into the physical hadronic states. This, as suggested earlier, may be accomplished by going over to a *vierbein* description of the strong field.[3] The gravitational symmetry, as extended by Weyl for the treatment of fermions, includes a group of local SL(2, *C*) transformations. This local group can easily be enlarged to include a color symmetry, SL(2, *C*) $\to$ SL(2, *C*)$\times$SU(3) $\to$ SL(6, *C*), etc. Instead of a vierbein field L_μ^a ($a=1,2,3,4$) of basic vectors, one takes an n-bein field L_μ^{ai}, where i is a color label.

To summarize, there are four aspects to the work reported here:

(1) A study of the Klein-Gordon equation in de Sitter space including eigenfunctions and energy values is set out in Sec. II.

(2) The embedding problem is expressed in terms of a two-tensor model in which one tensor represents the spacetime metric and the other the strong force. An exact spherically symmetric and static solution in the c-number theory is discussed in Sec. III.

(3) Color is incorporated in the form of a global SU(2) by enlarging the vierbein system $L_\mu^a \to L_\mu^{ai}$, $i=0,1,2,3$ [singlet plus triplet of SU(2)]. In Sec. IV we show that it is possible to set up a Lagrangian whose classical version yields a spherically symmetric solution with fixed "isospin". We argue that it is possible to arrange the parameters of the theory such that a source quark isodoublet will confine a test quark of isospin I if the total isospin is $I-\frac{1}{2}$ but not $I+\frac{1}{2}$. Color is essential if we wish to reproduce the observed hadronic structures.

(4) The problem of giving a dynamical role to the source is discussed in the simple case where the strong tensor is coupled to a scalar field (Sec. V).

II. KLEIN-GORDON EQUATION IN DE SITTER SPACE

The tensor field $f_{\mu\nu}(x)$ is assumed to satisfy the Einstein equations with a cosmological term, i.e., the Lagrangian is

$$\mathfrak{L}=\sqrt{-f}\left[\frac{1}{\kappa^2}R(f)+\lambda\right], \tag{2.1}$$

where $f=\det f_{\mu\nu}$, and $R(f)$ is the curvature scalar. The equations have a spherically symmetric and static solution given by the Schwarzschild–de Sitter expression

$$f_{\mu\nu}\,dx^\mu dx^\nu = \left(1-\frac{2\mu}{r}+\frac{r^2}{R^2}\right)dt^2 - \left(1-\frac{2\mu}{r}+\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\varphi^2\,, \tag{2.2}$$

where r, θ, φ are the usual spherical polar coordinates[4] and $R^{-2}=\kappa^2\lambda/6$. The parameter μ is an integration constant or, equivalently, the strength of the singularity at $r=0$ where the source is concentrated (and the equations break down). This singularity is an artifact of the empty-space approximation. In a more realistic formulation, other fields representing a matter distribution would be present in the Lagrangian and one would expect the singularity to be replaced by a smoothed-out source with finite spatial extension. (See the discussion of this program in Sec. V.) For the following analysis of the Klein-Gordon equation we shall make the simplification of setting $\mu=0$, in which limit the field $f_{\mu\nu}$ acquires O(3, 2) symmetry.

The Klein-Gordon equation for a spin-zero test particle ϕ of mass m takes the form[5]

$$0 = \frac{1}{(-f)^{1/2}}\,\partial_\mu(\sqrt{-f}\,f^{\mu\nu}\,\partial_\nu\,\phi) + m^2\phi \tag{2.3}$$

$$= \left(1+\frac{r^2}{R^2}\right)^{-1}\partial_t^2\phi - \frac{1}{r^2}\,\partial_r\left[r^2\left(1+\frac{r^2}{R^2}\right)\partial_r\phi\right] - \frac{1}{r^2\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta\,\phi) - \frac{1}{r^2\sin^2\theta}\,\partial_\varphi{}^2\phi + m^2\phi\,. \tag{2.4}$$

A complete set of solutions for this equation can be found in the form

$$\phi_{nlm}(x) = \langle f\,|\phi(x)\,|nlm\rangle = e^{-i\omega_{nl}t}u_{nl}(r)\,Y_{lm}(\theta,\varphi)\,. \tag{2.5}$$

The radial function u_{nl} satisfies the equation

$$0 = \frac{1}{r^2}\,\partial_r\left[r^2\left(1+\frac{r^2}{R^2}\right)\partial_r u\right] + \left[\frac{\omega^2}{1+r^2/R^2} - m^2 - \frac{l(l+1)}{r^2}\right]u\,, \tag{2.6}$$

which can be reduced to hypergeometric form. Write

$$u(r) = z^{l/2}(1-z)^{3/4+(9/4+m^2R^2)^{1/2}/2}F(z)\,, \tag{2.7}$$

where z is defined by

$$z = \frac{r^2}{r^2+R^2}\,. \tag{2.8}$$

The resulting equation for $F(z)$ takes the form

$$0 = z(1-z)F'' + [l+\tfrac{3}{2}-(\alpha_l+1)z]F' - \tfrac{1}{4}(\alpha_l{}^2-\omega^2R^2)F\,, \tag{2.9}$$

and has the solution

$$F = {}_2F_1\left(\frac{\alpha_l+\omega R}{2},\ \frac{\alpha_l-\omega R}{2};\ l+\tfrac{3}{2};\ z\right)\,, \tag{2.10}$$

where α_l represents the expression

$$\alpha_l = l+\tfrac{3}{2}+(9/4+m^2R^2)^{1/2}\,. \tag{2.11}$$

The hypergeometric function will be regular at $z=1$ ($r=\infty$) provided ω satisfies the eigenvalue condition

$$\tfrac{1}{2}(\alpha_l-\omega R) = -n\,, \tag{2.12}$$

where n takes the values $0, 1, 2, \ldots$. Hence the energies are discrete and are given by the formula

$$\omega_{nl} = \frac{1}{R}[2n+l+\tfrac{3}{2}+(\tfrac{9}{4}+m^2R^2)^{1/2}]\,. \tag{2.13}$$

Apart from the zero-point term, this is just the spectrum of the three-dimensional oscillator. The wave functions, however, are quite different. In terms of the original variables, they are given by

$$\langle f\,|\phi(x)|\,nlm\rangle = \frac{1}{N'_{nl}}\,e^{-i\omega_{nl}t}\left(\frac{r}{R}\right)^l\left(1+\frac{r^2}{R^2}\right)^{-\alpha_l/2}{}_2F_1\left(\alpha_l+n,\ -n;\ l+\tfrac{3}{2};\ \frac{r^2}{r^2+R^2}\right)Y_{lm}$$

$$= \frac{1}{N_{nl}}\,e^{-i\omega_{nl}t}\left(\frac{1-y}{2}\right)^{l/2}\left(\frac{1+y}{2}\right)^{3/4+(9/4+m^2R^2)^{1/2}/2}P_n^{(l+1/2,\,(9/4+m^2R^2)^{1/2})}(y)\,Y_{lm}(\theta,\varphi)\,, \tag{2.14}$$

where the functions P_n are Jacobi polynomials and y is given by

$$y = \frac{R^2-r^2}{R^2+r^2}\,. \tag{2.15}$$

The normalization requires some discussion. For a scalar field the norm should take the form

$$i\int d\sigma_\mu f^{\mu\nu}\langle nlm|\phi|f\rangle\overleftrightarrow{\partial}_\nu\langle f|\phi|n'l'm'\rangle=\delta_{nn'}\delta_{ll'}\delta_{mm'}, \tag{2.16}$$

where the choice of spacelike hypersurface is immaterial. With the simple choice $t=0$ the integral reduces to

$$\begin{aligned}(\omega_{nl}+\omega_{n'l'})\int dr\,d\theta\,d\varphi\, r^2\sin\theta\left(1+\frac{r^2}{R^2}\right)^{-1}Y^*_{lm}u_{nl}u_{n'l'}Y_{lm}&=\delta_{ll'}\delta_{mm'}(\omega_{nl}+\omega_{n'l})\int_0^\infty dr\,r^2\left(1+\frac{r^2}{R^2}\right)^{-1}u_{nl}u_{n'l}\\&=\delta_{ll'}\delta_{mm'}\tfrac{1}{2}(\omega_{nl}+\omega_{n'l})\,R^3\int_{-1}^{1}dy(1-y)^{1/2}(1+y)^{-3/2}u_{nl}u_{n'l}\\&=\delta_{ll'}\delta_{mm'}\tfrac{1}{2}(\omega_{nl}+\omega_{n'l})R^3 2^{-\alpha l}N_{nl}^{-1}N_{n'l}^{-1}\\&\quad\times\int_{-1}^{1}dy(1-y)^{l+1/2}(1+y)^{(9/4+m^2R^3)^{1/2}}\\&\qquad\times P_n^{(l+1/2,\sqrt{\cdot\cdot})}(y)\;P_{n'}^{(l+1/2,\sqrt{\cdot\cdot})}(y)\\&=\delta_{ll'}\delta_{mm'}\delta_{nn'}N_{nl}^{-2}R^2\frac{\Gamma(n+l+\frac{3}{2})\Gamma(n+1+(\frac{9}{4}+m^2R^2)^{1/2})}{n!\,\Gamma(l+\frac{3}{2}+n+(\frac{9}{4}+m^2R^2)^{1/2})}\,.\end{aligned} \tag{2.17}$$

Hence the normalization factor is given by

$$N_{nl}^2=R^2\frac{\Gamma(n+l+\frac{3}{2})\Gamma(n+1+(\frac{9}{4}+m^2R^2)^{1/2})}{n!\,\Gamma(n+l+\frac{3}{2}+(\frac{9}{4}+m^2R^2)^{1/2})}\,. \tag{2.18}$$

An improved approximation to the eigenvalue spectrum can be obtained by treating the source term $-2\mu/r$ as a perturbation. We estimate the lowest-order correction to ω_{nl} by elementary methods to obtain

$$\delta\omega_{nl}=\tfrac{1}{2}\int_0^\infty dr\,r^2\left(-\frac{2\mu}{r}\right)\times\left[|u'_{nl}|^2+\frac{\omega_{nl}^2}{(1+r^2/R^2)^2}|u_{nl}|^2\right], \tag{2.19}$$

where u_{nl} denotes the radial part of the zeroth-order wave function. This integral converges and has the form

$$\frac{\delta\omega_{nl}}{\omega_{nl}}=-\frac{\mu}{R}f(n,l) \tag{2.20}$$

in the limit of the test mass

$$m=0\,. \tag{2.21}$$

The factor $f(n,l)$ removes the degeneracy of the energy levels of (2.13) in the parameter $(2n+l)$.

For a spin-$\frac{1}{2}$ test particle of mass m, we must solve the Dirac equation $(L^{\mu a}\gamma_a\partial_\mu+m)\psi=0$. Here $L^{\mu a}$ are the vierbein fields which correspond to the $f_{\mu\nu}$ [see (4.12) and (4.13) for the explicit expression]. [The Dirac equation in general contains the derivative ∂_μ in the combination $\partial_\mu+iB_\mu$ (see Sec. IV for the definition of B_μ) but one can show that owing to spherical symmetry the B_μ term drops out.] This equation has been solved by Fronsdal and Haugen.[5]

To summarize, the exact eigenvalues for a system consisting of a test quark embedded in a constant f density microuniverse are given by (2.13). Both the quark and the f matter are confined. If a source quark of "mass" μ is present in addition, the approximate eigenvalues are given by (2.19). There is no dissociation if the source-quark mass μ is small (cf. end of Sec. V).

III. THE EMBEDDING PROBLEM

The de Sitter microuniverses—hadronic bags—consisting of concentrations of tensor fields plus quarks interacting through the intermediacy of the tensor field $f_{\mu\nu}$, must reside in spacetime. They must have an "outside." To describe forces in the outside world we need fields which are not subject to the strong de Sitter geometry of this microuniverse. Indeed there should appear a regime of ordinary flat-space geometry "outside" the hadron—to a very good approximation in hadron physics. To describe the geometry of the exterior world we need the metric tensor $g_{\mu\nu}$. Thus in a complete theory there must appear among all the fundamental fields at least two tensors, $f_{\mu\nu}$ and $g_{\mu\nu}$. The tensor $f_{\mu\nu}$ couples to colored fields such as the quark, while $g_{\mu\nu}$ couples to color-neutral fields.

A theory of two tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ can be formulated in a variety of ways but there are a number of criteria which must be satisfied. These are as follows:

(1) The theory must be generally covariant. This is essential for the existence of a massless helicity-2 graviton as an exact consequence of the theory. General covariance is necessary also for the validity of the gravitational equivalence principle.

(2) In order that colored hadrons, which interact directly with $f_{\mu\nu}$ but not $g_{\mu\nu}$, shall exhibit a gravitational coupling, there must be a mixing between f and g fields.

(3) It is desirable (though not essential) that the theory possess a stable flat-space solution where $\langle f_{\mu\nu}\rangle = \langle g_{\mu\nu}\rangle = \eta_{\mu\nu}$, the Minkowski tensor. Among the excitations on this solution one expects, in addition to the massless graviton, a massive spin-2 particle represented by appropriate orthogonal combinations of the two tensors $g_{\mu\nu}$ and $f_{\mu\nu}$.

(4) The theory should be capable of being generalized to include color. For such generalizations it will be necessary to employ a vierbein formalism—this is necessary in any case for the description of fermions. As is well known, in the vierbein approach of Weyl an additional local symmetry exists in the theory: SL(2, C) in the case of pure gravity and SL(2, $C)_g \times$SL(2, $C)_f$ in the case of f-g theory. The inclusion of color would generalize SL(2, $C)_f$ to a higher symmetry such as SL(6, $C)_f$.

(5) The mixing term between f and g tensors should ideally break this higher symmetry [e.g., SL(2, $C)_g \times$SL(6, $C)_f$] down to SL(2, C)$\times$SU(3)$_{\text{color}}$ *spontaneously*.

(6) One may wish to adopt a supersymmetric approach[6] towards gravity theory (i.e., include spin-$\frac{3}{2}$ fermions as well). In this case, without color, the symmetry (before spontaneous breaking) would be the graded OSp(4, 1)$_g\times$OSp(4, 1)$_f$. With the inclusion of SU(3)$_{\text{color}}$, this would need generalization to a still larger structure.

For the remainder of this section we shall ignore the vierbein and color aspects and consider only the rather simple system described by two neutral tensors, $f_{\mu\nu}$ and $g_{\mu\nu}$. The general form of the Lagrangian is

$$\mathcal{L} = \frac{1}{\kappa_g^2}\sqrt{-g}R(g) + \frac{1}{\kappa_f^2}\sqrt{-f}\,R(f) + \mathcal{L}_{fg}\,, \qquad (3.1\text{a})$$

where the first two terms are the respective Einstein Lagrangians for $g_{\mu\nu}$ and $f_{\mu\nu}$. The couplings appear explicitly: $\kappa_g \approx G_{\text{Newton}}^{1/2} \sim 10^{-19}$ GeV^{-1} and $\kappa_f \sim 1$ GeV^{-1}. The third term gives the mixing between f and g. For the present we treat this term in a phenomenological way (though, as stated above, ideally, it should arise from a *spontaneous* symmetry-breaking mechanism[6]). First, we assume that, in addition to being a scalar density, it has no derivatives. *In addition we require that it shall reduce to the form of a Pauli-Fierz mass term in the flat-space approximation.* There are a number of simple ways to satisfy these requirements, of which the following two are examples[3]:

$$\mathcal{L}_{fg}^{\text{I}} = -\frac{M^2}{4(\kappa_f^2+\kappa_g^2)}\sqrt{-g}\left(\frac{f}{g}\right)^c (f^{\mu\nu}-g^{\mu\nu}) \times (f^{\kappa\lambda}-g^{\kappa\lambda})(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\lambda}g_{\mu\nu})\,, \qquad (3.1\text{b})$$

where $g=\det g_{\mu\nu}$, $f=\det f_{\mu\nu}$ and $f^{\mu\nu}=(f^{-1})^{\mu\nu}$ denotes the matrix inverse of $f_{\mu\nu}$, and likewise $g^{\mu\nu} = (g^{-1})^{\mu\nu}$. In addition to the mass M an arbitrary parameter c has been introduced. Alternatively,

$$\mathcal{L}_{fg}^{\text{II}} = \lambda\sqrt{-g}+\lambda'\sqrt{-f}-(\lambda+\lambda')(-f)^{\alpha}(-g)^{\beta} \times \{-\det[xg^{-1}+(1-x)f^{-1}]\}^{\alpha+\beta-1/2}. \qquad (3.1\text{c})$$

In spite of its cosmological look, this term yields a stable flat spacetime provided the parameters are subject to two constraints:

$$\begin{gathered} 2[-x\alpha+(1-x)\beta](\lambda+\lambda') = -x\lambda'+(1-x)\lambda\,, \\ (\alpha+\beta-\tfrac{1}{2})x(x-1)(\lambda+\lambda')^2 = \tfrac{1}{4}\lambda\lambda'\,. \end{gathered} \qquad (3.2)$$

For both examples of $\mathcal{L}_{fg}$, in the linearized version one finds,[3] in addition to the graviton

$$\tilde{g}_{\mu\nu} = \left(\frac{1}{\kappa_g^2}+\frac{1}{\kappa_f^2}\right)^{-1}\left(\frac{1}{\kappa_g^2}g_{\mu\nu}+\frac{1}{\kappa_f^2}f_{\mu\nu}\right)\,,$$

a massive spin-2 particle associated with the orthogonal combination, $f_{\mu\nu}-g_{\mu\nu}$. The mass of this state is given by

$$M^2 = (\kappa_f^2+\kappa_g^2)\frac{\lambda\lambda'}{\lambda+\lambda'}\,.$$

It is possible to obtain exact solutions to the coupled equations derived from the above Lagrangian,

$$\begin{aligned} G_{\mu\nu}(g) &= -\frac{\kappa_g^2}{2}\frac{1}{(-g)^{1/2}}\frac{\partial}{\partial g^{\mu\nu}}\mathcal{L}_{fg}\,, \\ G_{\mu\nu}(f) &= -\frac{\kappa_f^2}{2}\frac{1}{(-f)^{1/2}}\frac{\partial}{\partial f^{\mu\nu}}\mathcal{L}_{fg}\,. \end{aligned} \qquad (3.3)$$

Consider first the "cosmological" example $\mathcal{L}_{fg}^{\text{II}}$ where a simple trick suffices for the extraction of nontrivial solutions. Thus suppose that the solution is such as to cause the coupling term to vanish together with its first derivatives with respect to f and g, i.e.,

$$\det[xg^{-1}+(1-x)f^{-1}] = 0 \qquad (3.4)$$

and $\alpha+\beta > \frac{3}{2}$. Then the equations reduce to

$$\begin{aligned} G_{\mu\nu}(g) &= -\frac{\kappa_g^2}{2}\lambda g_{\mu\nu}\,, \\ G_{\mu\nu}(f) &= -\frac{\kappa_f^2}{2}\lambda' f_{\mu\nu}\,. \end{aligned} \qquad (3.5)$$

Among the 20 equations here, only 12 are independent. We have 8 coordinate conditions (rather than just 4) at our disposal. The simplified equations (3.5) are effectively decoupled, and we can solve them by, for example, the de Sitter expressions

$$g_{\mu\nu}\,dx^\mu dx^\nu = \left(1+\frac{r^2}{R_g^2}\right)dt^2 - \left(1+\frac{r^2}{R_g^2}\right)^{-1} dr^2 - r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{3.6}$$

$$\begin{aligned} f_{\mu\nu}\,dx^\mu dx^\nu &= C(r)dt^2 - 2D(r)dt\,dr - A(r)dr^2 - B(r)(d\theta^2+\sin^2\theta d\varphi^2)\,,\\ &= \left(1+\frac{\bar r^2}{R_f^2}\right)d\bar t^2 - \left(1+\frac{\bar r^2}{R_f^2}\right)^{-1} d\bar r^2 - \bar r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \end{aligned} \tag{3.7}$$

where $R_g^{-2}=\kappa_g^2\lambda/6$ and $R_f^{-2}=\kappa_f^2\lambda'/6$. In the latter form, components are referred to a modified coordinate system,

$$\bar r^2 = B(r)\,,\quad \bar t = h(t,r)\,. \tag{3.8}$$

The coordinates t, r are fixed by assuming the standard de Sitter form for the components $g_{\mu\nu}$. Likewise, the coordinates $\bar t, \bar r$ correspond to the standard form for $f_{\mu\nu}$. We must now construct a transformation of (3.8)—an identification of coordinate patches—such that the constraint (3.4) is satisfied, i.e.,

$$0=\det[xg^{-1}+(1-x)f^{-1}]=(r^2\sin\theta)^{-2}\left(x+(1-x)\frac{r^2}{B}\right)^2\left\{x^2+\frac{x(1-x)}{\Delta}\left[\left(1+\frac{r^2}{R_g^2}\right)A+\left(1+\frac{r^2}{R_g^2}\right)^{-1}C\right]+\frac{(1-x)^2}{\Delta}\right\}. \tag{3.9}$$

There are many ways to satisfy this. Perhaps the simplest is by choosing

$$\bar r=\sqrt{B}=\left(\frac{x-1}{x}\right)^{1/2} r\,,\quad \bar t = t \tag{3.10}$$

so that

$$f_{\mu\nu}\,dx^\mu dx^\nu = \left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)dt^2 - \frac{x-1}{x}\left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)^{-1} dr^2 - \frac{x-1}{x}r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{3.11}$$

Alternatively, one may take

$$\bar r=\left(\frac{x-1}{x}\right)^{1/2} r\,,\quad \bar t=\left(\frac{x}{x-1}\right)^{1/2}\Delta[t+f(r)] \tag{3.12}$$

and choose $f(r)$ so as to make the last factor in (3.9) vanish. One finds

$$\begin{aligned} f_{\mu\nu}\,dx^\mu dx^\nu = \frac{x}{x-1}\Delta\Bigg\{&\left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)dt^2 + 2\left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)f'dt\,dr\\ &-\left[\frac{1}{\Delta}\left(\frac{x-1}{x}\right)^2\left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)^{-1} - \left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)f'^2\right]dr^2\Bigg\} - \frac{x-1}{x}r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \end{aligned} \tag{3.13}$$

where f' is given by

$$f'=\left[\left(1+\frac{r^2}{R_g^2}\right)^{-1} - \left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)^{-1}\right]^{1/2}\left[\left(1+\frac{r^2}{R_g^2}\right)^{-1} - \left(\frac{x-1}{x}\right)^2\frac{1}{\Delta}\left(1+\frac{x-1}{x}\frac{r^2}{R_f^2}\right)^{-1}\right]^{1/2}\,. \tag{3.14}$$

The method extends readily to give the de Sitter–Schwarzschild and de Sitter–Kerr solutions. The latter metric is given by Frolov[7] in the form

$$ds^2 = \left[1 - \frac{2mr}{r^2 + a^2\mathrm{cn}^2\theta} - \frac{\lambda}{3}(r^2 + a^2\mathrm{sn}^2\theta)\right] du^2 + 2du\,dr + 4a^2\mathrm{sn}^2\theta\left[\frac{mr}{r^2+a^2\mathrm{cn}^2\theta} + \frac{\lambda}{6}(r^2+a^2)\right] du\,d\varphi$$
$$- 2a\,\mathrm{sn}^2\theta\,dr\,d\varphi - \frac{r^2 + a^2\mathrm{cn}^2\theta}{1 + \lambda a2/3}\,d\theta^2$$
$$- \frac{1}{r^2 + a^2\mathrm{cn}^2\theta}\left[a^2\mathrm{sn}^4\theta\left(\frac{\lambda}{3}r^4 + \frac{\lambda}{3}a^2r^2 - r^2 + 2mr - a^2\right) + \mathrm{sn}^2\theta(a^2 + r^2)^2\left(1 + \frac{\lambda}{3}a^2\mathrm{cn}^2\theta\right) d\varphi^2\right., \qquad (3.15)$$

where $\mathrm{sn}\,\theta$ and $\mathrm{cn}\,\theta$ stand for the Jacobi elliptic functions $\mathrm{sn}(\theta, k)$ and $\mathrm{cn}(\theta, k)$, respectively, with k given by

$$k = \left(\frac{\Lambda a^2/3}{1 + \Lambda a^2/3}\right)^{1/2}. \qquad (3.16)$$

With the contravariant components

$$g^{\theta\theta} = -\frac{1 + \Lambda a^2/3}{r^2 + a^2\mathrm{cn}^2\theta}$$

and

$$\bar{f}^{\theta\theta} = -\frac{1 + \Lambda' a'^2/3}{\bar{r}^2 + a'^2\mathrm{cn}^2\bar{\theta}},$$

we can arrange to have

$$x g^{\theta\theta} + (1 - x) f^{\theta\theta} = 0$$

by choosing $\bar{\theta} = \theta$ and $\bar{r} = (a'/a)r$ with the ratio a'/a given by

$$\left(\frac{a'}{a}\right)^2 = \frac{x-1}{x}\,\frac{1 + \lambda' a'^2/3}{1 + \lambda a^2/3}.$$

The mixing term $\mathcal{L}^I$ also yields equations which can be solved exactly in the spherical static case.[8] One finds

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2\mu g}{r} + \frac{r^2}{R_g^2}\right)dt^2 - \left(1 - \frac{2\mu g}{r} + \frac{r^2}{R_g^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad (3.17)$$

$$f_{\mu\nu}dx^\mu dx^\nu = \tfrac{3}{2}\Delta\left(1 - \frac{2\mu_f}{r} + \frac{r^2}{R_f^2}\right)dt^2 - 2D\,dt\,dr - A\,dr^2 - \tfrac{2}{3}r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad (3.18)$$

with

$$D^2 = \Delta[1 - (1 + \tfrac{9}{4}\Delta)X + \tfrac{9}{4}\Delta X^2] \qquad (3.19)$$

and

$$A = \left(1 - \frac{2\mu_g}{r} + \frac{r^2}{R_g^2}\right)^{-1}[\tfrac{2}{3} + \tfrac{3}{2}\Delta(1 - X)], \qquad (3.20)$$

where

$$X = \left(1 - \frac{2\mu_f}{r} + \frac{r^2}{R_f^2}\right)\left(1 - \frac{2\mu_g}{r} + \frac{r^2}{R_g^2}\right)^{-1}. \qquad (3.21)$$

The parameters μ_g and μ_f are integration constants. The radii R_f^2 and R_g^2 are given by

$$\frac{1}{R_g^2} = \frac{M^2}{12}\frac{\kappa_g^2}{\kappa_f^2}\left(\frac{4}{9}\Delta\right)^c\left[\tfrac{3}{2}(c - \tfrac{1}{2}) - \frac{2}{\Delta}(c - \tfrac{3}{2})\right], \qquad (3.22)$$

$$\frac{1}{R_f^2} = \frac{M^2}{18}\left(\frac{4}{9}\Delta\right)^{c-1/2}\left[-\tfrac{3}{2}c + \frac{2}{\Delta}(c-1)\right], \qquad (3.23)$$

where Δ is a third integration constant. *The solution is surprising in that although the mixing term* $\mathcal{L}^I_{fg}$ *does not have the appearance of a cosmological term, it can be made to simulate one.* In effect, the equations for g and f reduce to the form

$$G_{\mu\nu}(g) = -\frac{3}{R_g^2}g_{\mu\nu}, \qquad G_{\mu\nu}(f) = -\frac{3}{R_f^2}f_{\mu\nu}, \qquad (3.24)$$

although the effective cosmological constants R_g^2 and R_f^2 are not independent: they are given in terms of the integration constant Δ. It is curious to note that for the special value

$$\Delta = \frac{4}{3}\,\frac{2c-3}{2c-1}, \qquad (3.25)$$

the constant $1/R_g^2$ vanishes so that the Minkowskian metric $g_{\mu\nu} = \eta_{\mu\nu}$ can be an exact solution. For this value of Δ one obtains

$$\frac{1}{R_f^2} = -\frac{M^2}{24}\,\frac{1}{3/2 - c}\left(\frac{27}{16}\,\frac{1/2 - c}{3/2 - c}\right)^{1/2-c}. \qquad (3.26)$$

Having obtained these exact solutions, one may raise the question, what is the "gravitational mass" of the objects described by the f equation? As we know[3] even in the flat background ($g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$) the weak-field approximation associates "true" gravity with the combination

$$\tilde{g} = \left(\frac{1}{\kappa_g^2} + \frac{1}{\kappa_f^2}\right)^{-1}\left(\frac{g}{\kappa_g^2} + \frac{f}{\kappa_f^2}\right). \qquad (3.27)$$

The orthogonal combination $\tilde{f}$ is associated with a short-range force. What the analogs of $\tilde{g}$ and $\tilde{f}$

are in the cosmological backgrounds determined above can, we think, only be discovered after the problem of perturbations on these backgrounds has been analyzed. We have not done this.

We summarize what we have done in this section: We have exhibited two alternative forms for the mixing terms $\mathcal{L}_{fg}^{I}$ and $\mathcal{L}_{fg}^{II}$ [Eqs. (3.1b) and (3.1c)]. Both were invented in the first place to describe in a linear approximation one massive plus one massless spin-2 object. For both one can also find an exact classical solution to the f-g system. And both (surprisingly for us) yield what prove to be essentially *de Sitter solutions* for the f and the g fields describing a micro- and a macrouniverse, respectively.

IV. INTRODUCTION OF COLOR

A. Background

In the work described so far there is no distinction between quarks or antiquarks, nor is there any selective restriction on the confinement of composites, formed from quark-quark or quark-antiquark systems. One must build color sensitivity into the confining mechanism. And, as a first step, one must generalize the Einstein equation to allow for the tensor—or vierbein fields $L^{\mu a}$—to carry color.

To keep to the essentials, we ignore once again the problem of the embedding of hadronic "micro-universes" into spacetime (i.e., set $\kappa_g=0$, $g_{\mu\nu}=\eta_{\mu\nu}$). Also for mathematical convenience, we shall work with SU(2) of color rather than SU(3). As we shall see later, the simpler structure of the group SU(2), as contrasted with the structure of SU(3) (or any of the higher Lie groups), permits us to obtain an explicit solution for our equations in this case.

B. The Lagrangian

A simple Lagrangian for the tensor fields which is invariant[3] under $SU(2)_{\text{global}} \times SL(2, C)_{\text{local}}$ as well as general coordinate transformations is given by

$$\mathcal{L}_1 = \frac{1}{16}\,\mathrm{Tr}[L^\mu, L^\nu]B_{\mu\nu}\,, \tag{4.1}$$

where $B_{\mu\nu}$ is a "field strength" made from the "gauge fields", B_μ,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]. \tag{4.2}$$

The fields B_μ and L^μ are 8×8 matrices which can be expanded as follows:

$$B_\mu = \tfrac{1}{4}B_{\mu ab}^{\alpha}\sigma_{ab}\tau_\alpha\,,$$

$$L^\mu = L_\alpha^{\mu a}\gamma_a\tau_\alpha\,,$$

where the Minkowski spacetime indices a, b and isotopic indices α take the values 0, 1, 2, 3.

Under the general coordinate transformations, $x^\mu \to \bar{x}^\mu$, we have

$$B_\mu(x) \to \bar{B}_\mu(\bar{x}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu}B_\nu(x)\,,$$

$$L^\mu(x) \to \bar{L}^\mu(\bar{x}) = \left|\det\frac{\partial x}{\partial\bar{x}}\right|^{1/2}\frac{\partial\bar{x}^\mu}{\partial x^\nu}L^\nu(x)\,,$$

so that $\mathcal{L}_1$ is a scalar density. Under SL(2, C), on the other hand,

$$B_\mu(x) \to B'_\mu(x) = \Omega(x)B_\mu(x)\Omega(x)^{-1} + \frac{1}{i}\,\Omega(x)\partial_\mu\Omega(x)^{-1}\,,$$

$$L^\mu(x) \to L'^\mu(x) = \Omega(x)L^\mu(x)\Omega(x)^{-1}\,,$$

where $\Omega(x)$ is an SL(2, C) matrix generated by $\sigma_{ab}\tau_0$.

To the kinetic terms provided by $\mathcal{L}_1$ may be added a mass term $\mathcal{L}_M$ for the tensor fields $L_\alpha^{\mu a}$. This term preserves SU(2) ×SL(2, C). Since we are interested in setting up an exact classical solution, we shall assume a form for the mass term so that it takes a manageable form in the class of solutions in question (see later).

The equations of motion are

$$[L^\mu, B_{\mu\nu}] + \frac{\partial\mathcal{L}_M}{\partial L^\nu} = T_\nu\,, \tag{4.3}$$

$$\partial_\mu[L^\mu, L^\nu] + i(B_\mu, [L^\mu, L^\nu]) = S^\nu\,, \tag{4.4}$$

where T_ν and S^ν denote the matter contributions to the stress (isostress) and torsion (isotorsion), respectively. We shall be concerned with the case where T and S are concentrated at a point and spherical symmetry is maintained. In this case a particularly simple solution emerges if we assume that the isovector components of B_μ and L^μ are constrained to have a fixed direction in isospace,

$$B_{\mu ab}^{i} = n^i B_{\mu ab}^{n}\,, \quad L_i^{\mu a} = n^i L_n^{\mu a}\,, \tag{4.5}$$

where n^i is independent of x. It is now possible to effect a separation of the equations of motion. Define the mixtures

$$L_\pm^\mu = L_0^\mu \pm L_n^\mu\,, \quad B_\mu^\pm = B_\mu^0 \pm B_\mu^n\,. \tag{4.6}$$

In terms of these combinations the kinetic term reduces to the form

$$\mathcal{L}_1 = \tfrac{1}{4}\,\mathrm{Tr}[L_+^\mu, L_+^\nu]B_{\mu\nu}^+ + \tfrac{1}{4}\,\mathrm{Tr}[L_-^\mu, L_-^\nu]B_{\mu\nu}^-\,, \tag{4.7}$$

where

$$B_{\mu\nu}^\pm = \partial_\mu B_\nu^\pm - \partial_\nu B_\mu^\pm + i[B_\mu^\pm, B_\nu^\pm]\,.$$

[The separation of $\mathcal{L}_1$ into two independent pieces occurs for SU(2). For SU(3) the problem would require a more elaborate treatment.]

In order to make use of the classical solution found in Sec. III we make a further change of variable. Define the "metric" tensors $f_{\pm\mu\nu}$ by

$$\tfrac{1}{4}\operatorname{Tr}(L_+^\mu L_+^\nu) = (-f_+)^{1/2} f_+^{\mu\nu}\,, \tag{4.8}$$

and likewise for $f_-^{\mu\nu}$. [In the usual fashion, the contravariant tensor $f_+^{\mu\nu}$ is defined as the matrix inverse of $f_{+\mu\nu}$ while $(-f_+)^{1/2}$ denotes the density $(-\det f_{+\mu\nu})^{1/2}$.] In terms of the tensors $f_{\pm\mu\nu}$ the kinetic term becomes the sum of two Einstein-type Lagrangians,

$$\mathcal{L}_1 = (-f_+)^{1/2} f_+^{\mu\nu} R_{\mu\nu}(f_+) + (-f_-)^{1/2} f_-^{\mu\nu} R_{\mu\nu}(f_-)\,. \tag{4.9}$$

We now choose the mass terms so as to recover the problem solved in Sec. III, i.e.,

$$\mathcal{L}_M = M^2(-g)^{1/2-c}(-f_+)^c (f_+^{\mu\nu} - g^{\mu\nu}) \times (f_+^{\kappa\lambda} - g^{\kappa\lambda})(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\nu} g_{\kappa\lambda}) \tag{4.10}$$

plus an identical term with $f_{+\mu\nu}$ replaced by $f_{-\mu\nu}$. On the face of it, this mass term seems not to be compatible with SU(2) invariance since f_+ and f_- are singlet-triplet mixtures. However, it can be shown that there do exist invariant forms obtained by replacing L_n^μ by $\underline{L}^\mu\cdot\underline{\tau}$, which reduce to this one when $L_i^\mu = n_i L_n^\mu$ provided $\det(L_0^{\mu a}) \neq 0$. [Since the order of factors in $\mathcal{L}_M$ is immaterial, in fact there are many different forms which, on replacing $\underline{L}^\mu\cdot\underline{\tau}$ by L_n^μ, reduce to (4.10).]

C. A solution

A spherically symmetric solution to the equations for $f_{+\mu\nu}$ (or $f_{-\mu\nu}$) in the approximation $g_{\mu\nu} = \eta_{\mu\nu}$ in (4.10),

$$R_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} R + \frac{1}{(-f)^{1/2}}\frac{\partial \mathcal{L}_M}{\partial f^{\mu\nu}} = 0\,,$$

has been previously obtained in Eqs. (3.17)–(3.21). Setting $\kappa_g = 0$ and writing $\Delta = \tfrac{4}{9}(1+\alpha)^{-1}$, the contravariant components are given by

$$f^{00} = \tfrac{3}{2}(1+\alpha+p)\,,\quad f^{0j} = -\tfrac{3}{2}[p(p+\alpha)]^{1/2}\,\frac{x^j}{r}\,,$$
$$f^{ij} = -\tfrac{3}{2}\delta^{ij} + \tfrac{3}{2}p\frac{x^i x^j}{r^2}\,, \tag{4.11}$$

where $i, j = 1, 2, 3$ and $r^2 = x^j x^j$. The function $p(r)$ is given by

$$p(r) = \tfrac{1}{2}M^2 r^2\left[\frac{81}{16}(1+\alpha)\right]^{1/2-c}\left[(1-c)(1+\alpha) + \frac{c}{3}\right],$$

where M and c are parameters appearing in the mass term and $1+\alpha = \tfrac{4}{9}\Delta^{-1}$ denotes an integration constant. It is restricted by the condition $1+\alpha > 0$.

Two such solutions are found independently for f_+ and f_- involving integration constants α_+ and α_-. The next step is to extract the square roots L_+^μ and L_-^μ. This can be done in a variety of ways but we shall choose schemes such that $L_\pm^{\mu a} = L_\pm^{a\mu}$. For L_+ one finds

$$L_+^0 = (\tfrac{2}{3})^{1/2}(1+\alpha_+)^{-1/4}\left\{\left((1+\alpha_+)^{1/2} - [1-(1+\alpha_+)^{1/2}]\frac{p_+}{\alpha_+}\right)\gamma^0 - [1-(1+\alpha_+)^{1/2}]\left[\frac{p_+}{\alpha_+}\left(1-\frac{p_+}{\alpha_+}\right)\right]^{1/2}\frac{x^i\gamma^i}{r}\right\}, \tag{4.12}$$

$$L_+^j = (\tfrac{2}{3})^{1/2}(1+\alpha_+)^{-1/4}\left\{\gamma^j + [1-(1+\alpha_+)^{1/2}]\frac{x^j x^k\gamma^k}{r^2} + [1-(1+\alpha_+)^{1/2}]\left[\frac{p_+}{\alpha_+}\left(1+\frac{p_+}{\alpha_+}\right)\right]^{1/2}\frac{x^j}{r}\gamma^0\right\}. \tag{4.13}$$

Two distinct solutions for L_- may be obtained from these expressions by the replacements $\alpha_+ \to \alpha_-$ and $(1+\alpha_+)^{1/2} \to -(1+\alpha_-)^{1/2}$ or $(1+\alpha_+)^{1/2} \to (1+\alpha_-)^{1/2}$. For later purposes we choose the first alternative and obtain

$$L_-^0 = (\tfrac{2}{3})^{1/2}(1+\alpha_-)^{-1/4}\left\{\left[-(1+\alpha_-)^{1/2} - [1+(1+\alpha_-)^{1/2}]\frac{p_-}{\alpha_-}\right]\gamma^0 - [1+(1+\alpha_-)^{1/2}]\left[\frac{p_-}{\alpha_-}\left(1+\frac{p_-}{\alpha_-}\right)\right]^{1/2}\frac{x^i\gamma^i}{r}\right\}, \tag{4.14}$$

$$L_-^j = (\tfrac{2}{3})^{1/2}(1+\alpha_-)^{-1/4}\left\{\gamma^j + [1+(1+\alpha_-)^{1/2}]\frac{x^j x^k\gamma^k}{r^2} + [1+(1+\alpha_-)^{1/2}]\left[\frac{p_-}{\alpha_-}\left(1+\frac{p_-}{\alpha_-}\right)\right]^{1/2}\frac{x^j}{r}\gamma^0\right\}. \tag{4.15}$$

Finally, by taking sums and differences one recovers the singlet and triplet vierbein components L_0^μ and L_n^μ.

If one wishes the singlet component not to exhibit the confining phenomenon of unrestricted growth at large r^2, then it is sufficient to correlate the two integration constants α_+ and α_- such that the r^2 dependence is canceled from L_0^{00}. Since

$$L_0^{00}=\tfrac{1}{2}(L_+^{00}+L_-^{00})=\frac{1}{\sqrt{6}}[(1+\alpha_+)^{1/4}-(1+\alpha_-)^{1/4}]$$
$$+\frac{1}{\sqrt{6}}\left[\frac{(1+\alpha_+)^{1/2}-1}{(1+\alpha_+)^{1/4}}\frac{p_+}{\alpha_+}-\frac{(1+\alpha_-)^{1/2}+1}{(1+\alpha_-)^{1/4}}\frac{p_-}{\alpha_-}\right], \tag{4.16}$$

it is necessary to choose α_+ and α_- to satisfy the equality

$$\frac{1}{\alpha_+}[(1+\alpha_+)^{1/2}-1](1+\alpha_+)^{1/4-c}\left[(1-c)(1+\alpha_+)+\frac{c}{3}\right]$$
$$=\frac{1}{\alpha_-}[(1+\alpha_-)^{1/2}+1](1+\alpha_-)^{1/4-c}$$
$$\times\left[(1-c)(1+\alpha_-)+\frac{c}{3}\right]. \tag{4.17}$$

Clearly, L_0^{00} reduces to a constant as do the L_0^{ij}. Only L_0^{0i} among the singlet components retains any r^2 dependence and this is of a relatively mild character; for large r^2 these components approach fixed values.

D. Interpretation

The solution given above is a vacuum solution whose isovector component maintains a fixed direction n_i independent of x. We would like to interpret it as an idealized version of the tensor fields generated by a point source (with spherical symmetry) which itself carries isospin. In other words, we would like to interpret the vector n_i in terms of the source isostress (or *isotorsion* since, presumably, these fix the same direction because of the equations of motion). For a quark doublet the isostress would be

$$T^j_{\mu a}\sim\bar{q}\,\tau^j\gamma_a(\partial_\mu+iB_\mu)q+\text{H.c.}$$

One may expect that in a quasistatic approximation the terms T^j_{00} factorize like $n^jT^n_{00}$ and together with T^0_{00} represent the dominant effect of the quark in generating the tensor components L_0^μ and L_j^μ. Thus in this semiclassical treatment of isospin,

$$L_j^{00}\sim\langle\tau_j\rangle T^n_{00}r^2,$$

where $\langle\tau_j\rangle\approx T^j_{00}/T^n_{00}$. In this approximation then, the action of the source on a test particle moving in this field would take the form

$$\langle\vec{\tau}_{\text{source}}\rangle\cdot\langle\vec{\text{I}}_{\text{test}}\rangle r_{12}{}^2,$$

at least if the particles are moving slowly and the nonrelativistic approximation has some meaning. (Clearly, the detailed correlation of the source to the tensor fields needs a solution of the coupled set of equations involving both the sources and the fields.)

To conclude, we have shown in the last section that it appears to be possible, by an adjustment of α_+ and α_- to arrange *that the singlet fields* L_0^μ *do not grow with* r^2 *and hence do not participate in the confinement mechanism*, and that only the triplet fields $L_0^{\mu i}$ can produce confinement—the hadronic composites are formed in the state where $(\vec{\tau}\cdot\vec{\text{I}})$ is negative—i.e., in the isotopic-spin state $I-\frac{1}{2}$. Assuming that both the source and the test quark possess I-spin-$\frac{1}{2}$, we find that quarks confine to form singlet composite hadrons ($\vec{\text{I}}=0$), and these composites themselves do not exhibit any confining r^2 "potential" towards each other ($\langle\vec{\text{I}}_1\rangle\cdot\langle\vec{\text{I}}_2\rangle=0$) or towards other quarks in the vicinity ($\langle\vec{\text{I}}\rangle\cdot\langle\vec{\tau}\rangle=0$).

We have not been able to discuss the case of SU(3) color, due to its mathematical complexity. Even in a simplified "classical" treatment of color, as in the sections above, SU(3) unitary spin would point in two directions (along λ_8 as well as λ_3) rather than just one direction [along τ_3 as assumed in (4.5) for the simpler case of SU(2)] making the discussion that much harder.

To distinguish between quarks and antiquarks, one can now introduce vector gluons in the conventional manner $[\partial_\mu\to\partial_\mu-ig(\tau/2)\cdot V_\mu]$. As is well known, from a generalization of the Reissner-Nordström type of solution for the Einstein equation, in all de Sitter expressions for $1+r^2/R^2$, one makes the replacement

$$1+\frac{r^2}{R^2}+\beta\frac{g^2}{r^2}, \tag{4.18}$$

where g is the Yang-Mills coupling parameter for vector mesons and β is the quadratic Casimir operator for the internal color symmetry.

Before we conclude this section, we would like to remark that we have introduced SU(2) color in the above treatment by extending Weyl's SL(2, C) internal gauge symmetry of the Einstein-Weyl vierbein equation to the symmetry SL(2, C)$_{\text{local}}$ $\times$SU(2)$_{\text{global}}$. A more satisfactory gauge extension would have been SL(4, C)$_{\text{local}}$, which includes SL(2, C)$_{\text{local}}\times$SU(2)$_{\text{color}}$ as a subgroup. This extended local symmetry has been considered[3] earlier and the appropriate Lagrangian exhibited. Such an SL(4, C) Lagrangian necessitates extended L^μ and B_μ multiplets with the content

$$L^\mu=L^{\mu a}_\alpha\gamma_a\tau_\alpha+L^{\mu a5}_\alpha i\gamma_a\gamma_5\tau_\alpha,$$
$$B_\mu=(B^\alpha_\mu+\tfrac{1}{2}B^\alpha_{\mu ab}\,\sigma_{ab}+B^\alpha_{\mu5}\gamma_5)\frac{\tau_\alpha}{2}.$$

The field equations of the restricted SL(2, C) $\times$SU(2) invariant Lagrangian of this paper are those solutions of the SL(4, C) Lagrangian where the extra components B_μ, $B_{\mu5}$, $L^{\mu a5}$, and the auxiliary fields all vanish.[9]

V. STRONG GRAVITY WITH A SCALAR SOURCE

As a first step towards a consistent treatment of the source distribution and in order to avoid the Schwarzschild singularity which appears when the source is concentrated at a point, we consider the equations which result when a real scalar field is coupled to the tensor $f_{\mu\nu}$. This real scalar represents hadronic matter and, in a more realistic description, would be replaced by a set of quark fields. At this point, however, our main concern is with the singularity rather than realistic quantum numbers.

To the pure f-g Lagrangian (3.1) add the scalar term

$$\mathcal{L}_\Phi = \tfrac{1}{2}\sqrt{-f}\,[f^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi - m^2\Phi^2]. \tag{5.1}$$

The equations of motion (3.3) are here replaced by

$$G_{\mu\nu}(g) = -\frac{\kappa_g^2}{2}\frac{1}{(-g)^{1/2}}\frac{\partial\mathcal{L}_{fg}}{\partial g^{\mu\nu}}, \tag{5.2}$$

$$G_{\mu\nu}(f) = -\frac{\kappa_f^2}{2}\left[\frac{1}{(-f)^{1/2}}\frac{\partial\mathcal{L}_{fg}}{\partial f^{\mu\nu}} + \partial_\mu\Phi\,\partial_\nu\Phi - \tfrac{1}{2}f_{\mu\nu}(f^{\kappa\lambda}\partial_\kappa\Phi\,\partial_\lambda\Phi - m^2\Phi^2)\right], \tag{5.3}$$

$$0 = \frac{1}{(-f)^{1/2}}\partial_\mu(\sqrt{-f}\,f^{\mu\nu}\partial_\nu\Phi) + m^2\Phi, \tag{5.4}$$

among which there are the usual identities resulting from general covariance. We fix the gauge and remove all degeneracy by choosing for $g_{\mu\nu}$ the flat-space form

$$g_{\mu\nu} = \eta_{\mu\nu}, \tag{5.5}$$

which is a solution of (5.2) in the approximation $\kappa_g = 0$ (recall, $\kappa_g \sim 10^{-19}\kappa_f$). It remains to solve (5.3) and (5.4) with $g_{\mu\nu}$ given by (5.5). Assume spherical symmetry and time independence,

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),\\ f_{\mu\nu}dx^\mu dx^\nu &= C(r)dt^2 - 2D(r)dt\,dr - A(r)dr^2 - B(r)(d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \tag{5.6}$$

The equations for A, B, C, and $\Delta = AC + D^2$ as well as $\Phi(r)$ can be derived from the effective Lagrangian density

$$\mathcal{L} = \frac{1}{\kappa_f^2}\left(2\sqrt{\Delta} + \frac{B'(\sqrt{B}\,C)'}{(B\Delta)^{1/2}}\right) + \frac{M^2r^2}{2\kappa_f^2}\left(\frac{B\sqrt{\Delta}}{r^2}\right)^{2r}\left(\frac{1}{\Delta} + \frac{r^4}{B^2} + 2\frac{r^2}{B}\frac{A+C}{\Delta} - 6\frac{r^2}{B} - 3\frac{A+C}{\Delta} + 6\right) - \frac{1}{2}B\sqrt{\Delta}\left(\frac{C}{\Delta}\Phi'^2 + m^2\Phi^2\right), \tag{5.7}$$

where differentiation with respect to r is denoted by a prime. (The mixing term $\mathcal{L}_{fg}^{\mathrm{I}}$ of Sec. III is used here.) The five functions A, B, C, Δ, and Φ are to be varied independently. Varying A yields the constraint

$$B = \tfrac{2}{3}r^2. \tag{5.8}$$

The other equations, after some rearrangement can be expressed in the form

$$0 = A + C - \tfrac{3}{2}\Delta - \tfrac{2}{3} + \frac{r}{2}\frac{\Delta'}{\Delta} \times[(1-c)(1-\tfrac{2}{3}c) + \tfrac{1}{4}c(1-2c)\Delta], \tag{5.9}$$

$$0 = -\frac{1}{r^2\sqrt{\Delta}}\left[r^2\frac{C}{(\Delta)^{1/2}}\Phi'\right]' + m^2\Phi, \tag{5.10}$$

$$\frac{\Delta'}{\Delta} = \frac{\kappa_f^2}{2}r\Phi'^2, \tag{5.11a}$$

$$\frac{1}{r^2\sqrt{\Delta}}\left[\left(\frac{rC}{\sqrt{\Delta}}\right)\right]' = \frac{3}{2}\frac{1}{r^2} + \frac{M^2}{2}\left[\frac{4}{9}\Delta\right]^{c-1/2}\times\left(\frac{c-1}{\Delta} - \tfrac{3}{4}c\right) - \frac{\kappa_f^2m^2}{4}\Phi^2. \tag{5.12a}$$

The latter two can be expressed in integral form,

$$\Delta = \Delta_\infty \exp\left(-\frac{\kappa_f^2}{2}\int_r^\infty dr\,r\Phi'^2\right), \tag{5.11b}$$

$$C = \frac{\sqrt{\Delta}}{r}\int_0^r dr\sqrt{\Delta}\left[\tfrac{3}{2} + \tfrac{1}{2}(\tfrac{4}{9}\Delta)^{c-1/2}\left(\frac{c-1}{\Delta} - \tfrac{3}{4}c\right)M^2r^2 - \frac{\kappa_f^2m^2r^2}{4}\Phi^2\right], \tag{5.12b}$$

where it is assumed that Δ approaches a constant Δ_∞ as $r\to\infty$, and that $C(r)$ is regular at $r = 0$. The absence of an arbitrary constant in (5.12b) signals the computability of the "mass" parameter μ mentioned in Sec. II. Here there is at least a chance of finding a perfectly regular solution. With Δ and C expressed in terms of Φ through (5.11b) and (5.12b), we can view the Klein-Gordon equation (5.10) as a nonlinear (and nonlocal) equation for Φ. We hope that it will have at least one regular solution but at present we can do no more than test the asymptotic regions $r\to\infty$ and $r\to 0$.

First, with $r\to\infty$, suppose that Φ and Φ' approach zero with sufficient rapidity to ensure the covergence of the integrals (5.11b) and (5.12b). Then we should have

$$\Delta \to \Delta_\infty \text{ and } C \to r^2/R^2, \tag{5.13}$$

where the scale R is given by

$$\frac{1}{R^2} = \tfrac{3}{8}(\tfrac{4}{9}\Delta_\infty)^{c+1/2}\left(\frac{c-1}{\Delta_\infty} - \tfrac{3}{4}c\right)M^2. \tag{5.14}$$

Inserting the asymptotic expressions (5.13) into the Klein-Gordon equation (5.10) gives the condition, for $r \to \infty$,

$$0 \approx -\frac{1}{r^2}(r^2\Phi')' + m^2R^2\Phi ,$$

which implies (5.15)

$$\Phi \approx r^{-[3/2+(9/4+m^2R^2)^{1/2}]} ,$$

i.e., Φ and Φ' do indeed vanish with sufficient rapidity. Thus the asymptotic forms (5.13) and (5.15) are self-consistent.

Next, with $r \to 0$ suppose Φ and Φ' are bounded so that $\Delta \approx \Delta_0$ and $C \approx \frac{3}{2}\Delta_0$. One easily sees that under these assumptions (5.10) has a regular solution

$$\Phi(r) \approx \Phi(0)(1 + \tfrac{1}{9}m^2r^2 + \cdots) .$$

We do not know whether such solutions can be extended out to $r = \infty$ and, in particular, whether the integral

$$\ln\frac{\Delta_\infty}{\Delta_0} = \frac{\kappa_f^2}{2}\int_0^\infty dr\, r\Phi'^2$$

is finite, but this appears to be a reasonable conjecture.

To summarize, we have made plausible the claim that the singular $1/r$ term in the Schwarzschild–de Sitter solution can be made soft for $r = 0$, when we consider an extended rather than the artificial point source at the origin.

Before closing this section it is pertinent to remark that the idealized Schwarzschild–de Sitter solution with $f_{00} = 1 - 2\mu/r + r^2/R^2$ may exhibit event horizons for values of $r > 0$ which satisfy $f_{00}(r) = 0$. The existence of such horizons depends on the respective values of μ and R, and would signal the decay of the hadronic composites, through the emission of Hawking's thermodynamic radiation in the form of liberated f field radiation as well as quarks. In this case the confinement would be partial rather than exact. We have not dwelt on this aspect of the formalism, since an extended (rather than a point) source described by the field Φ treated in this section is likely to influence these horizons rather profoundly. However, the fact is that this elegant situation exists within the theory to motivate partial confinement and also to give a quantitative estimate of the transmission probability for the liberated matter for escape from the hadronic bag.

VI. CONCLUSIONS

We realize that there is a general prejudice among the particle physics community against tensor fields and Einstein's equation, even though it is acknowledged that, in its richness, as a gauge equation, it has no peer. This prejudice stems mostly from the equation's perturbation-theoretic nonrenormalizability and its relative unfamiliarity and complexity. However, if confinement is indeed a physical law, the directness with which the de Sitter solution of Einstein's cosmological equation (with "strong" coupling constants) provides a formalism for confined "micro-universes" of hadronic bags is so striking that we believe both the Einstein equation and its de Sitter solution have a fundamental role to play in strong-interaction dynamics. The equation yields a λr^2 type of "potential"—we use quotes for "potential" because the background field $f_{\mu\nu}$ occurs in the Klein-Gordon or Dirac equations for test particles in a subtle multiplicative manner [see (2.3)]. Surprisingly, at least for us, the harmonic-oscillator spectrum (2.13) is still reproduced, though the wave functions are very different.[10]

How does this spectrum compare with the energy-level spectrum, for example, of charmonium? Before a confrontation is made with experiment, one must realize that the spectrum (2.13) is the spectrum of an f bag containing a test quark; it ignores color and also treats the source and the test quarks nonsymmetrically. Though a beginning has been made to include sources properly in strong gravity in Sec. V, the problem still remains to be solved completely. As an indication of the source corrections, (2.19) exhibits the source corrections due to the term $-2\mu/r$ in the "potential". Even more important, one must include a part of the vector-gluon corrections ($\beta g^2/r^2$)—i.e., those corrections which are given by the modification (4.18) to the f potential. These $1/r$ and $1/r^2$ corrections signaling the presence of the source singularity at the origin[11] already remove the n and l degeneracy in (2.13) of the pure harmonic-oscillator spectrum. Additional spin-dependent modifications are obtained by considering for the source quark the Kerr–de Sitter potential (3.15) and for a spin-$\frac{1}{2}$ test quark by solving a Dirac rather than a Klein-Gordon equation in the Kerr–de Sitter background.[5] All this is still at the f-field level. The additional effects of the colored vector gluons in distinguishing, for example, between quarks and antiquarks (introduced by replacing everywhere ∂_μ by $\partial_\mu - i\lambda^a A^a_\mu$) in the usual manner, are over and above the modifications mentioned above and need to be computed.

A unified complex of quarks, gluons and spin-2 particles has recently come into prominence through extended supersymmetric supergravity theories. *It is our opinion that if it is desired to make contact with reality, one should consider such* *theories as referring to $f_{\mu\nu}$ supergravity,* with internal symmetries so incorporated in the supersymmetric formulation as to produce an octet of colored f fields.[3,9] The g field may be the ninth member of a U(3) nonet in a unified $f-g$ approach, but one would then have to find an explanation for the vast numerical difference between the two couplings[12,13]—the strong $G_f \approx \kappa_f^2 \approx 1\ \text{GeV}^{-2}$ and the gravitational $G_g \approx \kappa_g^2 \approx 10^{-38}\ \text{GeV}^{-2}$.

To conclude on a topological note, this paper has been mainly concerned with the "metrical" aspects of the $f_{\mu\nu}$ microuniverse geometry. Following Wheeler, it has recently been suggested[14] that the internal symmetries such as SU(3) may be consequences of spacetime topology. In particular, Hawking and Pope have shown, by a consideration of the Atiyah-Singer theorem, how the very definition of spin in a spacetime manifold such as CP_2 appears to necessitate the introduction of a U(1) symmetry and of an electromagnetic field. It would seem to us natural that in the idealization when the $f-g$ mixing term vanishes [for example in (3.1c)] we are indeed dealing with two types of "universes"—the one type representing the "insides" of hadronic de Sitter microuniverses associated with $f_{\mu\nu}$'s, the other the "outside" macrouniverse associated with $g_{\mu\nu}$. Surely it is the topological structure of the $f_{\mu\nu}$ type of manifold which is the one directly relevant to the internal symmetries of the hadronic world and may provide an origin for strong symmetries such as SU(3) or SL(6, C) by an extension of the work of Hawking and Pope. The $f-g$ mixing term, which solves the problem of *embedding the microuniverses into the macrouniverse,* may, in this view, not only bring about the spontaneous symmetry breaking of structures such as SL(2, $C)_g \times$ SL(6, $C)_f$ down to SL(2, C) $\times$ SU(3) but also simultaneously lead to the spontaneous breaking of the strong symmetries such as SU(3) which are associated with the topology of the $f_{\mu\nu}$ microuniverses; this breakdown is brought about by the topological changes induced by the embedding process mentioned above.

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[2]Adbus Salam and J. Strathdee, Phys. Lett. 67B, 429 (1977); Phys. Rev. D 16, 2668 (1977). See also, E. W. Mielke, Phys. Rev. Lett. 39, 530 (1977); N. Baaklini, Dublin Institute for Advanced Study, Report No. DIAS TP-77-34; W. Drechsler and M. E. Mayer, *Fibre Bundle Techniques in Gauge Theories* (Springer, Berlin, 1977); P. Caldirola, M. Pavsic, and E. Recami, Report No. INFN/AE-77/14 (unpublished).

[3]C. J. Isham, Abdus Salam, and J. Strathdee, Phys. Rev. D 3, 867 (1971); 8, 2600 (1973); B. Zumino in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970).

[4]In the sequel we shall choose $\kappa^2 \simeq 1\ \text{GeV}^{-2}$, $\lambda \approx 1\ \text{GeV}^4$. The radius R of the microuniverse is thus $\approx 1\ \text{GeV}^{-1}$.

[5]An alternative formulation of (2.3) suggests the replacement of m^2 by $m^2 + R/6$, where R is the curvature. In a space of constant curvature, an approximation of R by $\langle R\rangle$ modifies the effective mass m^2 to $m^{*2} = m^2 + \langle R\rangle/6$. This, in essence, is the Archimedes effect—the effective mass m^* of the text quark inside the bag ($\langle R\rangle \neq 0$) is different from the mass outside ($\langle R\rangle \approx 0$) [see (3.24)]. For a solution to the Dirac equation in a de Sitter background, see C. Fronsdal and R. B. Haugen, Phys. Rev. D 12, 3810 (1975).

[6]A. Chamseddine, Abdus Salam, and J. Strathdee, Nucl. Phys. (to be published).

[7]V. P. Frolov, Lebedev Report No. 114 (unpublished).

[8]A solution in the approximation $\kappa_g = 0$ (i.e., a flat-space form for $g_{\mu\nu}$) was given by Abdus Salam and J. Strathdee [Phys. Rev. D 16, 2668 (1977)]. This solution was generalized to the form given here by C. J. Isham and D. Storey, Rev. D 18, 1047 (1978).

[9]The SL(4,C) local gauge symmetry—or more appropriately with SU(3) of color, the SL(6,C) local gauge symmetry—with nonvanishing components B_μ, $B_{\mu 5}$, $L^{\mu a 5}$, permits these extra components to be treated as vector (and axial-vector) gluons. For details see C. J. Isham, Abdus Salam, and J. Strathdee, in *Festchrift for I.I. Rabi*, transactions of New York Academy of Sciences, 1977, edited by Lloyd Motz, Ser. II, Vol. 38, 1977.

[10]At the very least one may say that if a field-theoretic formulation of the harmonic-oscillator "potential" is needed for physical applications one must use a tensor field satisfying an Einstein equation containing a cosmological parameter.

[11]Without such a singularity, the localization of the pure f bag is undefined, and such a pure bag may be unobservable. We are indebted to Dr. H. Romer for this remark.

[12]In this connection the question has been raised—should one really consider the f fields as fundamental fields to appear in a basic Lagrangian? Could one not think of these as composite fields, made up, for example, from a substratum of gluon fields with the f Lagrangians of this paper looked upon as local idealizations of an effective Lagrangian? In this sense, f-field exchanges represent effective two-gluon, four-gluon, six-gluon, $\cdots$ exchanges. Thus our work may be rephrased: even-gluon exchanges confine, while odd-gluon exchanges saturate (i.e., quark-antiquark composites bind more strongly than quark-quark composites). Now a similar question about the compositeness of gravity has recently been asked in respect to the gravitational field $g_{\mu\nu}$ by S. L. Adler, J. Lieberman, Y. J. Ng, and H.-S. Tsao [Phys. Rev.

D 14, 359 (1976)], who consider $g_{\mu\nu}$ as composed of photons. In a different context, P. Budini and P. Furlan [Nuovo Cimento 43A, 193 (1978)] have suggested that the neutral gauge field Z^0 in weak-interaction physics may be considered as a composite of a neutrino-antineutrino pair. Such (composite field) points of view are legitimate, except that it is not easy to see how one may reconcile these with, for example, the experimental universality of (the gauge) couplings of $g_{\mu\nu}$ or Z_μ^0, or, in the case of the $f_{\mu\nu}$ field, with the precise emergence of the Einstein-type Lagrangian (so necessary to give us the de Sitter confining solution) as an effective Lagrangian for a composite $f_{\mu\nu}$. Assuming that in a local Lagrangian context we need both the $f_{\mu\nu}$ and the $g_{\mu\nu}$ fields, the analogy of these with Z_μ^0 and the photon field A_μ naturally prompts one to think of a unification of strong with gravitational forces similar to the unification of weak and electromagnetic forces (see Abdus Salam, Marshak Festschrift, *Five Decades of Weak Interactions* (American Academy of Arts and Sciences, New York, 1977), Vol. 294, p. 12 and proceedings of the 1977 European Conference on Particle Physics, Budapest.

[13]The empirical fact that at present the ratio of the radius of the spacetime macrouniverse versus the radii of the microuniverses (10^{27} cm/10^{-13} cm $= 10^{40}$) is close to the ratio of the couplings G_f/G_g, implies in a de Sitter context that the respective cosmological parameters λ_f/λ_g must also be of order G_f/G_g. This universality is the well-known "law of coincidence of large numbers" of Dirac.

[14]S. Hawking and C. N. Pope, DAMTP, Cambridge, report, 1977 (unpublished); C. J. Isham, Imperial College, London, Report No. ICTP/77-78/3 (unpublished). Hawking and Pope have resolved the paradox of a factor of $\frac{1}{8}$ presented by $CP(2)$ in a purely gravitational context by invoking an additional electromagnetic field. It is amusing to note that they could alternatively have resolved this paradox by invoking eight gravitational f fields, corresponding to an internal SU(3) of color, thereby motivating this symmetry.