$(\bar{\psi}\psi)^2/N$ interaction in four dimensions: Nonlocal theory

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The Gross-Neveu version of the Thirring model is analyzed in four-dimensional space-time. In $2+\epsilon$ dimensions this model is nonrenormalizable by power counting; however, it is known that up to $4-\epsilon$ dimensions the $1/N$ summation produces the improved propagator of the collective $\bar{\psi}\psi$ field so that the theory requires no more subtractions than the theory in two dimensions. In four dimensions the situation deteriorates: There are two induced couplings of arbitrary strength, and the elastic unitarity is violated because the collective propagator decreases too fast. We show that the arbitrary coupling constant can be determined from the requirement that the effective potential has a minimum for zero values of the classical fermion and collective fields. One more reason for inconsistency is that a tachyon pole comes about even if one expands around the minimum of the potential. It is argued that one can get rid of inconsistencies by allowing the theory to be nonlocal. We construct the nonlocal form factor of the collective field from the requirements of unitarity, microcausality, correct spectral properties, and several assumptions about regularity properties of the nonlocal form factor.

I. INTRODUCTION

The model that we shall consider is the N-component spinor field theory with a quartic scalar interaction in four-dimensional space-time:

$$L_\psi = \bar{\psi}(i\not{\partial})\psi - \tfrac{1}{2}\lambda(\bar{\psi}\psi)^2 \tag{1.1}$$

(plus, possibly, the fermion mass term), where

$$\bar{\psi}\psi = \frac{1}{N}\sum_{n=1}^{N}\bar{\psi}_n\psi_n\,. \tag{1.2}$$

Following the usual procedure[1] we shall replace (1.1) with an equivalent Lagrangian involving the auxiliary collective field σ,

$$L_\sigma = \bar{\psi}(i\not{\partial})\psi - \frac{1}{2\lambda}\sigma^2 - \sigma\bar{\psi}\psi\,. \tag{1.3}$$

In more than two dimensions this model is nonrenormalizable by power counting; however, it has been recognized that up to dimensions less than four it can be renormalized without the need to introduce an infinite number of counterterms.[2] The essence of the method one has to use is that first one should sum up the contributions from the leading $1/N$-order self-energy diagrams (see Fig. 1) and then use the result as the "improved propagator" while evaluating higher-order contributions. Then only a few types of vertex functions turn out to be superficially divergent. The idea of the superpropagator method[3] is intimately built into the scheme of the $1/N$ expansion.

In order to demonstrate how the method works let us calculate the contribution from the single-fermion loop which enters in the graphs of Fig. 1. We obtain

$$\Pi(p^2) = -\int\frac{d^nk}{(2\pi)^n}\,\frac{\mathrm{Tr}[\not{k}(\not{k}+\not{p})]}{k^2(k+p)^2}\,. \tag{1.4}$$

Integration on the right-hand side reveals that

$$\Pi(p^2) = \frac{n-1}{(2\pi)^{n/2}}\,\Gamma(1-\tfrac{1}{2}n)B(\tfrac{1}{2}n,\tfrac{1}{2}n)(-p^2)^{(n-2)/2}\,. \tag{1.5}$$

Summing all the contributions by means of the formula for the sum of the geometrical series, we obtain for the improved propagator

$$D(p^2) = \frac{-i\lambda}{1-\lambda\Pi(p^2)}\,. \tag{1.6}$$

$D(p^2)$ asymptotically behaves like p^{2-n} and the convergence properties of the diagrams of the rearranged perturbation expansion are considerably improved. The tachyon pole which comes about in (1.6) is spurious and disappears by means of dynamical symmetry breaking. These two facts support the belief that the model is consistent in $4-\epsilon$ dimensions.

In four dimensions the situation deteriorates dramatically. Now $D(p^2)$ decreases too fast because of the term $p^2\ln p^2$ which arises in the finite part of (1.5). To make matters worse the ghost pole appears in the propagator even if we correctly expand Green's functions about the minimum of the effective potential.

In the sequel we shall try to attack the problem

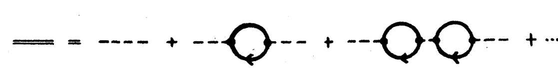

FIG. 1. The bubble chain contributing to the improved σ propagator.

in the framework of nonlocal quantum field theory. Textbook knowledge tells us that the renormalization of the perturbation expansion introduces nonlocality into Green's functions when the conventional technique is applied to nonrenormalizable interactions. This is because counterterms containing arbitrary powers of momenta should be added to any kind of Green's function. The $1/N$ expansion is essentially a rearranged and partially resummed perturbation expansion, and there is no principal reason why all these troubles should be absent from it. It possesses, however, one particularly attractive feature: The ambiguity due to the renormalization procedure fully concentrates in the leading $1/N$ order. Therefore all additional assumptions which are needed to fix nonlocal terms can be controlled and, we hope, interpreted.

This is not as unexpected a result as it sounds because nonlocal terms are produced by infinite summations, and in the $1/N$ expansion the bubble chain[4] of Fig. 1 is the only infinite summation which is needed to any order in $1/N$. Hence the single part of the argument which may require modifications is the formula (1.6) for the improved propagator which is the result of the asymptotic summation. It was the formula for the sum of the geometrical progression which led to (1.6). We have used this formula because then $D(p^2)$ is the analytic continuation of the result from the region where the series actually converges. In this sense the result (1.6) is unique. It corresponds to the simple form of the generating functionals: Equation (1.6) is generated by the effective action which contains $(1/\lambda)\sigma^2$ coupling and the standard tr log term.

The idea that one can get rid of the tachyon poles by a suitable modification of the method of summation is rather old.[5] In particular, one can remove the unwanted pole by replacing $D(p^2)$ with $D(p^2)+f(p^2)^{-1}$, i.e., by introducing an additional nonlocal $\frac{1}{2}\sigma f(-\Box)\sigma$ term into the effective action. The function $f(p^2)^{-1}$ should have a pole at the same place as $D(p^2)$ and the residue equals minus the residue of $D(p^2)$. If, moreover, the function f has a zero asymptotic expansion in powers of λ, then one can argue that the new improved propagator belongs to the equivalence class of the asymptotic sum for the series of Fig. 1 and therefore is as good an improved σ propagator as (1.6) was.[6] While applying this method one must take care for the effective potential; it may happen that the tachyon is removed, but the Green's functions are not expanded around the absolute minimum of the potential. We just know how to calculate the effective potential in different $1/N$ models.[1,7-9]

It is very convenient to construct $f(p^2)^{-1}$ in the form

$$\frac{W(p^2)}{p^2-p_0^{\,2}}\,. \tag{1.7}$$

Theories with nonlocal propagators of this type were rigorously elaborated by Efimov[12] and Alebastrov and Efimov.[13] They have demonstrated that if $W(z)$ is an entire function whose order of growth ρ satisfies

$$\tfrac{1}{2}\leqslant\rho<1\,, \tag{1.8}$$

then the unitarity condition for the S matrix is satisfied and if, moreover, $\rho=\frac{1}{2}$ then also the microcausality condition in Bogoliubov's sense[14] is fulfilled.

The principal aim of this paper is to construct the form factor $W(p^2)$ in such a form that the resulting theory possesses the ground state, has the correct spectral properties, and satisfies unitarity and microcausality conditions in the sense of Ref. 14. Under specific assumptions on the regularity properties of the nonlocal form factor and the behavior of the improved propagator in the Euclidean asymptotic region, it is possible to obtain the unique answer. To this end we use Evgrafov's method of constructing the entire function from its asymptotic behavior.[15] This method was applied in physics by Efimov and Mogilevsky[6] in the framework of nonlocal quantum electrodynamics.

We show that the resulting theory exhibits features predicted by Klauder for nonrenormalizable interactions.[16,17] In the limit of the vanishing coupling constant the quasifree instead of the free field theory is recovered. The form factor W does not have a zero asymptotic expansion. This result is not unexpected: In contradistinction to the renormalizable models, perturbation expansion reveals nonlocality of the interaction. Hence the additional nonlocal term should be considered as the "augmentation of the interaction" in the spirit of Ref. 17 rather than as a modification of the propagator (1.6) within the equivalence class of the asymptotic sum of the divergent series leading to the result (1.6).

II. THE EFFECTIVE POTENTIAL

The one-loop σ-self-energy diagram of the theory with massless fermions yields

$$\Pi(p)=-\int\frac{d^nk}{(2\pi)^n}\,\frac{\mathrm{Tr}[\not{k}(\not{k}+\not{p})]}{k^2(k+p)^2}\,. \tag{2.1}$$

Integrating and retaining only the finite part we obtain in the limit $n\to4$

$$D(p)=\frac{-i\lambda}{1-(\lambda/4\pi^2)p^2\ln(-p^2/\mu^2)}\,, \tag{2.2}$$

where several numerical factors were rearranged into the arbitrary scale parameter μ^2.

It is apparent that for positive values of λ the tachyon pole unavoidably comes about in (2.2). In that respect our model resembles the Gross-Neveu model.[1] For large negative values of λ ($\lambda<-4e\pi^2/\mu^2$) the pole disappears (unlike in the Gross-Neveu model), but for negative coupling the theory with the $\bar{\psi}\psi$ collective field does not have physical meaning,[1] so from now on we shall confine ourselves to the case $\lambda>0$.

If, instead of (1.1), we consider the Lagrangian with massive fermions then the analog of (1.3) is

$$L_\sigma=\bar{\psi}(i\not\partial-m)\psi-\frac{1}{2\lambda}\sigma^2-\sigma\bar{\psi}\psi\,, \tag{2.3}$$

and the contribution from the one-loop self-energy diagram is

$$\Pi(p,m)=\int\frac{d^nk}{(2\pi)^n}\frac{\mathrm{Tr}\{[-i(\not p+\not k)+m](-i\not k+m)\}}{(k^2-m^2)[(k+p)^2-m^2]}\,. \tag{2.4}$$

Expanding in powers of $4-n$ we obtain for the finite part,

$$\text{F.P. }\Pi(p,m)=\frac{1}{4\pi^2}\left[\left(3\gamma-7+3\ln\frac{m^2}{\mu^2}+2B(k^2,m^2)\right)m^2-\left(\tfrac{1}{2}\gamma-\tfrac{4}{3}+\tfrac{1}{2}\ln\frac{m^2}{\mu^2}+\tfrac{1}{2}B(k^2,m^2)\right)k^2\right], \tag{2.5}$$

where γ is the Euler's constant. While expanding (2.4) we have used the convention that we expand only the Euler's Γ function and integrals over the Feynman parameters. Everywhere else we set $n=4$. The function $B(k^2,m^2)$ is given by

$$B(k^2,m^2)=2\left(1-\frac{1}{x}\right)^{1/2}\ln[(1-x)^{1/2}+(-x)^{1/2}] \quad \text{for } x\leq 0\,,$$

$$B(k^2,m^2)=2\left(\frac{1}{x}-1\right)^{1/2}\arctan\left(\frac{x}{1-x}\right)^{1/2} \quad \text{for } 0\leq x\leq 1\,, \tag{2.6}$$

$$B(k^2,m^2)=\left(1-\frac{1}{x}\right)^{1/2}\{-i\pi+2\ln[(x-1)^{1/2}+x^{1/2}]\} \quad \text{for } x>1\,,$$

where

$$x=k^2/4m^2\,.$$

We cannot draw any reliable conclusions about the spectrum unless we eliminate the explicit dependence of (2.5) on the arbitrary scale parameter μ^2. This dependence cannot be rearranged into the parameters of the original Lagrangian. This manifest reflection of nonrenormalizability does not offer serious trouble. In the following we shall show that any change of μ^2 can be soaked up by a suitable redefinition of the induced σ^4 coupling constant. In the leading order there are only two induced couplings: quartic interaction of the composite mode and the kinetic quadratic term $(\partial\sigma)^2$. The lack of the induced couplings in higher orders is granted by power counting with a modified rule for the σ propagator.

We cannot continue our discussion unless we gain some information from the effective potential. We shall calculate the effective potential using the method proposed by Lee and Sciaccaluga.[18] This method seems to be perfectly suited for dimensionally regularized theories.

Shifting in (1.3) the field σ by its classical value u we obtain

$$L'=\bar{\psi}(i\not\partial-u)\psi-\frac{1}{2\lambda}\sigma^2-\sigma\bar{\psi}\psi-\frac{1}{\lambda}u\sigma\,. \tag{2.7}$$

In leading order there is only a single one-particle-irreducible nontrivial tadpole diagram (see Fig. 2). The contribution from this graph is

$$\Gamma^{(1)}(u)=-i\int\frac{d^nk}{(2\pi)^n}\frac{\mathrm{Tr}(-i\not k+u)}{k^2-u^2}$$
$$=\frac{1}{(2\pi)^{n/2}}\Gamma(1-\tfrac{1}{2}n)u^{n-1}\,. \tag{2.8}$$

At $n=4$ the finite part of (2.8) equals

$$\text{F.P.}\Gamma^{(1)}(u)=\frac{1}{4\pi^2}u^3\ln\frac{u^2}{s^2}\,, \tag{2.9}$$

where

$$\ln\frac{u^2}{s^2}=\ln\frac{u^2}{\mu^2}+\gamma-1\,. \tag{2.10}$$

There is a simple relation between $\Gamma^{(1)}(u)$ and the radiative part V_{rad} of the effective potential V_{eff},

$$\frac{\partial V_{\mathrm{rad}}(u)}{\partial u}=\Gamma^{(1)}(u)\,. \tag{2.11}$$

Hence the finite part of the radiative correction to the effective potential equals

FIG. 2. The σ-field tadpole diagram.

$$V_{\rm rad}=\frac{1}{16\pi^2}\sigma^4\left(\ln\frac{\sigma^2}{s^2}-\tfrac{1}{2}\right) \tag{2.12}$$

The full expression for the effective potential is

$$V_{\rm eff}=\frac{1}{2\lambda}\sigma^2+\sigma\bar\psi\psi+\frac{1}{16\pi^2}\sigma^4\left(\ln\frac{\sigma^2}{s^2}-\tfrac{1}{2}\right). \tag{2.13}$$

The condition $\partial V_{\rm eff}/\partial\sigma=0$ gives us the gap equation

$$\bar\psi\psi=-\frac{1}{\lambda}\sigma-\frac{1}{4\pi^2}\sigma^3\ln\frac{\sigma^2}{s^2}. \tag{2.14}$$

Eliminating the dependence of (2.13) on $\bar\psi\psi$ we find that the constrained effective potential is equal to

$$V_{\rm con}=-\frac{1}{2\lambda}\sigma^2-\frac{1}{16\pi^2}\sigma^4\left(3\ln\frac{\sigma^2}{s^2}+\tfrac{1}{2}\right). \tag{2.15}$$

Equating to zero the first derivative of $V_{\rm con}$ with respect to σ,

$$V'_{\rm con}=-\frac{1}{\lambda}\sigma-\frac{1}{4\pi^2}\left(3\ln\frac{\sigma^2}{s^2}+2\right)\sigma^3, \tag{2.16}$$

we observe that, except for $\sigma=0$, there is the second candidate for the minimum of $V_{\rm con}$, namely, $\sigma=\tilde\sigma$, where $\tilde\sigma$ is the solution of

$$\tilde\sigma^2\left(3\ln\frac{\tilde\sigma^2}{s^2}+2\right)=-\frac{4\pi^2}{\lambda}. \tag{2.17}$$

The second derivative of $V_{\rm con}$ equals

$$V''_{\rm con}=-\frac{1}{\lambda}-\frac{3}{4\pi^2}\left(3\ln\frac{\sigma^2}{s^2}+4\right)\sigma^2. \tag{2.18}$$

Substituting 0 and $\tilde\sigma$ for σ we obtain

$$V''_{\rm con}(0)=-\frac{1}{\lambda}\quad V''_{\rm con}(\tilde\sigma)=\frac{2}{\lambda}-\frac{3\tilde\sigma^2}{2\pi^2}. \tag{2.19}$$

Hence for $\lambda>0$ the point $\sigma=0$ is the maximum of the effective potential and we expect that the discrete symmetry

$$\psi\to\gamma_5\psi,\quad \sigma\to-\sigma \tag{2.20}$$

is dynamically broken.

The sign of $V''_{\rm con}(\tilde\sigma)$ depends on the actual value of $\tilde\sigma$ which, as implicitly defined by (2.17), is a function of the scale parameter s^2 [or μ^2, cf. (2.10)]. We cannot draw any conclusions from the above formulas unless all quantities are expressed in a μ-independent manner. We have renormalized the dimensionally regularized diagrams according to the 't Hooft–Veltman renormalization prescription,[19] i.e., we have subtracted only the pole parts of the divergent integrals. Using another prescription leads to different results, but in renormalizable theories all the differences can be absorbed by a suitable reparametrization of the Lagrangian.[20] When nonrenormalizable interactions are taken into account the ambiguity due to the choice of the renormalization procedure becomes essential. In consequence of this ambiguity an additional $\eta\sigma^4$ term with an arbitrary coupling strength η can be added to (2.12) and (2.13). Any change of η can be compensated for a suitable change of μ. We find it convenient not to introduce η but, alternatively, to treat μ^2 on an equal footing with other parameters of the model. Both formulations are manifestly equivalent. Another induced coupling creates the σ-field kinetic term. This term does not contribute to the effective potential and therefore will be discarded here.[21]

Equation (2.17) has zero, one, or two real positive solutions. We should choose μ^2 large enough to ensure that (2.17) has two such solutions, $\tilde\sigma_1<\tilde\sigma_2$, say. [We do not consider the case when (2.17) has one solution because then the potential has an inflection point instead of a minimum.] The gap equation defines the three-valued function $\sigma(\bar\psi\psi)$. The branch points are localized at the points which are solutions of the equation

$$d\bar\psi\psi(\sigma)/d\sigma=0$$

and coincide with the roots $\tilde\sigma_1,\tilde\sigma_2$ of (2.17). In that respect the situation resembles that which transpires in nonrenormalizable quartic scalar self-interaction in more than four dimensions (see the second paper of Ref. 8).

We have just noted that at $\sigma=0$ the constrained potential has a maximum. If σ increases then $V_{\rm con}$ decreases up to $\sigma=\tilde\sigma_1$ where we have a minimum and then begins to increase until $\sigma=\tilde\sigma_2$ (see Fig. 3). Inverting the gap function $\bar\psi\psi(\sigma)$ we have to choose the branch extending from $\tilde\sigma_1$ to $\tilde\sigma_2$. Then V has a minimum at $\bar\psi\psi(\tilde\sigma_1)$ and gets an imaginary part at $\bar\psi\psi(\tilde\sigma_2)$. Unfortunately we find that $\bar\psi\psi(\tilde\sigma_1)<0$. This result is independent of the value of the parameters λ and μ^2 and is highly unacceptable from the physical point of view.

If we want to have a minimum in the physical region we should somehow lift the plot $\bar\psi\psi(\sigma)$ upwards. This can be realized by letting the fermion field be massive from the beginning. If we start from the Lagrangian

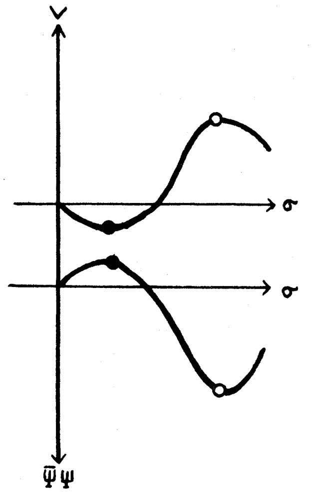

FIG. 3. The constrained effective potential versus σ versus $\bar\psi\psi$ in the massless case. The minimum (heavy dot) occurs for negative values of $\bar\psi\psi$.

$$L = \bar\psi(i\not\partial - m)\psi - \frac{1}{2\lambda}\sigma^2 - \sigma\bar\psi\psi\,, \tag{2.21}$$

then we obtain

$$V_{\rm rad} = \frac{1}{16\pi^2}(\sigma+m)^4\left[\ln\left(\frac{\sigma+m}{s}\right)^2 - \tfrac{1}{2}\right]. \tag{2.22}$$

The gap equation gets the form

$$\bar\psi\psi = -\frac{1}{\lambda}\sigma - \frac{1}{4\pi^2}(\sigma+m)^3\ln\left(\frac{\sigma+m}{s}\right)^2, \tag{2.23}$$

and the constrained effective potential is equal to

$$V_{\rm con} = -\frac{1}{\lambda}(\sigma+m) - \frac{1}{4\pi^2}(\sigma+m)^3\left[3\ln\left(\frac{\sigma+m}{s}\right)^2 + 2\right]. \tag{2.24}$$

Comparing (2.22) and (2.23) with their predecessors (2.12)–(2.15) we find that now the plot $V_{\rm con}$ versus σ is shifted to the left by m, and $\bar\psi\psi$ versus σ is shifted to the left by m and lifted upwards by m/λ (see Fig. 4). The position of the extrema (and of the branch points) is now $\sigma = \tilde\sigma_1 - m$ and $\sigma = \tilde\sigma_2 - m$.

We cannot expect the minimum of V to be the ground state unless two additional requirements are satisfied. The first of them is

$$\tilde\sigma_1 > m\,. \tag{2.25}$$

Then the minimum occurs for positive values of the classical field. Shifting σ by its classical value $\tilde\sigma_1 - m$ we make this field have zero expectation values at the minimum. This does by no means automatically imply that the same holds for $\bar\psi\psi$, therefore we have to assume that

$$\bar\psi\psi(\tilde\sigma_1 - m) = 0\,. \tag{2.25$'$}$$

This is a nontrivial restriction which we interpret as a consistency condition allowing us to fix the value of the (previously considered as arbitrary) induced σ^4 coupling strength. The question is how to calculate this coupling constant explicitly.

Condition (2.17) tells us that $\tilde\sigma$ satisfies

$$\tilde\sigma^2\ln\frac{\tilde\sigma^2}{s^2} = -\frac{4\pi^2}{3\lambda} - \frac{2}{3}\tilde\sigma^2\,. \tag{2.26}$$

Substituting this into the gap equation and using condition (2.25$'$) we obtain

$$\tilde\sigma^3 - \frac{4\pi^2}{\lambda}\tilde\sigma + \frac{6\pi^2 m}{\lambda} = 0\,. \tag{2.27}$$

For positive λ and m the product of the roots of (2.27) is negative, thus (2.27) has either one negative and two positive or one negative and two complex solutions. The condition that (2.27) has two real positive solutions plus the requirement that the smaller of these solutions satisfies (2.25) define the region on the (λ, m) plane where the ground state exists. For m and λ from this region Eq. (2.27) can be solved, e.g., using Cardan's formulas. Substituting the solutions back into (2.26) one can calculate s^2. Let us notice that the points close to the axis $\lambda = 0$ belong to the area on the (λ, m) plane where the solution exists. This allows us to discuss the limit $\lambda \to 0$ independently of the value of m. When $\lambda \to 0+$ then the positive roots of (2.27) approach the point $\tilde\sigma = \frac{3}{2}\pi m$ and satisfy (2.25). The negative root approaches minus in-

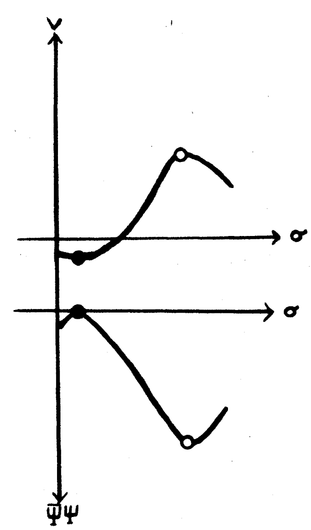

FIG. 4. The same as Fig. 3 except that the fermions are massive. The suitable choice of the induced σ^4 coupling strength allows the minimum to occur at $\bar\psi\psi = 0$.

finity faster than $-2\pi/\lambda^{1/2}$. Equation (2.26) reduces to

$$\tilde{\sigma}^2 \ln \frac{\tilde{\sigma}^2}{s^2} = -\frac{4\pi^2}{3\lambda} \to -\infty, \tag{2.28}$$

and therefore $s^2 \to 0+$ as $\lambda \to 0+$. This means that in the limit of small λ the induced coupling constant η becomes very large so the model does not reduce itself to the free one and offers a nice example of Klauder's phenomenon.[16]

Suppose that we have taken m and λ from the suitable region and we have calculated $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and s^2. Now formula (2.5) is a correct expression for the two-point Green's function provided we replace m^2 by $\tilde{\sigma}_1{}^2$ and we set

$$\ln \frac{\tilde{\sigma}_1{}^2}{\mu^2} = \ln \frac{\tilde{\sigma}_1{}^2}{s^2} + 1 - \gamma. \tag{2.29}$$

We are just in a position to examine what kind (if any) of poles occur in the σ propagator.

Let us observe that the function $B(k^2, m^2)$ which was defined by (2.6) increases monotonically when k^2 tends from zero to minus infinity (spacelike momenta). For $k^2 = 0$ we have $B = 2$, independently of the value of m^2. The condition for the absence of tachyon poles is

$$\lambda \Pi(p^2 = 0) > 1. \tag{2.30}$$

This, when combined with (2.5), implies

$$\tilde{\sigma}_1{}^2[\gamma - 1 + \ln(\tilde{\sigma}_1{}^2/\mu^2)] > \frac{4\pi^2}{3\lambda}. \tag{2.31}$$

Using (2.26) and (2.29) we obtain

$$\tilde{\sigma}_1{}^2 < -4\pi^2/\lambda.$$

Both λ and $\tilde{\sigma}_1{}^2$ are positive, hence a tachyon is unavoidable. Bound-state poles are absent from the σ propagator because $B(k^2, \tilde{\sigma}_1{}^2)$ decreases when k^2 tends from zero to $4\tilde{\sigma}_1{}^2$. To the right of the branch point $k^2 = 4\tilde{\sigma}_1{}^2$ B increases when k^2 tends to infinity, but now $-k^2 < -4\tilde{\sigma}_1{}^2$ and the term $-\frac{1}{2}k^2 B(k^2, \tilde{\sigma}_1{}^2)$ in (2.5) dominates $2\tilde{\sigma}_1{}^2 B(k^2, \tilde{\sigma}_1{}^2)$ therefore $\Pi(p^2, \tilde{\sigma}_1{}^2)$ still decreases (up to minus infinity). It is obvious that also the resonance poles are absent from the σ propagator.

III. CONSTRUCTION OF THE NONLOCAL IMPROVED σ PROPAGATOR

The aim of this section is to construct explicitly the additional term by which the effective action should be augmented in order to make the model have correct spectral properties and to satisfy the unitarity and microcausality conditions in the Bogoliubov sense. It is intuitively clear that the modification should affect only those Green's functions which involve infinite summations. To any finite order in $1/N$ there are no infinite summations except the bubble chain for the σ two-point function. The only possible refinement of the effective action can be realized by augmenting the generating functional by a term quadratic in the σ variable and having the form

$$\sigma f(\Box l^2)\sigma, \tag{3.1}$$

where l is a dimensional parameter which plays the role of the elementary length.

If we want the nonlocal term to cancel the pole at p_0 we should take f such that its negative of the inverse of the Fourier transform is of the form

$$\frac{W(p^2 l^2)}{p^2 - p_0{}^2}. \tag{3.2}$$

If the residue at the pole equals R_0, then

$$W(p_0{}^2 l^2) = -R_0. \tag{3.3}$$

Nonlocal models with propagators such as (3.2) were extensively studied in the past.[12,13] Rigorous proofs exist[13] that unitarity of the S matrix requires W to be an entire function whose order of growth ρ satisfies

$$\tfrac{1}{2} \le \rho < 1. \tag{3.4}$$

If $\rho = \frac{1}{2}$ then also Bogoliubov's microcausality condition[14] is satisfied. For the sake of clarity the construction will be performed for an arbitrary value of ρ from the interval (3.4). Putting $\rho = \frac{1}{2}$ at the end of the calculation we can compare the result with the case $\rho > \frac{1}{2}$ when microcausality is violated.

The requirement of Hermiticity leads to

$$[W(p^2 l^2)]^* = W((p^2 l^2)^*). \tag{3.5}$$

Our final assumption is that for $p^2 \to -\infty$,

$$W(p^2 l^2) = \frac{\text{const}}{\ln(-p^2 l^2)} + O\left(\frac{1}{\ln^2(-p^2 l^2)}\right) \tag{3.6}$$

in accord with our belief that diagrammatical calculations give the correct predictions for the asymptotic behavior. It is possible to construct $W(p^2 l^2)$ from the requirements (3.3)–(3.6). The general method of construction is due to Evgrafov.[15] We shall apply this method as we have learned it from the paper of Efimov and Mogilevsky,[6] where it was used to solve a similar problem in nonlocal QED.

The main hint is that our task can be alternatively stated as a task to construct a subharmonic function $\Phi(z)$ such that

$$\Phi(z) = \ln|W(z)|. \tag{3.7}$$

One can assume that in n infinite regions D_k of the z plane, obtained by dividing the plane by n curves L_k starting from one point, $\Phi(z)$ is equal to a certain harmonic function $U_k(z)$. On each L_k the continuity condition

$$U_k(z) = U_{k+1}(z) \quad \text{for } z \in L_k \tag{3.8}$$

is satisfied.

Denoting by $\sigma_k(s)$ the discontinuity of the normal derivative of U across L_k,

$$\sigma_k(s) = \left(\frac{\partial U_k(z)}{\partial n} - \frac{\partial U_{k+1}(z)}{\partial n} \right)\Bigg|_{z=z_k(s)}, \quad z_k(s) \in L_k, \tag{3.9}$$

one can write $\Phi(z)$ in the form

$$\Phi(z) = \sum_{k=1}^{n} \frac{1}{2\pi} \int_{L_k} \ln\left|1 - \frac{z}{z_k(s)}\right| \sigma_k(s)\, ds \tag{3.10}$$

or, equivalently,

$$\Phi(z) = \sum_{k=1}^{n} \int_{L_k} \ln\left|1 - \frac{z}{z_k(s)}\right| d\mu_k(s), \tag{3.11}$$

where

$$\mu_k(s) = \frac{1}{2\pi} \int_0^s \sigma_k(t)\, dt. \tag{3.12}$$

$W(z)$ can be represented in the form of the canonical product

$$W(z) = A \prod_{m=1}^{n} W_k(z), \tag{3.13}$$

$$W_k(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{\lambda_k(m)}\right), \tag{3.14}$$

where

$$\lambda_k(m) = z_k(s_k(m)). \tag{3.15}$$

We define $s_k(m)$ as the function inverse to $\mu_k(s)$.

Let us assume that the $z = p^2 l^2$ plane splits in two regions D_1 and D_2 separated by a trajectory of zeros L. In one region, involving the large-negative-x part of the real axis, $W(z)$ is expected to behave like $1/\ln(-z)$ [cf. (3.6)]. In the second region we suppose that W behaves like $\exp(\rho z)$ [cf. (3.4)]. The trajectory of zeros L is the curve on which the logarithms of the moduli of both functions are equal to each other [cf. (3.7) and (3.8)]. The form factor should be real on the real axis, therefore L should be symmetric with respect to the real axis [cf. (3.5)]. Hence L is defined by the following equation:

$$\ln\left|\frac{1}{\beta \ln(-z)}\right| = \ln\left|\exp(z^\rho e^{i\pi(1-\rho)})\right|, \tag{3.16}$$

where β is to be determined. Substituting $z = re^{i\varphi}$ we obtain

$$r^\rho \cos\rho(\varphi - \pi) = \ln\beta + \ln[(\ln r)^2 + (\varphi - \pi)^2]^{1/2}. \tag{3.17}$$

Now the symmetry of L with respect to the real axis is manifest, as (3.17) does not change when one replaces φ by $2\pi - \varphi$. The intersection point of L with the real axis remains unprecise. It is very natural to suppose that the intersection takes place at the site of the tachyon pole. Indeed, the normal derivative across the trajectory of zeros is discontinuous. If $z_0 = p_0^2 l^2$ is the intersection point, then the discontinuity of the derivative of the four-fermion elastic amplitude which occurs at p_0^2 can be understood as a reflection of the singular point of the (local) σ propagator. If the trajectory of zeros intersects the axis at $\tilde{z} \neq z_0$, then the momentum transfer $\tilde{p}^2 = \tilde{z}/l^2$ is singled out by the procedure for no reason whatever.

From (3.17) and the assumption that $z_0 = r_0$ is the intersection point we obtain (cf. Ref. 5)

$$\begin{aligned} r_0^\rho &= \ln\beta + \ln\ln r_0, \\ r_0^\rho \ln r_0^\rho &= 1. \end{aligned} \tag{3.18}$$

Solving (3.18) approximately we obtain

$$r_0^\rho \approx 1.76, \quad \beta \approx 10\rho. \tag{3.19}$$

From (3.9) we have

$$\sigma(s) = \frac{\partial}{\partial n}\left(\frac{1}{10\rho \ln(-z)}\right) - \frac{\partial}{\partial n}\left[\exp(z^\rho e^{i\pi(1-\rho)})\right]. \tag{3.20}$$

Calculating $\Phi(z)$ according to (3.10), we obtain on the real axis,

$$\Phi(x) = \begin{cases} \ln\dfrac{1}{\ln(-x)^\beta} & \text{for } x < x_0, \\ -|x|^\rho & \text{for } x_0 < x < -0, \\ x^\rho \cos\pi(1-\rho) & \text{for } x > +0, \end{cases} \tag{3.21}$$

where $x = l^2 p^2$, $x_0 = l^2 p_0^2$. We are now in a position to evaluate the remaining parameters l^2 and A. On the real axis,

$$W(x) = A \exp[\Phi(x)]. \tag{3.22}$$

Making use of (3.6), (3.19), and (3.21) we obtain

$$A = -R_0 \exp(-1.76). \tag{3.23}$$

From (3.20) and (3.21) we immediately anticipate that for $\frac{1}{2} < \rho < 1$ the form factor grows exponentially for timelike momenta. If we take $\rho = \frac{1}{2}$ then the difference is sharp: for $p^2 > 0$, $\Phi(x) = 0$, the form factor is constant in the timelike region and the region of growth is thrust away to the unphysical domain of complex momenta. For spacelike momenta W decreases, first exponentially then, from $p^2 = p_0^2$, logarithmically. The elementary length l is not an independent parameter. From (3.19) we obtain

$$l \approx \frac{1.76}{|p_0^2|}, \tag{3.24}$$

where we have substituted $\rho = \frac{1}{2}$, and p_0 is the solution of

$$\Pi(p_0^2) = 1/\lambda .$$

From (2.5), (2.6), and (2.28) we anticipate that p_0 tends to a finite value in the limit λ going to $0+$:

$$p_0^2 \to -(\tfrac{3}{2})^3 \pi^2 m^2 \quad (\lambda \to 0+). \tag{3.25}$$

The residue R_0 is equal to

$$R_0 = 1/\Pi'(p_0^2), \tag{3.26}$$

where the prime denotes differentiation with respect to p^2. In the limit $\lambda \to 0$, R_0 tends to zero like

$$R_0 \sim \lambda m^2/3 \quad (\lambda \to 0). \tag{3.27}$$

The normalization factor A is proportional to R_0 hence the form factor vanishes in the limit $\lambda \to 0$ but, as expected, its asymptotic expansion is not identically zero.

IV. SUMMARY AND CRITICISM

Section II was fully devoted to the calculation and examination of the effective potential. We have found that even if one evaluates Green's functions around the minimum of the effective potential then one finds a tachyon pole in the σ propagator. We have accepted this result quite unconcernedly; however, usually the presence of the tachyon pole means one of two things: Either the minimum was chosen erroneously or an important class of diagrams was overlooked in calculations. Of course, the theory may also be in error from the beginning. Here we have modestly confined ourselves to the leading order in $1/N$ and the requirement of the cancellation of the unwanted pole has even helped us to determine the nonlocal form in the σ propagator. We know nothing yet about higher orders. We can not exclude the possibility that a new kind of inconsistency, e.g., the complexity of the effective potential for all values of the classical fields, will appear in next to leading orders. Root[7] has demonstrated that such an effect takes place in the renormalizable $(1/N)(\Phi^4)_4$ model if one expands around the (incorrect) minimum where tachyons come about.

We have also rejected the possibility that λ may be negative. In fact considering the two-dimensional analog of our model one finds that for $\lambda < 0$ it is still consistent, but a collective mode does not exist in the $\bar{\psi} - \psi$ channel. Instead one finds the superconducting phase and collective modes in the $\psi - \psi$ channel.[22]

It would be instructive to examine this case also, as well as other nonrenormalizable models. Let us note that this procedure can be easily reproduced also in the case when spectral properties are correct from the beginning. The requirement of the cancellation of the tachyon pole has led us to a unique normalization condition for the nonlocal term. We have simply asked that the residue of the nonlocal term cancel the residue of the propagator obtained by conventional summation. Were we to deal with the bound-state pole, then the most natural condition would be to require the residue to be the same as that of the free scalar propagator ($-i$ in our conventions).

In Sec. III the nonlocal form factor was constructed. Obviously, such construction must be highly arbitrary. The task is hopeless if one is restricted only to diagrammatical considerations. Our conditions (3.3)–(3.6) determine a large class of possible solutions of the problem and by no means can be realized in a unique way. Let us specify additional requirements which have been used in constructing the result (3.21).

First of all we have rejected all but the first term in (3.6). There is no formal mathematical reason for this step; however, if we do it then the large-spacelike-momentum behavior of the nonlocal propagator is ("exactly" instead of "approximately" up to $1/\ln p^2$) the same as that obtained by conventional summation of diagrams. The modified condition (3.6) with amputated higher powers of the logarithm cannot be satisfied together with (3.4) by a completely regular function. This does not offer any problem because we actually need only the continuity of the form factor. Following Efimov and Mogilevsky[6] we have taken the simplest version: W is composed of two functions, one satisfying (3.4) and the second satisfying (3.6), continuously pieced together along the curve on which they are equal to each other. This line is the trajectory of zeros of the entire function W. The trajectory of zeros passes across the real axis through the point which was the position of the tachyon pole, otherwise the procedure would single out an additional point in the p^2 plane. The trajectory of zeros uniquely defines the entire function; hence the solution is found.

The resulting theory has very attractive features: On the real axis the form factor is a bounded function and is constant for timelike momenta. This implies that the leading-order elastic fermion-fermion scattering amplitude vanishes for timelike momenta such as $1/p^2$. In the local case it has vanished like $1/(p^2 \ln p^2)$ and the elastic unitarity was violated.

We have just warned that the modification of the effective action usually changes the properties of the effective potential and it may happen that a new vacuum has different features when compared with the old one. If we want to leave all the discussion of Sec. II unaltered (and, of course, we do) we must shift the σ-mass parameter $1/\lambda$ by the value at $p^2=0$ of the additional nonlocal term,

$$\frac{1}{2\lambda} \rightarrow \frac{1}{2\lambda} - \frac{p_0^{\,2}}{2R_0} e^{1.76}.$$

The remaining terms of the expansion in powers of p^2 do not contribute to the effective potential. Such redefinition does not change the general features of the $\lambda \rightarrow 0$ limit because then

$$p_0^{\,2}/2R_0 \rightarrow (\tfrac{3}{2})^4 \pi^2/\lambda\,,$$

and the shift reduces to the finite multiplicative renormalization of λ.

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