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Harmonic maps as models for physical theories

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Harmonic maps are an aesthetically appealing class of nonlinear field equations of which only a few nontrivial examples have as yet appeared in physical theories. These fields appear well suited for describing broken symmetries either in conjunction with or instead of the Yang-Mills equations. The harmonic mapping equation is quite similar in many respects to the Einstein equations for gravitation, although simpler in structure, and can describe any gauge symmetry group G broken to a subgroup H in a sense parallel to the way the gravitational (metric) field breaks general covariance [local $GL(4,R)$ invariance] down to local Lorentz invariance. This paper outlines the basis for a program of exploring harmonic mapping theories to see whether they may provide models of physical phenomena that either are not recognized, or are not well fitted to more familiar field theories.

I. INTRODUCTION

For about a decade before unified gauge theories of weak and electromagnetic interactions were introduced[1], Yang-Mills [2] theories continued to be investigated as model theories on two grounds more remote from phenomenology than their current uses in particle physics (Weinberg[3] gives a recent review). This paper proposes that another class of theories, those based on harmonic maps[4,5] deserve some attention from theorists on similar grounds, with a possibility open that direct applications in particle theory may also appear.

The two grounds that motivated studies of Yang-Mills theories in their esoteric period were (1) analogies to general relativity and (2) Pythagorean prejudices. I will appeal to both motives as favoring harmonic maps as well. The analogy of Yang-Mills theories to general relativity is widely appreciated (e.g., Utiyama[6]) and was explicitly the motivation for the detailed development of the Feynman rules for these theories (Feynman,[7] DeWitt,[8,9] Faddeev and Popov,[10] and Mandelstam[11]). By Pythagorean prejudices I mean a particular style of aesthetic or *thematic* (Holton[12,13]) considerations whose description I defer a few paragraphs so that uncommittedly skimming readers may be first shown some familiarizing examples of harmonic maps.

The wave equation or Laplace equation—depending on the signature of the metric $g_{\mu\nu}(x)$—reads, for a scalar field $\phi(x)$,

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}\right)=0\,, \tag{1.1}$$

and characterizes *harmonic functions* ϕ from which the class of harmonic maps takes its name. But the (typically) nonlinear geodesic equation,

$$G_{AB}(\phi)\frac{d^2\phi^B}{d\lambda^2}+\Gamma_{ABC}(\phi)\frac{d\phi^B}{d\lambda}\frac{d\phi^C}{d\lambda}=0\,, \tag{1.2}$$

is also a specialized subclass of the harmonic maps. The generic harmonic map combines aspects of both these equations in the nonlinear partial differential equation derived from the action integral

$$I=\frac{1}{2}\int\sqrt{|g|}\, d^nx g^{\mu\nu}(x)\frac{\partial\phi^A}{\partial x^\mu}\frac{\partial\phi^B}{\partial x^\nu}G_{AB}(\phi)\,. \tag{1.3}$$

The model theories of this class that I expect to be of most interest for physics are those where $g^{\mu\nu}(x)$ is the flat Minkowski metric or, in some quantum applications, its flat Euclidean continuation. A nontrivial example of this class of theories is the nonlinear σ model[14] where $G_{AB}(\phi)$ is the metric of a three-sphere and ϕ^A are three independent fields parametrizing four meson fields $\pi^k (k=1,2,3)$ and σ that satisfy

$$\pi^k\pi^k+\sigma^2=f^2=\text{const}$$

as an identity.

Note that when (as in the nonlinear σ model) the range of kinematically allowed field values ϕ do not form a vector space, so that $a\phi_1+b\phi_2$ is not even defined, there is no way to write (or define) a linear field equation, and the harmonic map equation is the simplest evident generalization of the wave equation. To write it requires that the space of field values have (or be given) a Riemannian or pseudo-Riemannian metric $G_{AB}(\phi)$. This is always possible, and introduces only a few arbitrary constants if the field values form a homogeneous space (Helgason[15]), i.e., a coset space G/H, the quotient of a Lie group G by some closed subgroup H as in $S^2=SU(2)/U(1)$. (When G/H is flat, a linear wave equation results.) A simple example from a class described more fully in another paper (Misner[16]) is a field of Hermitian projection matrices $\phi_a{}^b$ parametrizing $SU(n)/[U(p)\times SU(n-p)]$. The $\phi^2=\phi$ projection property

$$\phi_a{}^c\phi_c{}^b=\phi_a{}^b \tag{1.4}$$

provides the essential nonlinearity (and a condition $\mathrm{tr}\phi=p$ with $0<p<n$ for $n\times n$ matrices ϕ guarantees nontriviality). For these fields the natural harmonic mapping equation is

$$(1-\phi)(\Box\phi)\phi=0 \tag{1.5}$$

in a matrix notation or more explicitly

$$(\delta_a{}^b-\phi_a{}^b)(\partial_\mu\partial^\mu\phi_b{}^c)\phi_c{}^d=0\,, \tag{1.5$'$}$$

where $\Box=\partial_\mu\partial^\mu$ is the wave operator in flat spacetime. Note, in particular, that this equation can be naturally written with nonlinearities in the leading, second-derivative terms, in contrast to Yang-Mills equations where the leading terms are linear.

My acquaintance with harmonic maps began when Matzner and I[17] found that Einstein's gravitational equations in a highly specialized case derived from a Lagrangian

$$L=(\vec\nabla\beta)^2-\cosh^2\beta(\vec\nabla\alpha)^2\,,$$

that involved only flat-space gradient operators $\vec\nabla$ but implied a curved metric in the range space of the field values (spacetime metric parameters) α and β. Continued inquiry eventually revealed (Smale,[18] private communication) that this geometrically appealing class of nonlinear partial differential equations had, as I had presumed, received some study—a comparatively recent fundamental paper by Eells and Sampson.[5] Thereafter, Yavuz Nutku became infected from my interest in harmonic maps and has found useful applications of them in solutions of Einstein's equations (Nutku,[19] Eris and Nutku,[20] Eris,[21] and Nutku and Halil[22]). There appear to be no other deliberate attempts to find a rôle for them in physical theories.

I find four principal lines suggesting that harmonic maps deserve investigation and development by theoretical physicists. Each is described more fully in a later section of the paper. Section III: The harmonic maps model, in a simplified form, or a type of nonlinearity that occurs in the Einstein equations, but different from that modeled in Yang-Mills fields. Section IV: For any desired broken symmetry (from any group G to a subgroup H), a harmonically mapped bundle section is one conceivable mechanism. Section V: Gauge vector fields (connections in bundles) can be defined using solutions of the harmonic mapping equations, instead of the more familiar, inequivalent, Yang-Mills equations. Section VI: For many harmonic mapping theories, related ("relaxed") theories should be renormalizable, and there are suggestions for nonperturbative quantization attempts on the strict harmonic mapping theories.

Let us now return to the Pythagorean motivations for exploring harmonic maps. The thematic component of science, according to Holton (Ref. 12, p.13) consists of "unverifiable, unfalsifiable, and yet nonquite-arbitrary hypotheses" that "belong to a pool of specifically scientific ideas but spring from the more general ground of the imagination." One such theme is the hypothesis of atomism. Not, to be sure, the known atom of Bohr or Schrödinger, but the Holy Grail, the *ur-atom*, the fundamental discrete constituent underlying molecules, atoms, elementary particles, quarks, etc., from which these more accessible objects are to be constructed, and thus understood. The countervailing theme, the continuum, of course also lurks, and seeks a primary ether, less mechanistic than discarded electromagnetic ethers, more versatile than elastic, energetic, curved spacetime—an ether quantum mechanically stirring or resonating as more and less sharply defined particles. By contrast to these, the "Pythagorean" theme rejects or ignores the underlying physical substratum as a significant constituent of human insight into, or understanding of, Nature. It focusses instead on the program of reconstituting appearances from substructure, on the network of relationships and interaction among subentities that organize them into other significant structures. "God is a Geometer" could be the motto of this brotherhood. Note, however, that geometry is currently a term of esthetic approbation awarded selected architectures in a world of Bourbaki-style[23] mathematical structures.

It is useful to claim lineage for themes in current forms back through their historical antecedents, because the themes are valuable not for their truth values, but as guides to significance. A theory must explain Nature to Man, and no amount of truth is explanatory if its bases appear as capricious, disorganized, and inaccessible as the relevant observations themselves. Thus historical themata are a guide to lines of thought that would be satisfying (if found applicable) on the basis of a prejudice that is at least cultural and not merely personal. As Pythagorean styles in modern times one can cite the current elaboration of symmetries and conservation laws, Dirac's algebraic insights, Cartan's invention of affine connections to geometrize Newtonian gravity, Einstein's general covariance, Clifford's space theory of matter, Riemann's geodesy, etc. This list names only few and recent examples of many heros who were at least as concerned to hear the harmonies of the spheres as to see the instruments on which this celestial music was played, and the tradition continues back to Pythagorean ecstasies in discovering that pure numbers could engender things as diverse as geometrical shapes and musical chords. The beautifully balanced and simple variational principle (1.3) for harmonic maps may be only an unsophisticated bauble like the crystalline spheres that moved in the heavens before Kepler, but searching for its traces in Nature would seem to be an honorable quest.

II. HARMONIC MAPS

Let M and M' be two pseudo-Riemannian manifolds, with x^μ coordinates on M and ϕ^A coordinates on M'. We will normally think of M as space or spacetime and will in most examples specialize its metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{2.1}$$

to a flat Minkowski or Euclidean metric. The M' manifold is the set of possible values for some naturally nonlinear field ϕ. The nonlinearity enters because we think of the metric on M',

$$dL^2 = G_{AB}(\phi)d\phi^A d\phi^B\,, \tag{2.2}$$

as being curved, since a flat dL^2 would only lead to long familiar linear equations. A mapping

$$\phi:\ M\to M'\,,\ x\to\phi x\equiv\phi(x) \tag{2.3}$$

will be represented in coordinates as $\phi^A(x^\mu)$. It will be called a harmonic map if it satisfies the Euler-Lagrange equations of the variational principle $\delta I=0$ using the action I (called energy by Fuller[4]) of Eq. (1.3). These field equations read

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\,g^{\mu\nu}\frac{\partial\phi^A}{\partial x^\nu}\right) + \Gamma^A{}_{BC}(\phi)\frac{\partial\phi^B}{\partial x^\mu}\frac{\partial\phi^C}{\partial x^\nu}g^{\mu\nu}(x)=0\,. \tag{2.4}$$

Here $\Gamma^A{}_{BC}$ are the Christoffel symbols of the ϕ metric dL^2 on M'. We will later condense this equation to the form $\phi^{A\mu}{}_{;\mu}=0$ with a suitable covariant derivative, and with $\phi^A{}_\mu\equiv\partial\phi^A/\partial x^\mu$. Let us first study a simple but interesting example.

Let M be a flat Euclidean or Minkowski space, and for M' take the sphere S^2 with its usual metric and coordinates

$$dL^2 = d\Theta^2+\sin^2\Theta d\Phi^2\,. \tag{2.5}$$

A mapping is then a pair of fields $\Theta(x^\mu)$, $\Phi(x^\mu)$ satisfying differentiability requirements derived from the structure of S^2 and of the $\mathbb{R}^n$ spacetime. The action is

$$I=\frac{1}{2}\int d^nx\,[(\vec\nabla\Theta)^2+\sin^2\Theta(\vec\nabla\Phi)^2]\,, \tag{2.6}$$

which would give the sine-Gordon equation in case $(\vec\nabla\Phi)^2$ were constant. The resulting field equations are

$$\begin{aligned}&\Delta\Theta+\sin\Theta\cos\Theta(\vec\nabla\Phi)^2=0\,,\\ &\Delta\Phi-2\cot\Theta(\vec\nabla\Theta)\cdot(\vec\nabla\Phi)=0\,,\end{aligned} \tag{2.7}$$

where $\Delta=-\partial_\mu\partial^\mu$ is a Laplacian or a wave operator. Solutions have been known only in dimension $n=2$; they could bear static soliton interpretations in a three-dimensional Minkowski space, (see Duff and Isham[14]) or instanton interpretations for a two-dimensional spacetime.

For $n=2$ choose polar coordinates $r\phi$ in x space and assume $\Theta=\Theta(r)$, $\Phi=k\phi$. This proves consistent with equations (2.7) and gives a reduced action

$$\begin{aligned}I&=\pi\int_0^\infty r\,dr\left[\left(\frac{d\Theta}{dr}\right)^2+k^2\frac{\sin^2\theta}{r^2}\right]\\ &=|k|\pi\int_{-\infty}^\infty d\rho\left[\left(\frac{d\Theta}{d\rho}\right)^2+\sin^2\theta\right]\,,\end{aligned} \tag{2.8}$$

where, in the second form, $\rho=\ln(r/R)^k$. The corresponding "energy" integral is

$$E/\pi|k|=(d\Theta/d\rho)^2-\sin^2\Theta\,, \tag{2.9}$$

but only the zero-energy solution gives acceptable behavior for $\rho\to\pm\infty$. This solution, $\rho=\ln\tan(\frac{1}{2}\Theta)$, can be combined with $\Phi=k\phi$, to give the mapping in the form

$$(re^{i\phi}/R)^k=e^{i\Phi}\tan(\tfrac{1}{2}\Theta), \tag{2.10}$$

depending on a positive or negative integer k, the degree of the map, and a scale parameter R. The total action, $I=4\pi|k|$, is independent of R, however. Eells and Sampson[5] give essentially this example in the form of a harmonic map $S^2\to S^2$. But they do not succeed in finding harmonic maps $S^n\to S^n$ in higher dimensions $n>2$ for degree $k>1$, and they show that no absolute minimum is achieved by the action integral within the class of sphere maps with fixed $k>1$ for any $n>2$.

Eells and Sampson[5] give a number of classes of examples of harmonic maps that will be familiar to some physicists. I have already mentioned (1) harmonic functions, the case dim $M'=1$ and (2) geodesics, the case dim $M=1$. In addition (3) any isometry $M\to M'$ or covering of Riemannian manifolds $M\to M'$ is a harmonic map. Minimal (maximal) hypersurfaces are significant coordinate conditions in constructing solutions of Einstein's equations, and (4) Eells and Sampson show that any minimal immersion $M\to M'$ of Riemannian manifolds is a harmonic map. Further, (5) any homomorphism of compact semisimple Lie groups $G \to G'$ is a harmonic map. The final class (6) that I mention of Eells-Sampson examples is that of holomorphic maps of Kähler manifolds. This class of harmonic maps supplies a number of interesting examples in Appendix A.

Eells and Sampson[5] emphasize the importance of curvature of the space M' of field values in limiting the solutions of the harmonic mapping equation. Their remarks apply strictly to compact manifolds M with positive-definite (Riemannian) metrics. Without attempting to prove rigorous theorems, I shall try to indicate this curvature influence as it appears in cases that may arise in physical models. First the appropriate covariant derivative must be introduced.

In the covariant derivative ∇_ν familiar from general relativity [where I follow the notation and sign conventions of Misner, Thorne, and Wheeler (MTW)[24]] the connection coefficients $\Gamma^\alpha{}_{\beta\mu}$ are interpreted as arising from the use of nonconstant basis vectors

$$\nabla_\mu \vec{e}_\beta = \vec{e}_\alpha \Gamma^\alpha{}_{\beta\mu} \tag{2.11}$$

and give formulas like $v^\alpha{}_{;\mu} = \partial_\mu v^\alpha + \Gamma^\alpha{}_{\beta\mu} v^\beta$ for components of the covariant derivative of tangent vectors and their associated tensors. An ordinary differential operator $\partial_{\vec{v}} = v^\mu \partial_\mu$, evaluated at a point x (when acting on scalar functions) is a tangent vector $\vec{v}_x = (\partial_{\vec{v}})_x$. The totality of these tangent vectors $\vec{v}_x$ associated with a given manifold M forms the tangent bundle TM, and the projection π: $TM \to M, \vec{v}_x \to x$ is an important part of the bundle's structure. The space of all (tangent) vectors at a fixed point x is the fiber $M_x \equiv \pi^{-1}(x)$ or tangent space at x. The covariant derivative needed for discussions of harmonic maps generalizes this standard covariant derivative in ways familiar from gauge theories.

In a gauge theory the fields carry "isospin" or internal symmetry indices as in $\xi = \vec{e}_a \xi^a$ and the basis vectors $\vec{e}_a$ are not tangent vectors. To define their covariant derivatives

$$\nabla_\mu \vec{e}_b \equiv \vec{e}_a \Gamma^a{}_{b\mu} \equiv \vec{e}_a A^a{}_{b\mu} \tag{2.12}$$

one needs a field of connection coefficients $\Gamma^a{}_{b\mu}$ that are usually called $A^a{}_{b\mu}$ in this context and customarily obtained as solutions of the Yang-Mills equation. The matrix $A^a{}_{b\mu} dx^\mu$ represents an infinitesimal "rotation" in the internal-symmetry space, and objects carrying internal-symmetry indices such as ξ^a have covariant-derivative formulas of the form

$$\xi^a{}_{;\mu} = \partial_\mu \xi^a + A^a{}_{b\mu} \xi^b . \tag{2.13}$$

A field value $(\vec{e}_a \xi^a)_x = \vec{\xi}_x$ at a point $x \in M$ is a point in a bundle space E with projection π: $E \to M$, $\vec{\xi}_x \to x$ but the fiber $\pi^{-1}(x)$ consisting of all "isovectors" $\vec{\xi}_x$ at a single point x can be a vector space quite unrelated to the space of tangent vectors M_x.

With this review in mind we now consider a map f: $M \to M'$ into a pseudo-Riemannian manifold M'. A Riemannian covariant derivative is defined in the tangent bundle TM'. For instance, if $\vec{v} = v^A(\partial/\partial\phi_A)$ is a vector field on M', then

$$v^A{}_{;C} = \partial_C v^A + \Gamma^A{}_{BC}(\phi) v^B \tag{2.14}$$

are the components of its covariant derivative, where $\Gamma^A{}_{BC}$ are Christoffel symbols of $G_{AB}(\phi)$ in the ϕ^A coordinate patch on M'. We now want to introduce the induced bundle $f^*(TM')$ and see that it inherits a covariant derivative. A point in $f^*(TM')$ is a pair $(\vec{v}_{f(x)}, x)$ where $x \in M$ and $\vec{v}_{f(x)} \in M'_{f(x)}$, i.e., $\vec{v}_{f(x)}$ is a vector tangent to M' at a point $f(x)$ in M'. But $f^*(TM')$ is a bundle over M with projection π: $f^*(TM') \to M$, $(\vec{v}_{f(x)}, x) \to x$. An example may help (and will be useful in forming the "geodesic deviation," "Jacobi," or linear perturbation equation associated with a harmonic map). Let f_λ be a family of maps f_λ: $M \to M'$ represented in coordinates by $\phi^A = f^A(\lambda, x^\mu)$. Then $\vec{n}_x \equiv (\partial/\partial\lambda)_x \in f^*_\lambda(TM')$ is a differential operator that acts on functions ψ: $M' \to \mathrm{R}$ defined on M'. In coordinates, if $\psi(\phi^A)$ represents ψ on M', then

$$(\partial\psi/\partial\lambda)_x = (\partial f^A/\partial\lambda)_x \partial_A \psi ,$$

i.e.,

$$(\partial/\partial\lambda)_x = (\partial f^A/\partial\lambda)_x (\partial/\partial\phi^A)_{f(x)} . \tag{2.15}$$

Thus $\vec{n} = (\partial/\partial\lambda)$ is a vector field defined over M whose values are vectors tangent to M'. One can expand it in basis vectors as

$$\vec{n}(x) = n^A(x) \vec{e}_A(f(x)) \tag{2.16}$$

where, for this example, $n^A = \partial f^A/\partial\lambda$. Since covariant derivatives of the basis vectors $\vec{e}_A$ are defined on M' as

$$\nabla \vec{e}_B = \vec{e}_A \Gamma^A{}_{BC}(\phi) d\phi^C ,$$

given $\phi^A = f^A(x)$ one can easily decide to define

$$\nabla_\mu \vec{e}_B = \vec{e}_A \Gamma^A{}_{BC}(f(x)) \partial_\mu f^C(x) .$$

For the components of any field $n^A(x)$ [any local cross section of the bundle $f^*(TM')$] the formula then reads

$$n^A{}_{;\mu} = \partial_\mu n^A + \Gamma^A{}_{B\mu} n^B , \tag{2.17}$$

where

$$\Gamma^A{}_{B\mu}(x) = \Gamma^A{}_{BC}(f(x)) \partial_\mu f^C(x) \tag{2.18}$$

are then the connection coefficients on M for the bundle $f^*(TM')$. From

$$[\nabla_\mu, \nabla_\nu]\vec{e}_B = \vec{e}_A R^A{}_{B\mu\nu} \qquad (2.19)$$

one readily computes the curvature of this connection and finds its components to be

$$R^A{}_{B\mu\nu}(x) = R^A{}_{BCD}(f(x)) f^C{}_\mu f^D{}_\nu \qquad (2.20)$$

with

$$f^A{}_\alpha(x) \equiv \partial f^A / \partial x^\alpha \qquad (2.21)$$

and where $R^A{}_{BCD}(\phi)$ give the Riemann curvature of the metric $G_{AB}(\phi)$ on M'. In this induced connection one finds $G_{AB;\mu} = 0$ on M under the mapping f as a consequence of the Riemannian condition $G_{AB;C} = 0$ on M'.

Let us now further assume that there is a metric $g_{\mu\nu}(x)$ on M, then we can form the action integral for harmonic maps

$$I = \frac{1}{2}\int \sqrt{|g|}\, d^n x\, \phi^A{}_\mu \phi^\mu{}_A = \frac{1}{2}\int_M \|d\phi\|^2 * 1\,, \qquad (2.22)$$

where the indices on $\phi^A{}_\mu \equiv \partial\phi^A/\partial x^\mu$ are each raised and lowered by the appropriate metric. The quantities $\phi^A{}_\mu$ are components of a tensor

$$\phi_* \equiv d\phi \equiv \phi^A{}_\mu(x)\vec{e}_A(\phi x) \otimes dx^\mu \qquad (2.23)$$

of a mixed type that is an element of the tensor product bundle $\phi^*(TM') \otimes TM^*$ over M. The covariant derivatives of this tensor will reflect the motion of both of the bases $\vec{e}_A = (\partial/\partial\phi^A)_{\phi x}$ and $\omega^\mu = dx^\mu$. Thus for its covariant derivative we find

$$\phi^A{}_{\mu;\nu} = \partial_\nu \phi^A{}_\mu - \phi^A{}_\alpha \Gamma^\alpha{}_{\mu\nu} + \Gamma^A{}_{B\nu}\phi^B{}_\mu\,. \qquad (2.24)$$

Note that

$$\phi^A{}_{\mu;\nu} = \phi^A{}_{\nu;\mu} \qquad (2.25)$$

since $\partial_\nu \phi^A{}_\mu = \partial^2\phi^A/\partial x^\nu \partial x^\mu$ and $\Gamma^A{}_{B\nu}\phi^B{}_\mu = \Gamma^A{}_{BC}\phi^C{}_\nu\phi^B{}_\mu$. In this notation the harmonic mapping equation (2.4) reads simply

$$\phi^A{}_\mu{}^{;\mu} = 0 \qquad (2.26)$$

with the nonlinearities hidden in the fact that the same map ϕ appears both in $\phi^A{}_\mu$ and in the semicolon that forms the covariant derivative (i.e., in $\Gamma^A{}_{B\nu}$).

We may consider linear perturbations of the harmonic map $\phi(x)$ by embedding it in a family of harmonic maps $f_\lambda(x) = f(\lambda, x)$ with $\phi = f_0$. Then $\vec{n} = (\partial/\partial\lambda)$, as defined in Eq. (2.15), will satisfy an equation obtained by differentiating Eq. (2.26) with respect to λ. The result (see Appendix B for details) is the linear "Jacobi" equation

$$0 = n^A{}_{;\mu}{}^{;\mu} + (R^A{}_{CBD}\phi^C{}_\mu \phi^{D\mu})n^B\,. \qquad (2.27)$$

In many cases solutions of this equation are nonexistent or highly restricted if M' is a Riemannian manifold with negative sectional curvatures, since upon integrating and neglecting a boundary integral it yields

$$\int n^A{}_{;\mu} n_A{}^{;\mu}\sqrt{|g|}\, d^4x = \int n^A \phi^C{}_\mu R_{ACBD} n^B \phi^{D\mu}\sqrt{|g|}\, d^4x\,, \qquad (2.28)$$

where the left-hand side is non-negative for Riemannian (positive) metrics.

Eells and Sampson give an identity that does not refer to perturbations. Let $\mathfrak{L} = \frac{1}{2}\phi_A{}^\mu \phi^A{}_\mu$ be the Lagrangian in the action integral (2.22). Then in Appendix B we repeat their computations to show that

$$\mathfrak{L}_{;\mu}{}^{;\mu} = \phi_A{}^{\alpha;\beta}\phi^A_{\alpha;\beta} + \phi^\alpha{}_A(\delta^A{}_B R^\beta{}_\alpha - \delta^\beta{}_\alpha R^A{}_{CBD}\phi_\gamma{}^C \phi^{D\gamma})\phi^B{}_\beta + \phi^\alpha{}_A(\phi^A{}_\mu{}^{;\mu})_{;\alpha} \qquad (2.29)$$

and then show that for finite action in an asymptotically flat space M, the integral of each side must vanish. Under these circumstances, and for harmonic maps ϕ, one obtains the integral formula

$$\int \phi_A{}^{\alpha;\beta}\phi^A{}_{\alpha;\beta}\sqrt{|g|}\, d^n x = \int (\phi^A{}_\mu \phi^C{}_\nu R_{ACBD}\phi^{B\mu}\phi^{D\nu} - \phi_A{}^\alpha R_{\alpha\beta}\phi^{A\beta})\sqrt{|g|}\, d^n x\,. \qquad (2.30)$$

This shows that negative sectional curvatures for M' (with zero or positive Ricci curvature for M) leave the possibilities for harmonic maps of the strictly Riemannian manifolds $M \to M'$ very limited, e.g., constant maps $\phi(x) = p_0 \in M'$.

None of the above integral formulas have obvious consequences in Minkowski spacetimes M, but for Euclidean theories, i.e., if either static solitons or instantons are desired[25], they suggest that range spaces M' with negative curvature should be avoided.

The prime candidates for model theories involve M' that are not merely homogeneous spaces G/H, but those whose metrics involve the highest symmetry and therefore the least arbitrariness. These would be the Riemannian symmetric spaces (Helgason[15]). A simply connected globally symmetric space decomposes as the product of three factors, an uninteresting Euclidean type that is flat, a compact type that has non-negative sectional curvatures, and a noncompact type that has nonpositive sectional curvatures. The irreducible compact types that thus appear most interesting consist of the compact connected simple Lie groups and a number of coset spaces G/H listed in Table II, Chapter IX of Helgason. These include $SU(p+q)/SU(p)\times U(q)$, similar quotients of orthogonal and symplectic groups, a few mixtures like $Sp(n)/SU(n)$, and a dozen exceptional cases. Some, also listed by Helgason, are Hermitian symmetric spaces, i.e., Kähler manifolds for which holomorphic maps $C^2\to M'$ would be examples of harmonic maps, as in Appendix A. These Hermitian symmetric spaces include $SU(p+q)/SU(p)\times U(q)$ and $SO(2n)/U(n)$.

III. MODELING GRAVITATIONAL NONLINEARITIES

One reason for studying harmonic maps is to better understand some of the nonlinearities that occur in the more complicated Einstein equations for general relativity. The Yang-Mills equations have already played an important role as models for gravitation theory, particularly as concerns its quantization. I think that harmonic maps can play a similar role, but that something new will be learned, as the nonlinearities they model are of a different type.

To show the different character of the nonlinearities in these three cases, I list their field equations in highly schematic form:

$$g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}+(g^{-1}\partial g)^2=0\,, \tag{3.1}$$

$$\frac{\partial^2 A^b{}_{a\nu}}{\partial x^\alpha \partial x_\alpha}+A\partial A+A^3=0\,, \tag{3.2}$$

$$\frac{\partial}{\partial x^\alpha}\left(G_{AB}(\phi)\frac{\partial \phi^B}{\partial x_\alpha}\right)+G'(\phi)(\partial\phi)^2=0\,. \tag{3.3}$$

All three equations are highly nonlinear, and in all three cases no coupling constant appears in the "free" field equation, when no further fields are included. The entire structure of the nonlinearities is determined by symmetry considerations in each case. [For harmonic maps this is true only if M' is a Riemannian symmetric space whose metric $G_{AB}(\phi)$ is essentially unique. In more general homogeneous spaces the entire matrix $G_{AB}(\phi_0)$ at one point in M' might be a set of adjustable coupling constants.] The idea that one is dealing with "naturally" generated nonlinearities (rather than with self-interactions introduced *ad hoc* by adding some higher-than-quadratic polynomial to a free field Lagrangian) is the central theme that ties these three theories together. Differences between the theories can be seen by studying the leading second derivative terms in the field equations.

For Yang-Mills fields, the second derivative terms are linear, and nonlinearities occur only in the lower-order terms. In this it differs significantly from the Einstein equations where the second derivative terms contain and essential nonlinearity in coefficients like the $g^{\alpha\beta}$ shown explicitly in Eq. (3.1). The harmonic map equation (3.3) is intermediate between these two. It is written in equation (3.3) in a form that suggests its similarities to the Einstein equation, i.e., nonlinearities in the second-derivative term, and homogeniety in the total order of differentiation—exactly two derivatives in every term. The nonlinearities in the leading term are, however, removable in the classical field equations—it can be rewritten in the "geodesic" form of Eq. (2.4). Also, the characteristics on which waves propagate are determined *a priori* by the fixed Minkowski metric in the Yang-Mills and harmonic mapping equations, but are field dependent in the Einstein case.

In the case of quantum theory, the analogy between harmonic maps and the Einstein equation is even closer. Factor ordering problems may prevent the reduction of Eq. (3.3) to the form (2.4), so the nonlinearities in the second-derivative term are probably essential. Another way to see this is to consider the action integrals for the three theories. Again schematically, they read

$$I_{\rm GR}=-\frac{c^3}{G}\int g\left(\frac{\partial g}{\partial x}\right)^2 d^4x\,, \tag{3.4}$$

$$I_{\rm YM}=-\left(\frac{\hbar^2 c}{e^2}\right)\int(\partial A+A^2)^2 d^4x\,, \tag{3.5}$$

$$I_{\rm HM}=-\frac{c^3}{\gamma^2}\int G(\phi)\left(\frac{\partial\phi}{\partial x}\right)^2 d^4x\,. \tag{3.6}$$

In this form the notation has obliterated any distinction between the character of the Einstein action and that for harmonic maps. [The essential distinction is that a constant Minkowski metric $\eta_{\mu\nu}={\rm diag}(-1,1,1,1)$ is hidden in Eq. (3.6) but does not occur at all in Eq. (3.4).] It is quite clear that the leading term $(\partial\ {\rm field})^2$ is much simpler in the Yang-Mills case than in the more seriously nonlinear Einstein and harmonic mapping actions. The significance of this for quantum mechanics is most directly suggested in the Feynman path integral formulation, where the integrand, $\exp(-iI/\hbar)$, of the functional integral shows that the form of the action integral I is all important.

The dimensionality of the coupling constants again shows (and follows from) the analogous structure of the Einstein and harmonic mapping actions. If all coordinates, including $x^0=ct$, have dimensions of length (a presently convenient but not always natural choice in general relativity), then the metric field $g_{\mu\nu}$ and the harmonic mapping field ϕ are both dimensionless while the gauge connection A_μ has dimensions $({\rm length})^{-1}$. [Mapping coordinates ϕ^A will be anglelike coordinates in a symmetric space of compact type; see, for example, Eq. (1.4) where $\phi_a{}^b$ must clearly be dimensionless. For Yang-Mills fields $A^a{}_{b\mu}$ I am adopting geometrical units, where a gauge covariant derivative is written $\nabla_\mu\psi^a = \partial_\mu\psi^a + A^a{}_{b\mu}\psi^b$ with no coupling constant or other dimensional coefficient appearing in "minimal coupling" interactions. Thus the geometrically scaled electromagnetic potential A_μ would be the quantity that is usually written $(e/\hbar c)A_\mu$.] The harmonic mapping coupling constant γ^2 has the same dimensions as the Newton-Cavendish gravitational constant G. In the dimensionless action integrals (Feynman phases) $I/\hbar$, the coupling constants that appear are a dimensionless $(e^2/\hbar c)$ in the Yang-Mills case (e.g., the fine-structure constant for electromagnetism) and

$$(\hbar G/c^3) = l_p{}^2 \tag{3.7}$$

for gravity, and

$$(\hbar\gamma^2/c^3) = \lambda\!\!\!^-{}^2 \tag{3.8}$$

for harmonic mapping fields. Here $l_p \approx 10^{-33}$ cm is the Planck length for gravity, and $\lambda\!\!\!^-$ is a fundamental length associated with the harmonic mapping field.

The appearance of a dimensioned coupling constant $\lambda\!\!\!^-{}^2$ in the harmonic mapping action suggests that these theories, like gravitation, will be nonrenormalizable in a perturbation theory quantization. I hope that harmonic mapping theories contain examples so simple that a rigorous nonperturbative quantization can be achieved for some model theories.

IV. SYMMETRY BREAKING VIA HARMONIC SECTIONS

Broken symmetries are a central idea in particle theory. One way symmetries can be broken is by including in the action "small" terms that exhibit invariance under only a subgroup H of the group G under which the dominant terms are invariant. Alternatively, in spontaneously broken symmetries, the vacuum state has a smaller invariance group than the action itself. General relativity can be regarded as a theory that incorporates yet another symmetry-breaking mechanism by which the invariance of the action under arbitrary coordinate transformations is reduced to the physical invariance of local phenomena under local Lorentz transformations. The symmetry reduction scheme used in general relativity can be extended to general gauge theories as will be broadly sketched below, by introducing fields satisfying gauge-invariant harmonic mapping equations. A model theory of this type is described in another paper (Misner[16]). The generality of this method, and its central idea, appear in the following theorem from Husemoller's text (Ref. 26, p. 71), originally due to Steenrod (Ref. 27, p. 43).

Theorem. Let H be a closed subgroup of G. A principal G bundle $\xi = (E, p, M)$ (or p: $E \to M$) has a reduction to a principal H bundle $\eta = ({}^rE, q, M)$ if and only if ξ mod H (or $\xi[G/H]$) has a cross section.

In this theorem $\xi[G/H]$ is the G bundle associated with ξ whose fiber is the space G/H of left cosets of H in G. Rather than explain the statement (much less the proof) of this theorem (for which see Husemoller), I will simply describe gravitational fields as an example of it.

The Einstein equations are written on a four-manifold M that possesses no metric or other structure beyond its differentiable structure. Naturally associated with M are all the standard bundles of tensors. These bundles are all associated with, and derive their structure from the principal bundle, or frame bundle $\xi = (E, p, M)$. A frame $\vec{e} \in E$ is a set of four linearly independent differential forms $\vec{e}^a = e^a{}_\mu dx^\mu$ at a point $x \in M$; thus

$$\vec{e} = (\vec{e}^0, \vec{e}^1, \vec{e}^2, \vec{e}^3, x) = (\vec{e}^a, x),$$

with $p(\vec{e}) = x \in M$. The general linear group $G = \mathrm{GL}(4, R)$ acts freely on E according to

$$\vec{e}s = (\vec{e}^a, x)s = (\vec{e}^b s_b{}^a, x),$$

where $s \in G$ is any nonsingular 4×4 matrix.

Any metric field $ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$ allows one to select a closed subset of E consisting of the orthonormal frames ${}^rE \subset E$ defined by the condition that $\vec{e} \in {}^rE$ if and only if

$$ds_x{}^2 = (\eta_{ab}\vec{e}^a \otimes \vec{e}^b)_x, \tag{4.1}$$

where η_{ab} is the Minkowski matrix diag$(-1,1,1,1)$ and $x = p(\vec{e})$. This set ${}^rE$ becomes a bundle $\eta = ({}^rE, q, M)$ by defining its projection q by the condition $q(\vec{e}) = p(\vec{e})$ for $\vec{e} \in N$, i.e., $q = p|{}^rE$. But the Lorentz group $H = \mathrm{O}(1,3)$ acts freely on ${}^rE$, since if $\vec{e} = (\vec{e}^a, x) \in {}^rE$ so $\vec{e}$ satisfies (4.1), then $\vec{e}\Lambda = (\vec{e}^b\Lambda_b{}^a, x)$ also satisfies (4.1) for any Lorentz transformation matrix $\Lambda \in H$. Therefore η is a principal H bundle. This shows constructively how a globally defined Minkowski signature metric field ds^2 (which, see below, is precisely a cross section of the bundle $\xi[G/H]$ with fiber $G/H = \mathrm{GL}(4,R)/\mathrm{O}(1,3)$) gives rise to a reduction of the general frame bundle ξ to an orthonormal frame bundle η. But the converse is also true. Suppose that any reduction of the general frame bundle ξ is given to an "orthonormal frame" bundle η, i.e., to a principal bundle over M with group $H = \mathrm{O}(1,3)$. Then at any point $x \in M$, define $(ds^2)_x$ by equation (4.1), using for $\vec{e}_x$ any "orthonormal frame," i.e., any element of $q^{-1}(x)$. As these frames are all related by Lorentz matrices Λ under which η_{ab} is invariant, they all give the same metric ds^2, so a cross section of $\xi[G/H]$ has been constructed. Husemoller's proof of the quoted theorem shows similarly that a strong statement is justified, namely that there is a 1-1 correspondence between cross sections σ of $\xi[G/H]$ and reductions i: ${}^rE \to E$ of a G bundle $\xi = (E, p, M)$ to an H bundle $\eta = ({}^rE, q, M)$.

A point that may require clarification in the above discussion is the identification of a matrix $g_{\mu\nu}$ with a point in G/H, i.e., with a coset of the Lorentz subgroup H of $G = \mathrm{GL}(4, R)$. Given the matrix $g_{\mu\nu}$, there will be a matrix $s \in G$ that reduces it to its signature:

$$g_{\mu\nu} s^\mu{}_\alpha s^\nu{}_\beta = \eta_{\alpha\beta}. \qquad (4.2)$$

But s is not unique, for $(s\Lambda)^\mu{}_\alpha = s^\mu{}_\gamma \Lambda^\gamma{}_\alpha$ also satisfies (4.2) when Λ is any Lorentz transformation $\Lambda \in H = \mathrm{O}(1,3)$. The set of solutions s of Eq. (4.2) is therefore an entire coset sH of the Lorentz group, and any such coset determines $g_{\mu\nu}$ according to $e_{\mu\nu} = \eta_{\alpha\beta}(s^{-1})^\alpha{}_\mu (s^{-1})^\beta{}_\nu$ where every member s of a coset gives the same $g_{\mu\nu}$. Thus the matrices $g_{\mu\nu}$ of Minkowski signature are in 1-1 correspondence with cosets of $H = \mathrm{O}(1,3)$, and therefore may be regarded as coordinates on the fiber G/H of $\xi[G/H]$.

Steenrod's bundle reduction theorem (which we have quoted from Husemoller[26]) says that a reduction of local symmetry from $G = \mathrm{GL}(4,R)$ to $H = \mathrm{O}(1,3)$ is equivalent to defining a metric (G/H) field. How is the field to be defined? One method (*a priori* geometry, external currents) is to regard the metric field as fixed, given *a priori* or *ad hoc*. Thus if ds^2 is the flat Minkowski metric, then the G symmetry (general covariance) can only be introduced spuriously in the action functional, and only Lorentz invariance (H symmetry) will be physically significant. The same would be true for any other fixed ds^2 (such as the Schwarzschild geometry). No true general covariance would exist in such a theory, as its action would look simpler in some coordinate systems (as where $g_{\mu\nu}$ was time-independent) than others, yet local Lorentz invariance would be a significant symmetry. But if ds^2 is chosen dynamically, as in general relativity where a locally G-invariant action integral is employed, and determines a G-invariant field equation for the symmetry-breaking G/H (metric) field, both G and H play significant roles in the resulting theory.

The corresponding dynamic symmetry breaking in gauge theories requires a G-invariant action integral for a symmetry-breaking G/H field, but now G may be any semisimple Lie group. The harmonic map action of Eq. (1.3) is readily generalized to accomplish this by converting ordinary derivatives to gauge-covariant derivatives. Thus one writes

$$I_\phi = -\frac{c^3}{2\gamma^2} \int \sqrt{|g|}\, d^4x g^{\mu\nu}(x) G_{AB}(\phi) \nabla_\mu \phi^A \nabla_\nu \phi^B. \qquad (4.3)$$

Here one assumes that G is the gauge group of some local symmetry, and that a gauge vector field A_μ defines a covariant derivative in the principal G bundle ξ. Then ∇_μ is the gauge-covariant derivative in the associated bundle $\xi[G/H]$, and the symmetry-breaking field ϕ is a cross section of $\xi[G/H]$ satisfying the variational principle $\delta I_\phi = 0$. The metric $dL^2 = G_{ab}(\phi) d\phi^A d\phi^B$ on the prototype fiber G/H is assumed to be invariant under the transformations $\phi \to s\phi$ of G/H by any element s of the gauge group G. It will then give a well-defined metric in each fiber $p^{-1}(x)$ above a spacetime point x, whose components $G_{AB}(\phi)$ transform contragrediently to $\nabla\phi^A \otimes \nabla\phi^B$ under gauge transformations, so I_ϕ will be gauge invariant. Such an invariant metric on G/H can always be found when H is compact by taking an arbitrary (positive definite) metric at the point $\phi_0 = H \in G/H$, averaging it under the action of the isotropy group $H \subset G$ at that point, and then translating by the action of G on G/H. If G/H is an irreducible Riemannian symmetric space it has an essentially unique metric that is invariant under G.

Equation (4.3) of course only gives the $(\partial\phi)^2$ terms in the action. The total action would contain in addition a term I_A (such as a Yang-Mills action) with $(\partial A)^2$ terms to give the equations for the A_μ field, and possibly other terms I_ψ for various lepton or meson fields. The full action would be gauge invariant under the gauge group G, yet any term involving the field ϕ would have an H symmetry that would be stronger than its G symmetry because the H symmetry would persist even when ϕ was considered an externally fixed background field.

V. HARMONIC CONNECTIONS

Harmonic maps can be used to define gauge-covariant derivatives as an alternative to the usual Yang-Mills equation for gauge connections A_μ. As I am not sufficiently familiar with the mathematics of connections in other-than-vector bundles, I shall not attempt a general theory, but just give some highlights of models developed further elsewhere (Misner[16]). These theories are motivated by questions about *generalized* magnetic monopole charge, i.e., topological invariants of bundle structure. Let us, however, consider the electromagnetic example. On a spacetime manifold M consisting of Minkowski space minus a world line (source at the spacial origin) Maxwell's equations admit solutions A_μ with arbitrary electric charge

$$\frac{1}{2}\int {}^*F_{\mu\nu}dx^\mu \wedge dx^\nu = Q$$

but the magnetic charge

$$\frac{1}{2}\int F_{\mu\nu}dx^\mu \wedge dx^\nu = P$$

is always zero. This corresponds to a gauge symmetry in which the principal U(1) bundle ξ is a simple product bundle p: $U(1)\times M \to M$, $(u,x)\to x$ in which $\nabla_\mu \vec{e} = -iA_\mu \vec{e}$ for a basis $\vec{e}$: $M\to U(1)$, $x \to \vec{e}(x)$. With a twisted bundle structure (nonproduct) for ξ where no global basis exists, Maxwell's equations allow nonzero magnetic charge P, but it is again not a property of the solution A_μ one chooses—all solutions for a given bundle have the same magnetic charge. This seems unsatisfactory. The monopole charge P is built into the *a priori* bundle geometry that the theorist chooses before the field equation $F^{\mu\nu}{}_{;\nu}=0$ is well defined, it does not arise from any dynamics in the theory. In the examples that follow, the bundle structure is not assigned *a priori*, but determined by a bundle structure field ϕ that satisfies a (harmonic mapping) wave equation. One could then expect that quantum excitations from one state of the ϕ field to another could change the generalized monopole charge associated with the bundle, and that, classically, different monopole charges P (all integers) could arise as different solutions of the same equation for ϕ.

Consider for simplicity a theory with a global (rather than local or gauge) symmetry group $G = \mathrm{SU}(n)$. (There are no obvious difficulties in replacing ∂_μ by a G-gauge-covariant derivative in all that follows.) If $\phi = (\phi^a{}_b)$ is a Hermitian matrix, then the action

$$I_\phi = -\frac{c^3}{\gamma^2}\int d^4x\, \mathrm{tr}[(\partial_\mu\phi)^\dagger(\partial^\mu\phi)] \tag{5.1}$$

is invariant under the symmetry transformations $\phi\to s\phi s^\dagger$ for $s\in \mathrm{SU}(n)$ which also preserves the Hermitian condition $\phi^\dagger=\phi$. With only the linear constraint $\phi^\dagger=\phi$ imposed, the action (5.1) corresponds to n^2 real free scalar fields and gives linear field equations. But if we now impose the further, nonlinear, constraints

$$\phi^2=\phi\ , \tag{5.2}$$

$$\mathrm{tr}\phi = p\ , \tag{5.3}$$

then the action (5.1) is highly nonlinear in the independent parameters needed to specify a unique ϕ, and is a harmonic mapping action. With these constraints $\phi\in G/H$ is a point in the Riemannian symmetric space $\mathrm{SU}(p+q)/\mathrm{SU}(p)\times \mathrm{U}(q)$ where $p+q = n$. This is a space of real dimension $2pq$. The field equation corresponding to $\delta I_\phi = 0$ is (1.5). [All the properties of this model have obvious parallels, replacing complex numbers C by real numbers R or quaternions H, for the symmetric spaces $\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ or $\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$ obtained from orthogonal or symplectic groups.]

Now let us use the harmonic field ϕ to construct a bundle with an H-covariant derivative and a gauge group $H=\mathrm{SU}(p)\times\mathrm{U}(q)\equiv \mathrm{S}(\mathrm{U}_p\times\mathrm{U}_q)$ consisting of unit determinant matrices from $\mathrm{U}(p)\times\mathrm{U}(q)$. Let $\psi=(\psi^a)$ be set of $n=p+q$ spinless complex fields with the global symmetry $\psi\to s\psi$ for $s\in G=\mathrm{SU}(n)$. The action integral for the ψ field will be

$$I_\psi = -\frac{1}{2}\int d^4x (D_\mu\psi)^\dagger(D_\mu\psi)\ , \tag{5.4}$$

where

$$D_\mu = D'_\mu + D''_\mu \tag{5.5}$$

is a gauge-covariant derivative with

$$D'_\mu = \phi\partial_\mu\phi\ ,\quad \text{i.e.,}\ D'_\mu\psi = \phi\partial_\mu(\phi\psi)\ , \tag{5.6}$$

and

$$D''_\mu = (1-\phi)\partial_\mu(1-\phi)\ . \tag{5.7}$$

Since $(D'_\mu\psi)^\dagger(D''_\mu\psi)=0$ from $\phi(1-\phi)=0$, the action decomposes into two parts, $I_\psi = I' + I''$, one being

$$I'_\psi = -\tfrac{1}{2}\int (D'_\mu\psi)^\dagger(D'_\mu\psi)d^4x\ .$$

Writing $\psi=\psi^r\vec{e}_r$ in an orthonormal basis $\vec{e}_r$ that satisfies $\phi\vec{e}_r=\vec{e}_r$ for $1\le r\le p$ and $\phi\vec{e}_r=0$ for $p+1 \le r\le p+q$, one can put the covariant derivative D in the familiar form

$$(D_\mu\psi)^r = \partial_\mu\psi^r + A^r{}_{s\mu}\psi^s\ , \tag{5.8}$$

where the A_μ are traceless anti-Hermitian matrices in $(p+q)$ block-diagonal form. Given a basis (gauge choice), the A_μ are computable from ϕ, and in the original basis where $\partial_\mu\vec{e}_a=0$ one finds

$$A_\mu = \phi(\partial_\mu\phi)(1-\phi) - (1-\phi)(\partial_\mu\phi)\phi\ , \tag{5.9}$$

which is clearly traceless and anti-Hermitian but not block diagonal. [The identity $\phi(\partial_\mu\phi)\phi=0$ that follows from $\phi^2=\phi$ is useful.] The curvature is also readily computed, either from

$$F = dA + A\wedge A\ , \tag{5.10}$$

using (5.9) for $A = A_\mu dx^\mu$, or from equivalent definitions such as

$$[D'_\mu, D'_\nu]\psi = F'_{\mu\nu}\psi\ . \tag{5.11}$$

The result is

$$F_{\mu\nu} = F'_{\mu\nu} + F''_{\mu\nu} \tag{5.12}$$

with

$$F'_{\mu\nu}=\phi[(\partial_\mu\phi)(\partial_\nu\phi)-(\partial_\nu\phi)(\partial_\mu\phi)]\phi \tag{5.13}$$

and, in the alternative notation, similarly

$$F''=(1-\phi)d\phi\wedge d\phi(1-\phi). \tag{5.14}$$

Note that D'_μ and D''_ν commute, with $D'_\mu D''_\nu=0$.

A model theory in which a bundle structure field $\phi=\phi^\dagger$ provides interactions among $p+q$ (complex) meson fields ψ is defined by the constraints (5.2) and (5.3) together with the action $I=I_\phi+I_\psi$. This differs from a Yang-Mills interaction among the mesons in several respects. First there is the dimensional coupling constant that appears in (5.1), while for $I_{\rm YM}=I_A+I_\psi$ the action I_A of the Yang-Mills field would have dimensionless coupling constants. Secondly, although in both cases I_ψ can be written in the same form with $D_\mu=\partial_\mu+A_\mu$, the independent fields comprise dim $H=p^2+q^2-1$ real vector fields $A_\mu=A'_\mu+A''_\mu$ in the Yang-Mills case, but $\dim(G/H)=2\,pq$ real fields ϕ in the case with the harmonic connection. Thus there can be a different number of gluon fields in the two theories; the harmonic gluons are spinless, while the Yang-Mills gluons are vectors; the H-bundle structure must be given *a priori* for a Yang-Mills theory, but is dynamically determined by the harmonic gluons; and, finally, the SU(p) and SU(q) mesons interact only electromagnetically [via the U(1) in $\mathrm{S}(\mathrm{U}_p\times\mathrm{U}_q)$] in the Yang-Mills theory, but more strongly via ϕ for the harmonically connected theory.

The covariant derivative (or harmonic connection) $D_\mu=\phi\partial_\mu\phi+(1-\phi)\partial_\mu(1-\phi)$ can of course be used in Lagrangians for multiplets of quarks or other fermions as well as in the for scalar multiplets such as ψ in Eq. (5.4). This approach seems well adapted to giving strong interactions via gluons that do not show up as free particles. Vector gluons would not materialize because the A_μ connection vectors are merely derivatives [Eq. (5.9)] of the underlying scalar harmonic projection fields ϕ. But scalar particles corresponding to the ϕ fields would also not seem to appear, as one can always (and naturally) choose gauges—those that make A_μ block diagonal in Eq. (5.8)—in which ϕ is just the constant matrix with unity in the first p diagonal entries and zero elsewhere. A gauge-invariant way of stating this (see Misner[16]) is to note that when D_μ is extended in the usual way to act on gauge tensors $\phi^a{}_b$ as well as gauge vectors ψ^a, it gives $D_\mu\phi=0$ for the field ϕ used in the definition of D_μ. Thus ϕ is covariantly constant in the bundle structure and connection it defines, even though it gives interactions among other fields that live in this bundle, and may be an entirely nontrivial solution of its harmonic mapping wave equation.

VI. QUANTIZATION

I can offer three remarks concerning the quantization of harmonic mapping fields. One is a pious hope that *functional integration* methods of Euclidean field theory (see Glimm and Jaffe[28] for an introduction) might provide a genuine quantum field theory for some model in this class. The second is an indication that these are not the worst of nonrenormalizable theories since many of them have renormalizable relatives (*"relaxed theories"*), in the sense that the nonlinear σ model is related to the renormalizable σ model. The other comment, elaborated first below, is a suggestion that the compactness of the range space of ϕ values together with the uncertainty principle leads to an expectation that the quantum behavior of harmonic mapping fields at *short wavelengths* may be quite simple but very different from free fields.

Short wavelength behavior. What can one expect from the quantum theory of a ϕ field defined on four-dimensional Minkowski space with values in a compact Riemannian symmetric space of positive sectional curvature? For some clues consider making a lattice model with discrete spatial sites x_n whose nearest neighbors on a simple cubic lattice will be called $x_{n'}$. From the action (1.3) or (5.1) we can write $I=\int\mathcal{L}dt$ with

$$\mathcal{L}=\frac{1}{2}\frac{\hbar c}{\lambda\!\!\!^{-2}}\int d^3x\left(\frac{1}{c^2}\|\frac{\partial\phi}{\partial t}\|^2-\|\nabla\phi\|^2\right), \tag{6.1}$$

where

$$\|\partial\phi\|^2=G_{AB}(\phi)\partial\phi^A\partial\phi^B. \tag{6.2}$$

Approximated on a lattice this becomes

$$\mathcal{L}=\frac{1}{2}\frac{\hbar c}{\lambda\!\!\!^{-2}}\sum_n\Delta^3\left(\frac{1}{c^2}\|\phi_n\| -\frac{1}{2\Delta^2}\sum_{n'}\|\phi_{n'}-\phi_n\|^2\right), \tag{6.3}$$

where $\Delta=\Delta x=\Delta y=\Delta z$ is the lattice spacing, and the meaning of $\|\phi_1-\phi_2\|^2$ becomes a significant question. One possibility is to take $\|\phi_1-\phi_2\|=d(\phi_1,\phi_2)$ to be the Reimannian (least geodesic) distance between ϕ_1 and ϕ_2. A simpler, and analytic, choice that fits the projection operator examples like $G/H=\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p\times\mathrm{U}_q)$ of Sec. V is to take

$$\|\phi_1-\phi_2\|^2=2\,\mathrm{tr}[(\phi_1-\phi_2)^\dagger(\phi_1-\phi_2)], \tag{6.4}$$

but remember $\phi^2=\phi$. This choice, like d^2, reduces to Eq. (6.2) when ϕ_1 and ϕ_2 are infinitesimally close. What I am attempting to avoid is an approximation that ignores the finite volume or diameter of G/H and treats it as a vector space. In contrast, Eq. (6.4) gives a finite maximum, e.g.,

$\|\phi_1 - \phi_2\|^2 \leq 4p$ if $\phi_1, \phi_2 \in \mathrm{SU}(p+q)/S(\mathrm{U}_p \times \mathrm{U}_q)$ and $p \leq q$. This maximum uncertainty in the variable $\phi_1 - \phi_2$ gives rise to a minimum $\|p_{12}{}^2\|$ for its conjugate momentum by the uncertainty principle.

The Hamiltonian corresponding to (6.3) is

$$H=\frac{1}{2}\left(\frac{\hbar c}{\chi^2}\Delta\right)\sum_n\left[\left(\frac{\chi^2}{\Delta^2}\right)^2\|p_n\|^2 + \frac{1}{2}\sum_{n'}\|\phi_{n'}-\phi_n\|^2\right], \qquad (6.5)$$

where $p_n = \hbar^{-1}\partial\mathcal{L}/\partial\dot{\phi}_n$ is scaled to give dimensionless commutation relations

$$[\phi_n, p_m] = i\delta_{nm}, \qquad (6.6)$$

so that all dimensional quantities are manifest in Eq. (6.5). The collective modes of this Hamiltonian need careful investigation. Modes where all $\|\phi_i - \phi_j\|^2 \ll 1$ should behave as free field modes, but modes with a characteristic scale length near Δ should act qualitatively like a single nearest-neighbor pair for which the Hamiltonian is approximately

$$H_{12}=\frac{1}{2}\left(\frac{\hbar c}{\lambda^2}\Delta\right)\left[\left(\frac{\chi^2}{\Delta^2}\right)^2\|p\|^2+\|\phi-\phi_0\|^2\right] \qquad (6.7)$$

with ϕ_0 a constant. The eigenfunctions, and the spectrum of $H_{12}/(\Delta\hbar c/\chi^2)$—i.e., the pattern of eigenvalues—depend only on a dimensionless number $J^2 = \Delta^2/\lambda^2$. Since $\|\phi - \phi_0\|^2$ is bounded, we see that for $\chi^2/\Delta^2 \gg 1$ the interaction term can be neglected. This suggests that *for all modes with scale length $k^{-1} \ll \chi$ the space derivative terms in the action* (6.1) *are negligible.* The simplest example in which to attempt to verify this conjecture would be that with $G/H = \mathrm{SO}(2)/\mathrm{SO}(1)\times\mathrm{SO}(1) = S^1$. Then by (6.4)

$$\|\phi-\phi_0\|^2 = 4\sin^2(\tfrac{1}{2}\theta), \qquad (6.8)$$

where $\frac{1}{2}\theta$ is the angle between the two lines in R^2 onto which ϕ and ϕ_0 are the projections. One then readily verifies that, although many of the low-lying eigenstates of

$$H_{\mathrm{loop}} = \tfrac{1}{2}[J^{-2}p_\theta{}^2 + 4\sin^2(\tfrac{1}{2}\theta)] \qquad (6.9)$$

have $E_n \simeq J^{-1}(n+\frac{1}{2})$ and fit the harmonic-oscillator approximation $|\theta| \ll 1$ if $J^2 \gg 1$, those with $E_n \gtrsim 1$ do not. When $J^2 \ll 1$ one has $E_n \simeq 1 + (n^2/2J^2)$ for all n and even the ground state gives $\langle 2\sin^2(\frac{1}{2}\theta)\rangle = 1$. This Hamiltonian (6.9) of course just describes a simple pendulum with unit frequency $\omega_0{}^2 = g/l = 1$, but with an adjustable unit of action $J = ml^2\omega_0/\hbar$. A strongly quantum pendulum with $J \ll 1$ does "uncertainty principle loop-the-loops" even in its ground state.

In the harmonic projection field of equation (6.1), the harmonic oscillator limit $J \gg 1$, or $k\lambda \ll 1$, gives free field behavior, while the highly quantum limit $J \gg 1$ or $k\chi \ll 1$ suggests some kind of nonpropagating or at least nonrelativistic ($v \ll c$) excitation in the harmonic projection field. The limiting case for $k\chi \gg 1$ would appear to be a form of *ultralocal* field theory. Such field theories, whose action integrals involve time derivatives but no space derivatives, have been studied by Klauder[29-31] and others whom he references, and give exactly soluble models. However, ultralocal quantum field theory models with compact range spaces have apparently not been constructed.

Relaxed theories. As we have seen in Sec. V, for many significant examples (projection operators on R^n, C^n, or H^n) the harmonic mapping action can be written as a free field action

$$I_\phi = -\frac{\hbar}{\chi^2}\int \mathrm{tr}[(\partial_\mu\phi)^\dagger(\partial_\mu\phi)]\,d^4x \qquad (6.10)$$

supplemented by a constraint $\phi^2 = \phi$. Although this action is not expected to give a renormalizable perturbation expansion, it is closely related to a theory that is.

To relax the $\phi^2 = \phi$ constraint without removing it completely we add a constraining term to the free action (6.10),

$$I = I_\phi + I_\beta = I_\phi - \frac{\beta^2\hbar}{2\chi^4}\int d^4x\,\mathrm{tr}[(\phi^2-\phi)^2]. \qquad (6.11)$$

Now all n^2 compontents of $\phi = \phi^\dagger$ can be independently varied, allowing $\phi^2 \neq \phi$. If $\beta^2 > 0$, the "vacuum state" of this action will satisfy $\phi^2 = \phi$, since we have added a term in the Hamiltonian that is non-negative and vanishes only when $\phi^2 = \phi$. (We continue to assume $\phi^\dagger = \phi$, and the other linear constraint trace $\phi = p$ also causes no trouble.) Then $p^2 + q^2 - 1$ fields ϕ have mass $\beta(\hbar/\chi c)$ while the other $2pq$ fields ϕ remain massless if ϕ was an $n\times n$ matrix with $n = p+q$.

Because the added term is merely quartic, the theory defined by Eq. (6.11) should be renormalizable. This conclusion remains valid if we replace ∂_μ by $\nabla_\mu = \partial_\mu + A_\mu$ and add a Yang-Mills term

$$I_A = -(\hbar^2 c/8e^2)\int \mathrm{tr}(F^\dagger_{\mu\nu}F^{\mu\nu})\,d^4x \qquad (6.12)$$

to the action (6.11). The fully constrained theory and the quartic theory differ by the presence of the $p^2 + q^2 - 1$ Higgs mesons of mass $\beta(\hbar/\chi c)$, and the constrained theory would, one hopes, inherit some good manners from the $\beta \to \infty$ limit of S-matrix elements computed in the renormalizable theories.

Functional integration. My third comment on quantization of harmonic maps is an attempt to suggest by entirely formal analogies that the functional integral for a Feynman sum-over-histories quantization of a harmonic mapping theory is very close to the quadratic integrals for free fields. The first few pages of Glimm and Jaffe's[28] introduction to Euclidean field theory include the basic structures I will use in these formal parallels, namely measures on function spaces, and their Fourier transforms. I find it quite remarkable that the formal structure of harmonic mappings into Riemannian symmetric spaces is so rich that one sees not just the first steps in beginning to define the functional integral, but a number of touchstones along the way, and even suggestions for the form of the answer. Among the tools available are spherical functions of positive type. These functions χ: $M' \to \mathbb{C}$ play the role on a symmetric space M' that $\chi = \exp(ik\cdot\phi)$ plays when M' is a vector space, and a Fourier transformation theory is available using them. The positivity condition (stated in Helgason, Ref. 15, Chap. X, Sec. 4) implies $\chi(\overline{\phi}) = \overline{\chi(\phi)}$ where the bar is complex conjugation in $\mathbb{C}$, but on M' it is the involution characteristic of Riemannian symmetric spaces σ: $M' \to M'$, $\phi \to \sigma\phi \equiv \overline{\phi}$, with $\sigma^2\phi = \phi$. For M' a vector space one has $\overline{\phi} = -\phi$ for the involution, and, of course, $e^{-ik\phi}$ being the complex conjugate of $e^{ik\phi}$ is familiar. This positivity property appears well adapted to giving the Osterwalder-Schrader positivity condition on generating functionals in field theory.

It will be convenient at times to replace the map ϕ: $M \to M'$ by its graph $\tilde{\phi}$: $M \to M' \times M$, $x \to (\phi(x), x)$, which can be regarded as a cross section of the product bundle $N = M' \times M \to M$, $(\phi, x) \to x$. Generalizations to a nontrivial bundle N are often apparent. Let f: $N \to \mathbb{R}$ be any scalar function on N. Then for any cross section $\tilde{\phi}$ of N, $f\circ\tilde{\phi}$ is a function on M, and

$$f[\phi] = \int d^4x\, f\circ\tilde{\phi} = \int d^4x\, f(\phi(x), x) \qquad (6.13)$$

is meaningful and defines a functional of ϕ. If a probability measure $d\mu(\phi)$ were available on the space of cross sections of N, one could define a kind of Fourier transform of $d\mu$ by

$$S[f] = \int e^{if[\phi]} d\mu(\phi)\,. \qquad (6.14)$$

The problem of defining $d\mu(\phi)$ would then be equivalent to that of defining an $S[f]$ having appropriate behaviors.

The formula (6.14) is heuristic in that the class of functions f is not specified reasonably. I expect that one really wants to have $\chi = e^{if}$, when restricted to a single fiber of N, involve the spherical functions of positive type on M'. Not all such functions admit a representation of the form e^{ifx} —the Bessel functions are an example—but many admit integral representations of an exponential type that may suffice. (See Helgason, Ref. 15, Chap. X, Theorem 6.16.) What equation (6.14) means then is $S[\chi] = \int \chi[\phi]\, d\mu(\phi)$ where $\chi[\phi]$ are functionals of cross sections ϕ: $M \to M' \times M$ that derive from a complete set of functions for Fourier analysis in the fibers, incorporating positivity conditions that would imply, for example, $\chi[\overline{\phi}] = \overline{\chi[\phi]}$. This last is sufficient to prove that $S[\chi]$ is real when the measure has the symmetry $d\mu(\overline{\phi}) = d\mu(\phi)$ under Reimannian symmetric involutions in each fiber of $N = M' \times M$.

I continue to use Eq. (6.14) for heuristic purposes, lacking an adequate development of "Fourier functionals" $\chi[\phi]$. Another, more familiar, heuristic notation is to write

$$d\mu(\phi) = \exp\left(-\frac{1}{2}\int d^4x\, \phi^A{}_\alpha \phi^\alpha_A\right)\mathfrak{D}\phi\,, \qquad (6.15)$$

where the action of the harmonic mapping ϕ (continued to Euclidean spacetime) appears in the exponential, and

$$\mathfrak{D}\phi = \prod_{x\in M} d\phi_x\,,$$

where $d\phi_x$ is a volume element on the fiber $\pi^{-1}(x) \approx M'$. To the extent that the measure $d\mu$ behaves like a Wiener measure, one can evaluate the functional integral by the method of steepest descents and is forced to consider the variational problem

$$\delta\int d^4x\left(-\tfrac{1}{2}\phi^A_\mu\phi^\mu_A + if\circ\tilde{\phi}\right) = 0 \qquad (6.16)$$

for which the Euler-Lagrange equations are $\phi^\mu_{A;\mu} + if_{,A}(\phi) = 0$. Here $\phi^\mu_{A;\mu}$ is the expression that appears in the harmonic mapping equation $\phi^\mu_{A;\mu} = 0$, and $f_{,A}$ is the vertical derivative of f (tangent to the fiber M') at fixed x, and $f_{,A}(\phi)$ means $f_{,A}\circ\tilde{\phi}$. The imaginary i in this equation arises because the method of steepest descent applies only when analytic continuation in ϕ is defined. I regard it as a reminder to use a correct sign in the exponent of Eq. (6.18) below where ϕ^2 occurs. The equation that needs attention is really

$$\phi^\mu_{A;\mu} + f_{,A}(\phi) = 0 \qquad (6.17)$$

or some better equation with $\chi = e^{if}$ in its source term.

Equation (6.17) is a well-posed (covariant) elliptic equation, and by analogy to linear equations it would most likely have a unique solution for given f if $\phi^\mu_{A;\mu} = 0$ had *no* nontrivial solutions. As discussed in Sec. II and Appendix A, this is most plausible when M' has nonpositive sectional curvatures. But the classically attractive harmonic mapping theories involve fibers M' that are Riemannian symmetric spaces of compact type for which the sectional curvature is non-negative. This contrariness impells one to consider whether the continuation of the action from its Minkowski (classical) form to a ("quantum") form with a Euclidean metric on the spacetime M will also involve the duality (Helgason, Ref. 15, Chap. V, Sec. 2) by which compact-type symmetric spaces are associated with unique (dual) partners of noncompact type.

Could the evaluation of the Fourier transform (6.14) lead to a simple result? I would hope, by analogy to simple linear free fields, that the result would be

$$S[f]=\exp(-\tfrac{1}{2}\|f\|_{\rm HM}{}^2)\,, \tag{6.18}$$

where, in spite of the notation, $\|f\|_{\rm HM}$ is not a simple linear norm but is defined by

$$\|f\|_{\rm HM}{}^2=\int d^4x\,\phi^{\mu}_{A}(f)\phi^{A}_{\mu}(f) \tag{6.19}$$

and $\phi(f)$ is the cross section defined by the nonlinear equation (6.17) for ϕ, with the function f: $N=M'\times M\to \mathrm{R}$ given as a "source."

To lend some support to this long series of conjectures, let us verify that they lead to standard results in the case where M' is a vector space. Then for e^{if} to be a "positive-definite spherical function" of ϕ for each fixed x, under the translation group of the fiber, the function f must be a simple linear function on each fiber. Thus

$$f(\phi,x)=f_A(x)\phi^A \tag{6.20}$$

on $N=M'\times M$. The functional $f[\phi]$ defined in Eq. (6.13) is then the usual linear form

$$f[\phi]=\int d^4x f_A(x)\phi^A(x) \tag{6.21}$$

and (6.14) is the usual functional Fourier transform. With the choice (6.20) for f, and with $G_{AB}(\phi)=\delta_{AB}$ for a flat vector space M', Eq. (6.17) reads

$$\Delta\phi_A=f_A\,, \tag{6.22}$$

where $\Delta=-\partial_\mu\partial_\mu$ is a Laplacian with a positive spectrum. Then the "norm" $\|f\|_{\rm HM}$ from Eq. (6.19) becomes

$$\begin{aligned}\|f\|_{\rm HM}{}^2&=\int d^4x\,\partial_\mu\phi^A\partial_\mu\phi^A\\&=\int d^4x(\Delta\phi^A)\phi^A\\&=\int d^4x f_A\phi^A\\&=\int d^4x f_A\frac{1}{\Delta}f_A\\&=\left\langle f_A,\frac{1}{\Delta}f_A\right\rangle,\end{aligned}$$

which gives the standard formula for $S[f]$ for a Euclidean free field.

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APPENDIX A: EXAMPLES OF HOLOMORPHIC MAPS

The example given in Eq. (2.10) of a harmonic map $\mathrm{E}^2\to S^2$ can be recognized as an holomorphic mapping $\mathrm{C}^1\to \mathrm{P}^1$, $(x+iy)=z\to(z^k,R^k)$ as follows. A point in complex projective space P^n is an equivalence class of nonzero vectors $\xi=(\xi_1,\xi_2,\ldots,\xi_{n+1})\in \mathrm{C}^{n+1}$ under the equivalence $\xi\equiv\lambda\xi$ where λ is any nonzero complex number. To define Θ, Φ coordinates in $\mathrm{P}^1=S^2$ we write

$$\begin{aligned}(\xi_1,\xi_2)&=\lambda(e^{i\Phi/2}\sin\tfrac{1}{2}\Theta,\ e^{-i\Phi/2}\cos\tfrac{1}{2}\Theta)\\&\equiv(e^{i\Phi}\tan\tfrac{1}{2}\Theta,\ 1)\end{aligned} \tag{A1}$$

(the equivalence holds only for $\Theta\neq\pi$), where (ξ_1,ξ_2) is any representative of its equivalence class. In this holomorphic map form, the above example can be generalized to give a solution of Eqs. (2.7) on Euclidean four-space E^4. Identify $\mathrm{E}^4=\mathrm{C}^2$ by $(u,v)=(x+iy,z+it)\in \mathrm{C}^2$. Then define a holomorphic map

$$\phi:\ C^2 \to P^1,\ (u,v) \to (u^k v^l, R^{k+l}) \qquad (A2)$$

or view it equivalently, as $\phi:\ E^4 \to S^2$,

$$(x+iy)^k (z+it)^l / R^{k+l} = e^{i\Phi} \tan\tfrac{1}{2}\Theta\,. \qquad (A2')$$

This mapping does not have a simple asymptotic behavior near ∞ in E^4, but for $k, l \geq 2$ the total action (1.3) or (2.6) is finite. Its possible significance as an instanton solution for a nonlinear σ-type field theory $M^4 \to S^2$ is unclear to me. In the usual mathematical sense the map (A2) is topologically trivial, i.e., homotopic to a constant. To see this, one sets $R = \lambda^{-1}$ and considers the map

$$\phi:\ [0,1] \times C^2 \to P^1,\ (\lambda, u, v) \to (\lambda^{k+l} u^k v^l, 1)\,,$$

which is continuous (even analytic), with $\phi_0 = \text{const}$ for $\lambda = 0$. For application to physical model theories, however, this may not be relevant, as this homotopy does not keep the action integral I_λ continuous (and hence bounded) on $0 \leq \lambda \leq 1$. In fact one can easily see, by a change of coordinates absorbing the λ into u and v, that

$$I_\lambda = \lambda^{-2} I_1$$

for this family, so $I_\lambda \to \infty$ as $\lambda \to 0$. It is thus not a "bounded action homotopy." (Note: the homotopy attempt that uses (A2) for $0 \leq R \leq 1$ keeps the action finite, $I(R) = R^2 I(1)$, but cannot be defined so that the mapping is continuous at the point $(0,0,0)$ of $[0,1] \times C^2$.)

Another set of harmonic maps, obtained from holomorphic maps, that may provide interesting model theories are maps $P^2 \to P^2$. One obtains Euclidean four-space $E^4 = C^2 \subset P^2$ by identifying

$$(x+iy, z+it, 1) = (u, v, 1) \in P^2$$

as a point in E^4 or C^2. The range space P^2 could occur in models of symmetry breaking since $P^2 = SU(3)/SU(2) \times U(1)$. The following is an example of a holomorphic map ϕ: $P^2 \to P^2$ that can be restricted (require $w \neq 0$) to a holomorphic map $C^2 \to P^2$. Let $(u,v,w) \equiv (\lambda u, \lambda v, \lambda w)$ represent a point in the domain space. Then the map

$$\phi:\ P^2 \to P^2, (u,v,w) \to ((u-av)^k (v-bw)^l, (w-au)^k (u-bv)^l, (v-aw)^k (w-bu)^l) \qquad (A3)$$

is holomorphic for $k \geq l \geq 1$ if a and b are distinct nonzero complex numbers that are not cube roots of unity. For $l = 0$ the map is holomorphic if $a \neq 1$, and homotopic to the case $l = 0$, $a = 0$.

APPENDIX B: IDENTITIES AND INTEGRAL FORMULAS

In this appendix I derive the Jacobi (perturbation) equation (2.27) (see also Duff and Isham[14]), the identity (2.29), and the integral formula (2.30). Start from the harmonic mapping equation $f^{A;\mu}_{\mu} = 0$ for a family of maps $f(\lambda, x)$ and apply the derivative $n = (\partial/\partial\lambda)_x$. One has $\partial f^A_\mu / \partial\lambda = \partial^2 f^A / \partial\lambda \partial x^\mu = \partial_\mu n^A$ since $n^A = \partial f^A / \partial\lambda$ as in Eq. (2.15). It is convenient to use normal coordinates so that $\Gamma^\alpha{}_{\beta\mu}(x_0) = 0$ and $\Gamma^A{}_{BC}(f(\lambda_0, x_0)) = 0$. Then $\partial_\nu \Gamma^\alpha{}_{\beta\mu}$ appears in neither $n^A{}_{;\mu}{}^{;\mu}$ nor $\partial(f^{A;\mu}_\mu)/\partial\lambda$ and the $\partial\Gamma^A{}_{BC}$ terms can be identified with $R^A{}_{BCD}$ at this point, yielding the Jacobi equation (2.27).

Eells and Sampson,[5] along lines indicated by Bochner,[32] consider $\mathcal{L}_{;\mu}{}^{;\mu}$ where $\mathcal{L} = \frac{1}{2}\phi^\mu_A \phi^A_\mu$ is the harmonic mapping Lagrangian. One first uses Eq. (2.25), then the Ricci identities such as $n^A{}_{;\mu\nu} = n^A{}_{;\nu\mu} + n^B R^A{}_{B\nu\mu}$, to find

$$\phi^A_{\alpha;\mu}{}^{;\mu} = \phi^A_{\mu;\alpha}{}^{;\mu} = \phi^{A;\mu}_\mu{}_{;\alpha} + \phi^B_\mu R^A{}_B{}^\mu{}_\alpha - \phi^A_\nu R^\nu{}_\mu{}^\mu{}_\alpha\,. \qquad (B1)$$

This can be rewritten as

$$\phi_{A\alpha;\mu}{}^{;\mu} = (\phi_{A\mu}{}^{;\mu})_{;\alpha} + (G_{AB} R_{\alpha\beta} - R_{ACBD} \phi^C_\mu \phi^{D\mu} g_{\alpha\beta}) \phi^{B\beta}\,, \qquad (B2)$$

$$\mathcal{L}_{;\mu}{}^{;\mu} = (\tfrac{1}{2}\phi_{A\alpha}\phi^{A\alpha})_{;\mu}{}^{;\mu} = (\phi_{A\alpha;\mu}\phi^{A\alpha})^{;\mu} = \phi^{A\alpha}(\phi_{A\mu}{}^{;\mu})_{;\alpha} + \phi_{A\alpha;\beta}\phi^{A\alpha;\beta} + \phi^{A\alpha}\phi^{B\beta}(G_{AB}R_{\alpha\beta} - R_{ABCD}\phi^C_\alpha \phi^D_\beta) \qquad (B3)$$

or, equivalently, Eq. (2.29). For a harmonic map, with $\phi_{A\mu}{}^{;\mu} = 0$, in an asymptotically flat Riemannian n-manifold M, suppose the integral of the right-hand side of (B3) were finite but nonzero. Then as $r \to \infty$ in M one must have $\mathcal{L} \sim (\text{const}) r^{2-n}$ (or $\mathcal{L} \sim \ln r$ for $n = 2$) from the monopole term in $\mathcal{L}$ as a solution of this scalar Laplace equation. But then $I = \int \mathcal{L}\sqrt{|g|}\, d^n x$ diverges. We conclude, that *for finite action solutions of the harmonic mapping equation on asymptotically flat Riemannian manifolds, Eq.* (2.30) *holds*, that is, the integral of the right-hand side of Eq. (B3) must vanish. We can therefore draw similar conclusions in this asymptotically flat case to those Eells and Sampson drew for a compact M. For instance, suppose the Ricci curvature of M is non-negative, and the

Riemannian (sectional) curvature of M' are nonpositive. Then a finite action map ϕ: $M \to M'$ is harmonic if and only if it is "totally geodesic" (i.e., satisfies $\phi_{A\alpha;\beta}=0$). Furthermore, if the Riemannian sectional curvatures of M' are everywhere negative, then every finite action harmonic map f: $M \to M'$ is either constant or maps M onto a geodesic of M'.

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[1]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity* (*Nobel Symposium No. 8*), edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[2]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[3]S. Weinberg, in *Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies, Hamburg, 1977*, edited by F. Gutbrod (DESY, Hamburg, 1977), p. 619.

[4]Harmonic maps were defined and named by F. B. Fuller, Proc. Natl. Acad. Sci. 40, 987 (1954), who also gave important examples and asked significant questions. For a recent review see J. Eells and L. Lemaire, Bull. London Math. Soc. 10, 1 (1968).

[5]I have primarily studied the early paper of J. Eells, Jr., and J. H. Samson, Am. J. Math. 86, 109 (1964).

[6]R. Utiyama, Phys. Rev. 101, 1597 (1956).

[7]R. P. Feynman, Acta Phys. Pol. 24, 697 (1963).

[8]B. S. DeWitt, Phys. Rev. Lett. 12, 742 (1964).

[9]B. S. DeWitt, Phys. Rev. 162, 1195 (1967).

[10]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).

[11]S. Mandelstam, Phys. Rev. 175, 1580 (1968); 175, 1604 (1968).

[12]G. Holton, *Thematic Origins of Scientific Thought, Kepler to Einstein* (Harvard Univ., Cambridge, Mass., 1973).

[13]G. Holton, *The Scientific Imagination* (Cambridge Univ. Press, Cambridge, England, 1978).

[14]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 53 (1960). For a recent study see, e.g., M. J. Duff and C. J. Isham, Phys. Rev. D 16, 3047 (1977).

[15]S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962), Vol. XII.

[16]C. W. Misner (unpublished).

[17]R. A. Matzner and C. W. Misner, Phys. Rev. 154, 1229 (1967).

[18]S. Smale, private communication.

[19]Y. Nutku, Ann. Inst. Henri Poincaré A21, 175 (1974).

[20]A. Eris and Y. Nutku, J. Math. Phys. 16, 1431 (1975).

[21]A. Eris, J. Math. Phys. 18, 824 (1977).

[22]Y. Nutku and M. Halil, Phys. Rev. Lett. 39, 1379 (1977).

[23]N. Bourbaki, *Theorie d'Ensembles, Fasicule de resultats* (Hermann, Paris, 1939).

[24]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[25]For an introduction see, e.g., R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[26]D. Husemoller, *Fibre Bundles*, 2nd ed. Vol. 20 in *Graduate Texts in Mathematics* edited by P. R. Halmos and C. C. Moore (Springer, New York, 1975).

[27]N. Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, N. J. 1951), Corollary 9.5 and Theorem 9.4, p. 43.

[28]J. Glimm and A. Jaffe, in proceedings of the 1976 Cargèse Summer School (unpublished).

[29]J. R. Klauder, Commun. Math. Phys. 18, 307 (1970).

[30]J. R. Klauder, in *Relativity*, edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970).

[31]J. R. Klauder, Acta Phys. Austriaca, Suppl. VIII, 227 (1971).

[32]S. Bochner, Trans. Am. Math. Soc. 47, 146 (1940).