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Euler-Lagrange equation and conservation laws for bilocal fields

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A Lagrangian bilocal field theory is presented as an extended meson model. The Euler-Lagrange equation is found for the fields which vanish outside of a finite internal domain. For such bilocal fields the proper energy-momentum and angular momentum tensors can be defined.

I. INTRODUCTION

It has been over twenty-five years since Yukawa first proposed his bilocal field theory as a theoretical framework for hadrons.[1] Since then, several workers have attempted to develop a physically meaningful theory using the concept of bilocal fields.[2-8] Some of the workers concerned themselves with the general properties of bilocal fields and others tried to apply it to hadron physics in a concrete manner. Yukawa's original idea of the bilocal field assumed no local constituents. The field was introduced to describe a nondecomposable, elementary system containing a variety of particles with different masses, spins, and other intrinsic properties.[4]

Recently Preparata and Craigie proposed a massive quark model wherein the fields which depend on two space-time coordinates of quarks were used.[9] As pointed out by Chiang,[10] this model possesses features equivalent to Yukawa's bilocal field theory. In an attempt to give a more precise mathematical framework to the approach of Preparata and Craigie, Capri and Chiang suggested a different model of extended meson fields.[11]

The intention of the present article is to examine some of the dynamical properties of bilocal field theory, such as the Euler-Lagrange equation and the conservation laws associated with space-time symmetry. The formalism can be viewed as a direct generalization of local Lagrangian field theory. Following the usual local theory, we shall use the variational principle to find the Euler-Lagrange equation for the bilocal field. When the field vanishes outside of a finite internal domain, the Euler-Lagrange equation appears to be a simple generalization of the local equation. For such bilocal fields, there exist proper conservation laws of energy-momentum and angular momentum.

II. BILOCAL FIELD AND ITS FIELD EQUATION

Consider a scalar field ϕ depending on two points x_1 and x_2 in Minkowski space. We will regard these points as the space-time coordinates of the constituents (quark and antiquark pair) of the mesons. The Lagrangian density in general may be written as

$$\mathfrak{L} = \mathfrak{L}\left[x_1, x_2; \phi(x_1,x_2), \frac{\partial}{\partial x_1^\mu}\phi(x_1,x_2), \frac{\partial}{\partial x_2^\mu}\phi(x_1,x_2)\right]. \quad (1)$$

The action is defined by the integral

$$S = \int\int d^4x_1\, d^4x_2 \mathfrak{L}. \quad (2)$$

To treat the bilocal field properly, we introduce the center-of-mass (external) coordinate X and the relative (internal) coordinate x

$$X = \alpha_1 x_1 + \alpha_2 x_2, \quad \alpha_1 + \alpha_2 = 1$$
$$x = x_1 - x_2. \quad (3)$$

The field is now denoted as $\phi(X,x)$ and the Lagrangian density may be rewritten as

$$\mathfrak{L} = \mathfrak{L}[x; \phi(X,x), D_\mu\phi(X,x), d_\mu\phi(X,x)], \quad (4)$$

where $D_\mu = \partial/\partial X^\mu$ and $d_\mu = \partial/\partial x^\mu$. Note that the explicit dependence on X is eliminated from the Lagrangian density thereby ensuring translational invariance.

The infinitesimal volume element is unchanged by the redefinition of the coordinates and the action is given by

$$S = \int_\Omega d^4X \int_\omega d^4x \mathfrak{L}. \quad (5)$$

Here the external integration domain Ω may be considered to be the same as that of the usual local field theory. We now impose on the bilocal field the requirement that the quarks are permanently confined. As suggested by Preparata and Craigie,[9] a simple way to achieve this is to assume

$$\phi(X,x) = 0, \quad \text{for } x \notin \omega, \quad (6)$$

where ω is a finite space-time domain. For Lagrangian densities composed of products of ϕ, $D_\mu\phi$, and $d_\mu\phi$, the above condition implies that the dominant contribution to the action integral S comes from a bounded internal integration domain ω. We will refer to the hypersurface enclosing ω as the "cell" and denote it by C. Admittedly the cell is not a Lorentz-invariant notion. The form of the cell depends on the momentum of the bilocal system.

In the present discussion, the quark and antiquark are not associated with separate fields. Their presence is represented by the space-time coordinates of the bilocal field. A question may arise as to what binds the quarks and causes the relative coordinate x to be always finite. In this model, use is not made of a mediating field between the constituents. Instead, the interaction between the constituents is directly expressed in terms of the internal coordinate x.[12] The explicit dependence on x appearing in the Lagrangian density represents this direct interaction.

Let us now find the appropriate bilocal field equation from the action principle. As in the usual local theory, the field variation $\delta\phi$ is assumed to vanish on the boundary surface of Ω. The change of the action integral S due to the variation with respect to the field ϕ is given by

$$\delta S = \int_\Omega d^4X \int_\omega d^4x \left[\frac{\delta\mathcal{L}}{\delta\phi} - D_\mu \frac{\delta\mathcal{L}}{\delta(D_\mu\phi)} - d_\mu \frac{\delta\mathcal{L}}{\delta(d_\mu\phi)}\right]\delta\phi + \int_\omega d^4x \int_{\substack{\text{surface of}\\ \Omega}} ds_\mu \frac{\delta\mathcal{L}}{\delta(D_\mu\phi)}\delta\phi + \int_\Omega d^4X \int_C d\sigma n_\mu \frac{\delta\mathcal{L}}{\delta(d_\mu\phi)}\delta\phi, \tag{7}$$

where n_μ is the unit four-vector normal to the cell C. The second term in Eq. (7) vanishes since $\delta\phi = 0$ on the boundary surface of Ω. Owing to the condition expressed by Eq. (6), the third term in Eq. (7) also vanishes. Then the principle of stationary action $\delta S = 0$ demands

$$\frac{\delta\mathcal{L}}{\delta\phi} - D_\mu \frac{\delta\mathcal{L}}{\delta(D_\mu\phi)} - d_\mu \frac{\delta\mathcal{L}}{\delta(d_\mu\phi)} = 0 \tag{8}$$

for arbitrary $\delta\phi$.

III. CONSERVATION LAWS

To examine the conservation laws,[13] let us define the Lagrange functional[14]

$$\mathcal{K} = \int_\omega d^4x \mathcal{L}. \tag{9}$$

This is an implicit function of X and a functional of ϕ, $D_\mu\phi$, and $d_\mu\phi$. Thus we may write

$$\mathcal{K} = \mathcal{K}[(X); \phi, D_\mu\phi, d_\mu\phi]. \tag{10}$$

The functional $\mathcal{K}$ will replace the Lagrangian density $\mathcal{L}$ in the consideration of conservation laws for the bilocal field.

With the simultaneous, infinitesimal displacements of x_1 and x_2,

$$x_{r\mu} \to x'_{r\mu} = x_{r\mu} + \epsilon_\mu, \quad r = 1, 2 \tag{11}$$

and

$$X_\mu \to X'_\mu = X_\mu + \epsilon_\mu, \tag{12}$$

the field transforms as

$$\phi(X, x) \to \phi(X+\epsilon, x) = \phi(X, x) + \delta\phi(X, x) \tag{13}$$

and the variation of the field is given by

$$\delta\phi(X, x) = D_\nu\phi(X, x)\epsilon^\nu. \tag{14}$$

The translationally invariant Lagrange functional does not have an explicit dependence on X. To first order in ϵ_μ, the change in $\mathcal{K}$ is given by

$$\delta\mathcal{K} = D_\nu\mathcal{K}\epsilon^\nu = \left\{D_\mu\left[\frac{\delta\mathcal{K}}{\delta(D_\mu\phi)}\cdot D_\nu\phi\right] + \int_C d\sigma n_\mu \frac{\delta\mathcal{L}}{\delta(d_\mu\phi)}\cdot D_\nu\phi\right\}\epsilon^\nu. \tag{15}$$

The second line is due to the variation of the field and the Euler-Lagrange equation is used to obtain this expression. The dot denotes an implicit integration process

$$A\cdot B = \int\int d^3x\, d^3y A(x, y)B(x, y) \tag{16}$$

for arbitrary functions A and B. In view of condition (6), the surface integral over the cell vanishes and Eq. (15) can be rearranged and put into the form

$$D^\mu T_{\mu\nu} = 0 \tag{17}$$

with the energy-momentum stress tensor $T_{\mu\nu}$ defined by

$$T_{\mu\nu} = \frac{\delta\mathcal{K}}{\delta(D^\mu\phi)}\cdot D_\nu\phi - g_{\mu\nu}\mathcal{K}. \tag{18}$$

The above two equations express the differential conservation law of the energy-momentum of the system. The quantities given by

$$P_\nu = \int d^3X\, T_{0\nu} \tag{19}$$

are constants of motion. This may be regarded as the energy-momentum four-vector associated with the bilocal field.

Let us now turn to an examination of the conservation law under homogeneous Lorentz transformations. We perform simultaneous, infinitesimal Lorentz transformations of x_1 and x_2,

$$x_{r\mu} \to x'_{r\mu} = \Lambda_{r\nu}x^\nu_r, \quad r = 1, 2 \tag{20}$$

where

$$\Lambda_{\mu\nu}=g_{\mu\nu}+\epsilon_{\mu\nu}, \quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu}. \tag{21}$$

Then both the external and the internal coordinates are transformed as in Eq. (20)

$$X_\mu \to X'_\mu = \Lambda_{\mu\nu}X^\nu, \quad x_\mu \to x'_\mu = \Lambda_{\mu\nu}x^\nu. \tag{22}$$

For the scalar field, the transformation matrix is unity and

$$\phi'(X', x') = \phi(X, x). \tag{23}$$

When the Lorentz invariance of the Lagrangian density is assumed, the Lagrange functional defined by Eq. (9) is also Lorentz invariant.

A convenient test of the Lorentz invariance is to rewrite Eq. (23) as

$$\phi'(X, x) = \phi(X', x') \tag{24}$$

and require

$$\mathcal{K}[(X);\phi', D_\mu\phi', d_\mu\phi'] = \mathcal{K}[(X');\phi, D'_\mu\phi, d'_\mu\phi]. \tag{25}$$

To first order in $\epsilon_{\mu\nu}$, the right-hand side of the above equation can be expanded as

$$\mathcal{K}[(X');\phi, D'_\mu\phi, d'_\mu\phi] = \mathcal{K} + D_\nu\mathcal{K}\epsilon^{\nu\lambda}X_\lambda \tag{26}$$

and the left-hand side as

$$\mathcal{K}[(X);\phi', D_\mu\phi', d_\mu\phi'] = \mathcal{K} + D_\mu\left[\frac{\delta\mathcal{K}}{\delta(D_\mu\phi)}\cdot\delta\phi\right] + \int_C d\sigma n_\mu \frac{\delta\mathcal{L}}{\delta(d_\mu\phi)}\cdot\delta\phi, \tag{27}$$

where the variation of the field is given by

$$\delta\phi(X, x) = D_\nu\phi(X, x)\epsilon^{\nu\lambda}X_\lambda + d_\nu\phi(X, x)\epsilon^{\nu\lambda}x_\lambda. \tag{28}$$

Because of Eq. (6), the surface integral over the cell in Eq. (27) vanishes. We now equate Eqs. (26) and (27) and rearrange the terms to obtain

$$D^\mu\mathfrak{M}_{\mu\nu\lambda} = 0, \tag{29}$$

where the tensor $\mathfrak{M}_{\mu\nu\lambda}$ is given by

$$\mathfrak{M}_{\mu\nu\lambda} = (X_\nu T_{\mu\lambda} - X_\lambda T_{\mu\nu}) + \frac{\delta\mathcal{K}}{\delta(D^\mu\phi)}\cdot\Sigma_{\nu\lambda}\phi \tag{30}$$

and

$$i\Sigma_{\nu\lambda} = x_\nu p_\lambda - x_\lambda p_\nu. \tag{31}$$

Here $p_\mu = id_\mu$, and the energy-momentum stress tensor has been previously defined by Eq. (18). The first term in Eq. (30) expresses the orbital angular momentum of the overall motion. The second term is due to the relative motion of the constituents and we may regard $i\Sigma_{\nu\lambda}$ as the "spin tensor" of the mesons. In this model the spin of the constituents is not taken into consideration. The conserved angular momentum is given by

$$M_{\nu\lambda} = \int d^3X\mathfrak{M}_{0\nu\lambda}. \tag{32}$$

Having examined the Lagrangian formalism and established the conservation laws associated with space-time symmetry, we may quantize the bilocal field by defining a suitable canonical conjugate momentum. An appropriate choice of a Lagrangian density may give rise to the proper meson spectroscopy. These points will be discussed elsewhere.

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