Construction of the functional-integral representation for fermion Green's functions

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A direct derivation is given of the functional-integral representation for the Green's functions of quantum theories built from a finite number of canonical fermion operators $Q_\alpha, Q_\alpha^\dagger$ with $\{Q_\alpha, Q_\beta^\dagger\} = \delta_{\alpha\beta}$.

The Green's functions of a quantum field theory are often represented as functional integrals. For example, in a theory with a scalar field Φ and a Dirac field $\Psi(x)$, one has

$$\langle 0| T\{\Psi_\alpha(x)\overline{\Psi}_\beta(y)\Phi(z)\}|0\rangle = Z^{-1}\int d[\phi]d[\overline{\eta}]d[\eta]e^{iS[\phi,\overline{\eta},\eta]}\,\eta_\alpha(x)\overline{\eta}_\beta(y)\phi(z)\,, \quad (1)$$

where

$$S[\phi,\overline{\eta},\eta] = \int d^4x[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}M^2\phi^2 - \overline{\eta}(i\partial\!\!\!/ - m)\eta + \mathcal{L}_I(\phi,\overline{\eta},\eta)]$$

and Z is a normalization factor. The integral is over all classical field configurations $\phi(x)$ of the boson field and over anticommuting field variables $\eta(x)$ and $\overline{\eta}(x)$ at each point in space-time for the fermion field.

Integral representations such as (1) have proved to be useful for performing formal manipulations on the theory, including the derivation in a simple way of the Feynman rules for perturbation theory. In the case of boson fields, the integral representation is also useful as the starting point for various semiclassical approximations.

In most interesting field theories, $\mathcal{L}_I$ is quadratic in the fermion field variables. Thus in many applications one can perform the η and $\overline{\eta}$ integrations exactly. This leaves only the ϕ integral, which is susceptible to approximation schemes. Indeed, the paper[1] that first introduced the fermion functional integral also showed how to eliminate it in this way.

The derivation of the functional-integral representation for quantum theories with a finite number of canonical boson operators Φ_α, Π_α with $[\Phi_\alpha, \Pi_\beta] = i\delta_{\alpha\beta}$ is well known and very simple.[2] From there, one passes easily to the limit of an infinite number of field operators and thus to a boson quantum field theory. (That is, this step is easy if one *assumes* the existence of this limit.)

The functional-integral representation for a quantum theory with a finite number of canonical fermion operators Q_α, $Q_\alpha^\dagger$ with $\{Q_\alpha, Q_\beta^\dagger\} = \delta_{\alpha\beta}$ is also well known. It has been established by showing that it leads to the Schwinger action principle,[1] that it solves the field equations of motion written as functional differential equations for the generating function of Green's functions, or that it leads to the correct Feynman rules for perturbation theory. In the boson case, however, one constructs the functional-integral representation directly from the underlying quantum mechanics, without reference to perturbation theory or the functional differential equations of motion. One would like to have a similar direct construction available for fermions.

Two recent papers, one by Halpern, Jevicki, and Senjanović[3] and one by Samuel,[4] point the way to such a direct construction. (These papers are briefly discussed in Appendix C.) My purpose in the present paper is to generalize the methods of these authors to cover the construction of the Green's functions for fermion field theories with a general Hamiltonian. I hope to illuminate questions of operator ordering, antiperiodic boundary conditions, and the precise definition of the functional integral as the limit of a certain lattice approximation. In addition, I prove the existence of the limit in the case of a finite number of degrees of freedom.

One can easily combine the boson and fermion constructions so as to cover the case of a finite number of boson and fermion operators, then pass to the limit of an infinite number of operators and thus to a general quantum field theory. In the interest of simplicity, these steps are not discussed in this paper.

The objects for which we construct a functional-integral representation are the finite-temperature imaginary-time Green's functions

$$\mathrm{Tr}\{e^{-\beta H}T[Q_\beta(-it_B)Q_\alpha^\dagger(-it_A)\cdots]\}/\mathrm{Tr}\{e^{-\beta H}\}\,.$$

These Green's functions are directly relevant to statistical mechanics. To obtain the usual Euclidean Green's functions used in particle physics, one simply takes the limit $\beta \to \infty$. This leaves only the contribution from the vacuum state,

$$\langle \text{vac}| T[Q_\beta(-it_B)Q_\alpha^\dagger(-it_A)\cdots]|\text{vac}\rangle .$$

Finally, if one wants the real-time (Minkowski-space) Green's functions, one analytically continues back to real times using the Wick rotation $-it \to e^{-i\theta}t \to t$ with $\pi/2 \geq \theta \geq 0$. The resulting Green's function can be written formally as a functional integral with $i\int dt$ in place of $\int dt$ and $-i\partial/\partial t$ in place of $\partial/\partial t$, but, in addition to making these replacements, one must remember the direction of the Wick rotation by occasionally inserting $i\epsilon$ terms in calculations.

We begin the construction by considering a quantum system described by a set of operators Q_α with $\alpha = 1, \ldots, d$ and their adjoints $Q_\alpha^\dagger$. The operators are assumed to satisfy the canonical anticommutation relations

$$\{Q_\alpha, Q_\beta^\dagger\} = \delta_{\alpha\beta}, \qquad \{Q_\alpha, Q_\beta\} = \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0. \tag{2}$$

The Hilbert space describing the states of the system has the usual Fock basis

$$\begin{aligned} &|0\rangle ,\\ &|\alpha\rangle = Q_\alpha^\dagger|0\rangle ,\\ &|\alpha,\beta\rangle = Q_\alpha^\dagger Q_\beta^\dagger|0\rangle ,\\ &|1,2,\ldots,d\rangle = Q_1^\dagger Q_2^\dagger\cdots Q_d^\dagger|0\rangle , \end{aligned} \tag{3}$$

with $Q_\alpha|0\rangle = 0$. Of course, the ground state of the system need not be the state $|0\rangle$.

We consider the most general possible self-adjoint Hamiltonian H for the system, subject to the condition that H be unchanged under the transformation $Q \to -Q$, $Q^\dagger \to -Q^\dagger$. We choose to write H in the form of a normal-ordered polynomial

$$\begin{aligned} H = H(Q^\dagger, Q) = A &+ B^{\alpha\beta}Q_\alpha Q_\beta + C^{\alpha\beta}Q_\alpha^\dagger Q_\beta + D^{\alpha\beta}Q_\alpha^\dagger Q_\beta^\dagger\\ &+ E^{\alpha\beta\gamma\delta}Q_\alpha Q_\beta Q_\gamma Q_\delta\\ &+ F^{\alpha\beta\gamma\delta}Q_\alpha^\dagger Q_\beta Q_\gamma Q_\delta + \cdots . \end{aligned}$$

[The choice of which operators of the theory are Q's and which are $Q^\dagger$'s is arbitrary, but once we have made this choice we stick to it and write $H(Q^\dagger, Q)$ with all of the Q's to the right of the $Q^\dagger$'s. In the case of a Dirac field, the most convenient choice is to let each component of $\Psi_\rho(\vec{x}, 0)$ at each point in space be a Q_α.]

Our object is to write a functional-integral representation for the finite-temperature imaginary-time Green's functions

$$G(t_L, \ldots, t_A) = Z^{-1}\text{Tr}\{e^{-\beta H}T[Q_\lambda(-it_L)\cdots Q_\alpha(-it_A)]\},$$

where $Z = \text{Tr}\{e^{-\beta H}\}$, T denotes time ordering, and it is assumed that $-\beta/2 < t_n < \beta/2$.[6] In order to keep the notation simple, we will consider a definite example, the two-point function for $t_B > t_A$:

$$G_{\beta\alpha}(t_B, t_A) = Z^{-1}\text{Tr}\{e^{-(\beta/2 - t_B)H}Q_\beta e^{-(t_B - t_A)H} \times Q_\alpha^\dagger e^{-(t_A + \beta/2)H}\}. \tag{4}$$

Let us divide the "time" interval $(-\frac{1}{2}\beta, \frac{1}{2}\beta)$ into a large number N of small intervals (t_J, t_{J-1}) with $-\frac{1}{2}\beta = t_0 < t_1 < t_2 < \cdots < t_N = +\frac{1}{2}\beta$. We choose the lattice points t_J so that two of them fall at the values $t_{J_A} = t_A$ and $t_{J_B} = t_B$. The evolution operators can then be written as products of small-interval evolution operators $\exp[-(t_J - t_{J-1})H]$.

We now make the essential approximation of the present derivation. We replace $\exp(-\Delta t_J H)$ by $:\exp(-\Delta t_J H):$, where the colons denote normal ordering. This gives the "lattice Green's function"

$$\tilde{G} = \tilde{Z}^{-1}\text{Tr}\{:e^{-H\Delta t_N}:\cdots Q_\beta\cdots Q_\alpha^\dagger\cdots \times :e^{-H\Delta t_2}::e^{-H\Delta t_1}:\}, \tag{5}$$

where $\tilde{Z} = \text{Tr}\{:e^{-H\Delta t_N}:\cdots :e^{-H\Delta t_1}:\}$. Since $H(Q^\dagger, Q)$ is already normal ordered, the normal-ordering instruction in $:\exp(-\Delta t_J H):$ affects only the terms of second or higher order in the small quantity Δt_J. Thus the approximate Green's function $\tilde{G}$ will approach the exact Green's function G as the maximum lattice spacing Δt_J tends to zero.[7]

The rest of the analysis is purely algebraic: we write an exact representation of the lattice Green's function G as an "integral" over a set of anticommuting objects $\eta_\alpha(J), \eta_\alpha^*(J)$, with $\alpha = 1, \ldots, d$ and $J = 1, \ldots, N$. Each of these objects anticommutes with each other object; they are the generators of the Grassmann algebra of order $2Nd$.

Let us briefly review[8] the definition of integration on a Grassmann algebra $\mathcal{G}_{2Nd}$. For this purpose we denote the generators by $Z_1, Z_2, \ldots, Z_{2Nd}$. The algebra $\mathcal{G}_{2Nd}$ consists of all polynomials

$$P(Z) = A + B^j Z_j + C^{jk}Z_j Z_k + \cdots + LZ_1 Z_2\cdots Z_{2Nd} \tag{6}$$

together with self-evident rules for addition and multiplication of polynomials (using $Z_j Z_k + Z_k Z_j = 0$). Differentiation on $\mathcal{G}_{2Nd}$ is defined by

$$\frac{\partial}{\partial Z_j}Z_k = \delta_k^j, \quad \frac{\partial}{\partial Z_j}1 = 0,$$

together with the rules

$$\frac{\partial}{\partial Z_j}[aP(Z) + bQ(Z)] = a\frac{\partial}{\partial Z_j}P(Z) + b\frac{\partial}{\partial Z_j}Q(z),$$

$$\frac{\partial}{\partial Z_j}Z_k P(Z) = \delta_k^j P(Z) - Z_k\frac{\partial}{\partial Z_j}P(Z).$$

The standard definition of "integration" on $\mathcal{G}_{2Nd}$ can be stated very simply:

$$\int dZ_j P(Z) \equiv \frac{\partial}{\partial Z_j}P(Z).$$

(It is only for historical reasons that one uses the integration symbol for this operation.) In particular, if one integrates $P(Z)$, [Eq. (6)] over all of the anticommuting variables, one obtains an ordinary number, the coefficient of the highest-order term in $P(Z)$:

$$\int dZ_{2Nd}\cdots dZ_1 P(Z) = L.$$

We make use of the Grassmann algebra by mapping the space of states of the quantum system onto the space of polynomials $P(\eta^*)$ in N anticommuting variables η^*_α. Let the basis states $|0\rangle$, $|\alpha\rangle$, $|\alpha,\beta\rangle,\ldots$ correspond to the monomials

$$\begin{aligned} &|0\rangle \to 1\,,\\ &|\alpha\rangle \to \eta^*_\alpha\,,\\ &|\alpha,\beta\rangle \to \eta^*_\alpha\eta^*_\beta\,,\\ &|1,2,\ldots,N\rangle \to \eta^*_1\eta^*_2\cdots\eta^*_N\,. \end{aligned} \tag{7}$$

More generally, let the polynomial $P(\eta^*)$ correspond to the state $P(Q^\dagger)|0\rangle$.

Now consider a quantum operator F, which we write as a normal-ordered polynomial

$$F(Q^\dagger, Q) = A + B^\alpha Q_\alpha + \cdots + LQ_1^\dagger\cdots Q_N^\dagger Q_1\cdots Q_N.$$

The corresponding operator on the space of polynomials $P(\eta^*)$ can be written as an integral operator: if $FP(Q^\dagger)|0\rangle = \hat{P}(Q^\dagger)|0\rangle$, then

$$\hat{P}(\hat{\eta}^*) = \int\left(\prod_{\alpha=1}^{d} d\eta^*_\alpha d\eta_\alpha\right)\exp\left(\sum_\alpha\;(\hat{\eta}^*_\alpha - \eta^*_\alpha)\eta_\alpha\right) \times F(\hat{\eta}^*,\eta)P(\eta^*)\,. \tag{8}$$

The proof of Eq. (8) is straightforward and is given in Appendix A.

Evidently the relation (8) can be used repeatedly to write a product of operators,

$$F = F_N(Q^\dagger, Q)\cdots F_2(Q^\dagger, Q)F_1(Q^\dagger, Q)\,,$$

as an integral operator on polynomials $P(\eta^*)$, in the form of Eq. (8). The combined kernel $F(\eta^*,\eta)$ then corresponds to the normal-ordered form $F(Q^\dagger, Q)$ of the operator F. One finds that $FP(Q^\dagger)|0\rangle = \hat{P}(Q^\dagger)|0\rangle$, where

$$\hat{P}[\eta^*(N)] = \int\left(\prod_\alpha d\eta^*_\alpha(0)d\eta_\alpha(0)\right)e^{[\eta^*(N)-\eta^*(0)]\eta(0)}F[\eta^*(N),\eta(0)]P[\eta^*(0)]\,,$$

and the new kernel $F(\eta^*,\eta)$ is

$$F[\eta^*(N),\eta(0)] = \int\left(\prod_{J=1}^{N-1}\prod_\alpha d\eta^*_\alpha(J)d\eta_\alpha(J)\right)\exp\left(\sum_{J=2}^{N}[\eta^*(J)-\eta^*(J-1)]\eta(J-1)\right)e^{[\eta^*(1)-\eta^*(N)]\eta(0)} \times F_N[\eta^*(N),\eta(N-1)]\cdots F_1[\eta^*(1),\eta(0)]\,. \tag{9}$$

Here we have used the notation

$$\eta^*(J')\eta(J) \equiv \sum_\alpha \eta^*_\alpha(J')\eta_\alpha(J)\,. \tag{10}$$

To complete the construction, we need to write the trace of F as an integral. The relevant relation is

$$\mathrm{Tr}\{F(Q^\dagger, Q)\} = \int\left(\prod_\alpha d\eta_\alpha d\eta^*_\alpha\right)\exp\left(2\sum_\alpha \eta^*_\alpha\eta_\alpha\right)F(\eta^*,\eta) \tag{11}$$

for a normal-ordered polynomial $F(Q^\dagger, Q)$. This relation is proved in Appendix B.

When the relations (9) and (11) are combined, one obtains

$$\mathrm{Tr}\{F_N(Q^\dagger, Q)\cdots F_1(Q^\dagger, Q)\} = \int d\eta^* d\eta\,\exp\left(-\sum_{J=1}^{N}\eta^*(J)[\eta(J)-\eta(J-1)]\right)F_N[\eta^*(N),\eta(N-1)]\cdots F_1[\eta^*(1),\eta(0)]\,, \tag{12}$$

where

$$\int d\eta^* d\eta \equiv \int\prod_{J=1}^{N}\prod_{\alpha=1}^{d} d\eta^*_\alpha(J)d\eta_\alpha(J)\,,$$

and where, when $\eta(0)$ occurs, it means

$$\eta(0) = -\eta(N)\,. \tag{13}$$

The integral representation (12) for the trace of a product of operators can now be applied to the lattice Green's function (5) with the result

$$\tilde{G}_{\beta\alpha}=\tilde{Z}^{-1}\int d\eta^* d\eta\, e^{-S[\eta^*,\eta]}\eta_\beta(J_B)\eta^*_\alpha(J_A)\,,$$
$$\tilde{Z}=\int d\eta^* d\eta\, e^{-S[\eta^*,\eta]}\,, \tag{14}$$

where

$$S(\eta^*,\eta)=\sum_{J=1}^{N}\Delta t_J\left\{\eta^*(J)\frac{\eta(J)-\eta(J-1)}{\Delta t_J}+H[\eta^*(J),\eta(J-1)]\right\}.$$

In the limit in which the maximum time interval Δt_J goes to zero, the lattice Green's function $\tilde{G}$ approaches the comtinuum Green's function G. It is suggestive to write G in the standard form

$$G_{\beta\alpha}(t_B,t_A)=Z^{-1}\int d\eta^* d\eta\, e^{-S[\eta^*,\eta]}\eta_\beta(t_B)\eta^*_\alpha(t_A)\,,$$
$$Z=\int d\eta^* d\eta\, e^{-S[\eta^*,\eta]}\,, \tag{15}$$

where

$$S(\eta^*,\eta)=\int_{-\beta/2}^{\beta/2}dt\left\{\eta^*(t)\frac{d}{dt}\eta(t)+H[\eta^*(t),\eta(t)]\right\}.$$

A few comments are in order

(i) The functional integrals in Eq. (15) are *defined* as the $\Delta t\to 0$ limits of the finite-dimensional integrals in Eq. (14). We have proved[7] that this limit exists.

(ii) The representation (15) was derived under the assumption that $t_B>t_A$, but the same representation evidently applies for $t_A<t_B$.

(iii) The integral representations for Green's functions with more operators $Q_\gamma(-it_C)Q^\dagger_\delta(-it_D)\cdots$ have additional factors $\eta_\gamma(t_C)\eta^*_\delta(t_D)\cdots$. All of these Green's functions can be obtained by differentiating the generating functional

$$Z(J,J^*)=Z^{-1}\int d\eta^* d\eta\,\exp\Big(-S[\eta^*,\eta]+\int dt\,[\eta^*(t)J(t)+J^*(t)\eta(t)]\Big)\,,$$

where J and J^* are additional anticommuting variables.

(iv) The antiperiodic boundary condition (13) implies that the operator d/dt in the continuum action should be defined with antiperiodic boundary conditions and that the Green's functions are antiperiodic:

$$G(t_L,\ldots,t_C,+\tfrac{1}{2}\beta,t_A)=-G(t_L,\ldots,t_C,-\tfrac{1}{2}\beta,t_A).$$

(v) The methods used here also work for bosons $[Q_\alpha,Q^\dagger_\beta]=\delta_{\alpha\beta}$ if the η's are interpreted as complex numbers and

$$\int d\eta^*_\alpha(J)d\eta_\alpha(J)\equiv\frac{1}{\pi}\int_{-\infty}^{\infty}d\mathrm{Re}\eta\int_{-\infty}^{\infty}d\mathrm{Im}\eta\,.$$

Equation (8) is unchanged, while the exponential factor in Eq. (11) is to be omitted for bosons. As a consequence, the results (14) and (15) hold for bosons as well as fermions except that the boundary condition (13) becomes a *periodic* boundary condition $\eta(0)=+\eta(N)$ for bosons.

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APPENDIX A: PROOF OF EQ. (8)

Let $F(Q^\dagger,Q)$ be an operator written in the form of a normal-ordered polynomial and let $F(Q^\dagger,Q)P(Q^\dagger)|0\rangle=\hat{P}(Q^\dagger)|0\rangle$. Define $R(Q^\dagger)$ by

$$R(\hat{\eta}^*)=\int\Big(\prod_\mu d\eta^*_\mu d\eta_\mu\Big)\exp\Big(\sum_\nu(\hat{\eta}^*_\nu-\eta^*_\nu)\eta_\nu\Big)\times F(\hat{\eta}^*,\eta)P(\eta^*)\,.$$

We seek to prove that $R(\hat{\eta})=\hat{P}(\hat{\eta})$.

It suffices to prove this in the case that F and P are monomials

$$F(Q^\dagger,Q)=Q^\dagger_{\alpha_1}\cdots Q^\dagger_{\alpha_A}Q_{\beta_B}\cdots Q_{\beta_1}\,,$$

$$P(Q^\dagger)=Q^\dagger_{\gamma_1}\cdots Q^\dagger_{\gamma_C}\,.$$

Then,

$$R(\hat{\eta}^*)=\hat{\eta}^*_{\alpha_1}\cdots\hat{\eta}^*_{\alpha_A}\int\Big(\prod_\mu d\eta^*_\mu d\eta_\mu\Big)\prod_\rho(1+\eta_\rho\eta^*_\rho)\times\prod_\sigma(1+\hat{\eta}^*_\sigma\eta_\sigma)\times\eta_{\beta_B}\cdots\eta_{\beta_1}\eta^*_{\gamma_1}\cdots\eta^*_{\gamma_C}\,.$$

Evidently, $R(\hat{\eta}^*)=0$ if the values of the indices β are not a subset of the values of the indices γ. In this case $\hat{P}(Q^\dagger)=0$ also, so $\hat{P}=R$. In the contrary case $\{\beta_j\}\subset\{\gamma_j\}$, one can assume without loss of generality that $\beta_1=\gamma_1$, $\beta_2=\gamma_2$, $\ldots$, $\beta_B=\gamma_B$. Then

$$R(\hat{\eta}^*)=\hat{\eta}^*_{\alpha_1}\cdots\hat{\eta}^*_{\alpha_A}\int\Big(\prod_\mu d\eta^*_\mu d\eta_\mu\Big)\prod_\rho(1+\eta_\rho\eta^*_\rho)\eta_{\gamma_1}\eta^*_{\gamma_1}\cdots\eta_{\gamma_B}\eta^*_{\gamma_B}\hat{\eta}^*_{\gamma_{B+1}}\eta_{\gamma_{B+1}}\eta^*_{\gamma_{B+1}}\cdots\hat{\eta}^*_{\gamma_C}\eta_{\gamma_C}\eta^*_{\gamma_C}$$

$$=\hat{\eta}^*_{\alpha_1}\cdots\hat{\eta}^*_{\alpha_A}\hat{\eta}^*_{\gamma_{B+1}}\cdots\hat{\eta}^*_{\gamma_C}\int\Big(\prod_\mu d\eta^*_\mu d\eta_\mu\Big)\prod_\rho(1+\eta_\rho\eta^*_\rho)\eta_{\gamma_1}\eta^*_{\gamma_1}\cdots\eta_{\gamma_C}\eta^*_{\gamma_C}.$$

The remaining integral equals 1, so

$$R(\hat{\eta}^*)=\hat{\eta}^*_{\alpha_1}\cdots\hat{\eta}^*_{\alpha_A}\hat{\eta}^*_{\gamma_{B+1}}\cdots\hat{\eta}^*_{\gamma_C}.$$

In this case,

$$F(Q^\dagger,Q)P(Q)|0\rangle=Q^\dagger_{\alpha_1}\cdots Q^\dagger_{\alpha_A}Q_{\gamma_B}\cdots Q_{\gamma_1}Q^\dagger_{\gamma_1}\cdots Q^\dagger_{\gamma_C}|0\rangle$$
$$=Q^\dagger_{\alpha_1}\cdots Q^\dagger_{\alpha_A}Q^\dagger_{\gamma_{B+1}}\cdots Q^\dagger_{\gamma_C}|0\rangle.$$

Thus, $\hat{P}=R$ in this case also, and the theorem is proved.

APPENDIX B: PROOF OF EQ. (11)

Let $F(Q^\dagger,Q)$ be an operator written in the form of a normal-ordered polynomial and define

$$I=\int\Big(\prod_{\mu=1}^{d}d\eta_\mu d\eta^*_\mu\Big)\exp\Big(2\sum_\nu\eta^*_\nu\eta_\nu\Big)F(\eta^*,\eta).$$

We seek to prove that $I=\mathrm{Tr}\{F(Q^\dagger,Q)\}$. It suffices to prove this in the case that F is a monomial

$$F(Q^+,Q)=Q^\dagger_{\alpha_1}\cdots Q^\dagger_{\alpha_A}Q_{\beta_B}\cdots Q_{\beta_1}.$$

Evidently $\mathrm{Tr}F=I=0$ if the set $\{\alpha_j\}$ does not equal the set $\{\beta_j\}$. In the case $\{\alpha_j\}=\{\beta_j\}$, it suffices to assume that $\alpha_1=\beta_1,\ldots,\alpha_A=\beta_A$. In this case

$$I=\int\Big(\prod_{\mu=1}^{d}d\eta_\mu d\eta^*_\mu\Big)\prod_{\nu=1}^{d}(1+2\eta^*_\nu\eta_\nu)\eta^*_{\alpha_1}\eta_{\alpha_1}\cdots\eta^*_{\alpha_A}\eta_{\alpha_A}$$
$$=2^{d-A}.$$

But in this case $\mathrm{Tr}F=2^{d-A}$ also, so the theorem is proved.

APPENDIX C: COMPARISON WITH OTHER METHODS

The method used in this paper is a generalization of that of Samuel,[4] who constructed the functional-integral representation for $\langle 0|Q_\alpha \mathrm{T}e^{\int H dt}Q_\beta|0\rangle$, where $H=Q^\dagger_\alpha\Gamma(t)_{\alpha\beta}Q_\beta$ and $Q_\beta|0\rangle=0$. (Thus, scattering is included but pair creation is not.) The fermion number equal to one part of the isomorphism (7) between polynomials in the Grassmann variables and states is implicit in Samuel's construction, as is the representation (8) of operators as integral operators.

Another recent paper by Halpern, Jevicki, and Senjanović[3] constructs the functional-integral representation for the same object by using fermion coherent states:

$$|\eta(J)\rangle=e^{Q^\dagger_\alpha\eta_\alpha(J)}|0\rangle,$$

$$\langle\eta^*(J)|=\langle 0|e^{\eta^*_\alpha(J)Q_\alpha}.$$

Their construction is swift and elegant, as is the corresponding construction in the Bose case. They introduce the coherent states at each time interval using

$$1=\int\prod_\alpha d\eta^*_\alpha(J)d\eta_\alpha(J)e^{-\eta^*_\alpha(J)\eta_\alpha(J)}|\eta(J)\rangle\langle\eta^*(J)|.$$

Then they evaluate the matrix elements using

$$\langle\eta^*(J+1)|F(Q^\dagger,Q)|\eta(J)\rangle=F[\eta^*(J+1),\eta(J)]e^{\eta^*_\alpha(J+1)\eta_\alpha(J)}.$$

One can also introduce the trace by using

$$\mathrm{Tr}F(Q^\dagger,Q)=\int\prod_\alpha d\eta_\alpha(N)d\eta^*_\alpha(N)e^{+\eta^*_\alpha(N)\eta_\alpha(N)}\times\langle\eta^*(N)|F(Q^\dagger,Q)|\eta(N)\rangle.$$

Notice that the fremion coherent states do not belong to the original fermion Fock space, but to a larger vector space in which the role of scalars is played by the elements of a Grassmann algebra. The creation and annihilation operators on the space, $Q^\dagger_\alpha$ and Q_α, anticommute with the Grassman vraiables η^*_β and η_β. The construction of this Grassmann-Fock space and the proof of the formulas listed above are, if done carefully, not difficult but not entirely trivial.[5] The principal advantage of Samuel's method in comparison to the coherent-states method is that the Grassmann-Fock space need not be introduced.

[1]P. T. Matthews and A. Salam, Nuovo Cimento 2, 120 (1955).

[2]E. Nelson, J. Math. Phys. 5, 332 (1964); R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).

[3]M. B. Halpern, A. Jevicki, and P. Senjanović, Phys. Rev. D 16, 2476 (1977). See also the similar construction in Ref. 5.

[4]S. Samuel, Univ. of Calif., Berkeley, report, 1977 (unpublished).

[5]Y. Ohnuki and T. Kashiwa, Nagoya Univ., report, 1978 (unpublished).

[6]The definition is extended outside of this region using the antiperiodicity relation $G(\ldots, t_n+\beta, \ldots) = -G(\ldots, t_n, \ldots)$.

[7]Let $U \equiv \exp[-H(t_F-t_I)]$. Divide the interval (t_I, t_F) into small segments (t_{J-1}, t_J) and define $\tilde{U} = \Pi_J :\exp(-H\Delta t_J):$. Evidently

$$\| \exp(-H\Delta t_J) - :\exp(-H\Delta t_J): \| < C(\Delta t_J)^2$$

for some constant C. From this bound one easily derives

$$\| U - \tilde{U} \| < \exp[-E_0(t_F-t_I)]\, \{\exp[C(t_F-t_I)\max\Delta t_J] - 1\},$$

where E_0 is the smallest eigenvalue of H or zero, whichever is smaller. Thus, $\| U - \tilde{U} \| \to 0$, and therefore $|G - \tilde{G}| \to 0$, as $\max\Delta t_J \to 0$.

[8]For further details see F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[9]*Footnote added in proof.* A similar construction has been given independently by L. Faddeev in *Methods in Field Theory*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).