Einstein equations, supersymmetry, and flat limits of curved superspace

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An affine theory of unified interactions is reviewed. Einstein's free gravitational equation emerges as a special case of the proposed equations of motion on superspace. For this example the vacuum is spontaneously broken to be globally Lorentz invariant. Flat superspace is defined and the corresponding algebraic properties studied. Next the graded de Sitter group is introduced and various contractions performed. From these we obtain and discuss supersymmetry as well as some spin-3/2 algebras. Finally, form invariance under Fermi displacements is employed to develop a powerful calculational technique, an example of which is presented in the Appendix.

I. INTRODUCTION

In the past few years there has been a good deal of activity in developing affine theories in superspace[1–5] (i.e., a space containing commuting as well as anticommuting coordinates). Local invariance on superspace demands the inclusion of both Bose and Fermi fields as gauge fields, thus providing a natural framework for unifying various or all interactions. In Refs. 4 and 5 we also analyzed the modes of symmetry breakdown at the vacuum level. Depending upon the conditions, we obtained the Lorentz or various supersymmetric metrics. Most of these metrics were noninvertible and/or theories with torsion.

In this paper we further pursue the formalism developed in Ref. 4. In Sec. II. we review some of this material. In Sec. III we show how the spontaneously broken Lorentz line elements lead us to an exact solution to our proposed equation of motion. While we do introduce a cosmological constant into the superspace equations, it disappears and we retrieve Einstein's free gravitational equation (with no reference to the internal-symmetry group). We then proceed in Sec. IV to examine the "flat" limit of superspace. The purpose is to obtain the analog of the Poincaré group on flat superspace (FS). In Sec. V we consider various contractions of the graded de Sitter group, so as to obtain and discuss supersymmetry and generic spin-$\frac{3}{2}$ algebras. Sections IV and V allow us to briefly discuss the mass and spin of FS multiplets in Sec. VI. Section VII discusses the consequences of form invariance under Fermi displacements. In particular, it leads to a calculational device which considerably reduces the effort needed in obtaining the desired field equations. An example of this procedure is presented in the Appendix for a space consisting of one Bose and two Fermi coordinates.

II. NOTATIONS AND REVIEW

We shall try as far as possible to keep this paper self-contained. Thus a brief summary of the formalism developed in Ref. 4 for curved superspace is presented below.[6,7]

We use z^A to denote collectively Bose (x^μ) as well as Fermi (θ^α) coordinates. The index μ runs over the Bose dimensions (which need not be the traditional 4 of space-time) and the index α runs over Fermi dimensions (including internal-symmetry indices). Internal-symmetry indices will be explicitly exhibited, when needed.

Since

$$[x^\mu, x^\nu]=0,\quad [x^\mu,\theta^\alpha]=0,\quad \text{and } \{\theta^\alpha,\theta^\beta\}=0\,,$$

we can write

$$z^A z^B-(-1)^{ab} z^B z^A=0\,, \tag{2.1}$$

where the Grassmann parity $a=0$ or 1 according to whether A is Bose or Fermi, respectively.

Let us define the set of (coordinate-independent) operators $a_r{}^A$, $\bar a^s{}_B$ ($r,s=1,2,\ldots,\infty$) with the following generalized commutation relations:

$$a_r{}^A a^B{}_s-(-1)^{ab} a^B{}_s a_r{}^A=0\,, \tag{2.2a}$$

$$\bar a^r{}_A \bar a^s{}_B-(-1)^{ab}\bar a^s{}_B \bar a^r{}_A=0\,, \tag{2.2b}$$

$$a_r{}^A \bar a^s{}_B-(-1)^{ab}\bar a^s{}_B a_r{}^A=\delta_r{}^s\delta^A{}_B\,. \tag{2.2c}$$

In terms of these creation and annihilation operators, the generators $G_A{}^B$ of the graded pseudo-Lie group GGL (N_B, N_F, R) of real, general linear transformations, can be written as

$$G_A{}^B=\sum_{r=1}^{\infty}\bar a_A{}^r a_r{}^B\,. \tag{2.3}$$

Then, the required algebra is

$$G_A{}^B G_C{}^D-(-1)^{(a+b)(c+d)}G_C{}^D G_A{}^B = \delta^B{}_C G_A{}^D-(-1)^{(a+b)(c+d)}\delta^D{}_A G_C{}^B\,. \tag{2.4}$$

On superspace with coordinates z^A, local gauge transformations $U(z)$ are defined as

$$U(z)=\exp[G_A{}^B\omega_B{}^A(z)]\,, \qquad (2.5)$$

where $\omega_B{}^A(z)$ are the "gauge parameters." It was shown in Ref. 4 that the gauge transformations generated by $U(z)$ are in 1:1 correspondence with general coordinate transformations $z'^A=z'^A(z)$. Thus, for a contravariant vector V^A, one considers

$$V^s\equiv\bar{a}^s{}_A V^A$$

and its transformation law is then given by

$$V^{s\prime}(z')=U(z)V^s(z)U^{-1}(z)\,.$$

On the other hand, there are two types of covariant vectors W_A and $W_{\hat{A}}$, which transform, respectively, as $\partial_R\varphi/\partial z^A$ and $\partial_L\varphi/\partial z^A$, where $\varphi(z)$ is a scalar, i.e., $\varphi'(z')-\varphi(z)$ and R and L mean right and left derivatives. (Thus, $V^AW_{\hat{A}}$ is a scalar and so is W_AV^A.) For W_A and W_A, one forms

$$W_s=W_Aa^A{}_s\,,$$

$$W_{\hat{s}}=a^A{}_sW_{\hat{A}}\,,$$

which transform according to

$$W_s'=U\,W_sU^{-1}\,,$$

$$W_{\hat{s}}'=U\,W_{\hat{s}}U^{-1}\,.$$

Now let us consider more complicated tensors, for example, $G^A{}_{\hat{B}CD}$. We require that it transforms as if it were in the factorized form

$$G^A{}_{\hat{B}CD}=(G_1{}^A(z))(G_{2\hat{B}}(z))(G_{3C}(z))(G_{4D}(z))\,.$$

We now define

$$G^r{}_{\hat{s}tu}(z)=(\bar{a}^r{}_AG_1{}^A(z))(a^B{}_sG_{2\hat{B}}(z))(G_{3c}(z)a^C{}_t)(G_{4D}(z)a^D{}_u)$$
$$=\bar{a}^r{}_Aa^B{}_s[(-1)^{ab}G^A{}_{\hat{B}CD}(z)]a^D{}_ua^C{}_t\,.$$

Under the group element U,

$$UG^r{}_{\hat{s}tu}(z)U^{-1}=(\bar{a}^r{}_AG'^A{}_1(z'))(a^B{}_sG_{2B}'(z'))\times(G_{3C}'(z')a^C{}_t)(G_{4D}'(z')a^D{}_u)$$
$$=\bar{a}^r{}_Aa^B{}_s[(-1)^{ab}G'^A{}_{\hat{B}CD}(z')]a^D{}_ua^C{}_t$$
$$=G'^r{}_{\hat{s}tu}(z')\,.$$

This again illustrates the one-to-one correspondence between gauge transformations and coordinate transformations.

It is worth remarking that all functions of the type $G^r{}_{\hat{s}tu}$, in which all Bose and Fermi indices $(A,B,\ldots)$ have been contracted, have zero Grassmann parity. We shall call such quantities "reduced tensors."

To first order, the gauge parameter $\omega_B{}^A(z)$ is related to the coordinate transformation,

$$z'^A=z^A+\xi^A(z)\,,$$

by

$$\omega_B{}^A\simeq\frac{\partial_L\xi^A}{\partial z^B}\,. \qquad (2.6)$$

The Lorentz metric is chosen as $\eta_{\mu\nu}=(-,+,+,+)$. We work in the Majorana representation with all γ's real and

$$\{\gamma_\mu,\gamma_\nu\}=2\eta_{\mu\nu}\,, \qquad (2.7)$$

$\gamma_5{}^2=-1$ and the metric in Dirac space, $\eta_{\alpha\beta}$, is chosen to be $\eta_{\alpha\beta}=-(C^{-1})_{\alpha\beta}$, where C is the charge conjugation matrix. We define $\sigma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]$.

III. SUPERSPACE EQUATIONS OF MOTION AND EINSTEIN'S FREE GRAVITATIONAL EQUATION

In this section we shall review a previously proposed equation of motion on superspace,[4] and show how it contains Einstein's free gravitational equation. The first step is to discuss the covariant derivative and the resulting Yang-Mills connection. From the latter, we may then construct the field strengths on superspace, which finally leads us to a very simple equation of motion.

The connections that are defined on the superspace are to be expanded in a power series in the Fermi coordinates. This power series terminates, and the coefficients are to be taken as the dynamical fields of the theory.

Consider a reduced tensor $T^{\{s\}}{}_{\{\hat{t}\}\{u\}}$ which transforms under U as

$$T'^{\{s\}}{}_{\{\hat{t}\}\{u\}}=UT^{\{s\}}{}_{\{\hat{t}\}\{u\}}U^{-1}\,, \qquad (3.1)$$

where $\{s\}=s_1s_2\ldots s_n$, etc., then as shown in Ref. 4 we may define a right covariant derivative

$$D_rT^{\{s\}}{}_{\{\hat{t}\}\{u\}}=D_AT^{\{s\}}{}_{\{\hat{t}\}\{u\}}a^A{}_r\,, \qquad (3.2)$$

which will also transform according to Eq. (3.1) provided

$$D_AT^{\{s\}}{}_{\{\hat{t}\}\{u\}}=\frac{\partial_R}{\partial z^A}T^{\{s\}}{}_{\{\hat{t}\}\{u\}}+[\Gamma_A,T^{\{s\}}{}_{\{\hat{t}\}\{u\}}] \qquad (3.3)$$

and $\Gamma_A(z)$ transforms according to

$$\Gamma_A'(z')=\left[U\Gamma_B(z)U^{-1}+U\frac{\partial_R}{\partial z^B}U^{-1}\right]\frac{\partial_Rz^B}{\partial z'^A}\,. \qquad (3.4)$$

Thus, $\Gamma_A(z)$ plays the role of a connection as in the standard Yang-Mills theories. One writes

$$\Gamma_A(z)=G_B{}^C\Gamma_{\hat{C}}{}^B{}_A\,. \qquad (3.5)$$

In theories with no torsion

$$\Gamma_{\hat{C}}{}^B{}_A=(-)^{(b+1)(a+c)+ac}\Gamma_{\hat{A}}{}^B{}_C\,. \qquad (3.6)$$

In terms of Γ_A two field strengths [transforming according to Eq. (3.1)] may now be constructed of which a particular linear combination is R_{rs}, where

$$R_{rs} = a^A{}_r R_{\hat{A}B} a^B{}_s \tag{3.7}$$

and

$$R_{\hat{A}B} = -(-1)^c \Gamma_{\hat{A}C,B}^{C} + (-1)^{c(b+1)} \Gamma_{\hat{A}B,C}^{C} + (-1)^{c+be} \Gamma_{\hat{A}}{}^E{}_B \Gamma_{\hat{E}}{}^C{}_C - (-1)^{ce} \Gamma_{\hat{A}}{}^E{}_C \Gamma_{\hat{E}}{}^C{}_B . \tag{3.8}$$

In a space of only Bose coordinates, $R_{\hat{A}B}$ would be the contracted curvature tensor.

The simplest equation of motion one can choose which has nontrivial solutions is

$$D_t R_{rs} = 0. \tag{3.9}$$

Since, to this point we are dealing with a purely affine theory, one might wish to consider instead of Eq. (3.9) Schrödinger's variational principle[8] extended to superspace. Another attractive possibility is an extension of the Einstein-Strauss theory.[9] However, we should remark that the failure of Eq. (3.9) on a purely Bose space does not apply to superspace, and one can acommodate Fermi as well as Bose fields. Here, we shall show how one recovers Einstein's equation for the free gravitational field.

We first exhibit the Lorentz metric as a spontaneously broken solution of Eq. (3.9).[4] In particular we look for a solution in which $\Gamma_{\hat{B}}{}^A{}_C$ has the symmetry of Eq. (3.6) and is independent of all the coordinates. The nonvanishing components of the connection are

$$\Gamma^{(\beta n)}_{(\alpha m)\mu} = \Gamma^{(\beta n)}_{\mu(\alpha m)} = (\gamma_\mu)^\beta{}_\alpha (M_1)^n{}_m + (\gamma_5\gamma_\mu)^\beta{}_\alpha (M_2)^n{}_m , \tag{3.10a}$$

and

$$\Gamma^{\mu}_{(\alpha m)(\beta n)} = -\Gamma^{\mu}_{(\beta n)(\alpha m)} = (\eta\gamma^\mu)_{\alpha\beta}(M_3)_{mn} + (\eta\gamma_5\gamma^\mu)_{\alpha\beta}(M_4)_{mn} , \tag{3.10b}$$

where M_i are matrices in the internal-symmetry space with $M_3 = -M_3^T$ and $M_4 = M_4^T$. Inserting Eq. (3.10) into (3.8) and that into Eq. (3.9) we find that Eq. (3.9) is satisfied provided $M_3 = M_4 = 0$. Hence from Eq. (3.8) we obtain only one nonvanishing component of $R_{\hat{A}B}$:

$$R_{\mu\nu} = 4\eta_{\mu\nu}\,\mathrm{tr}(M_1{}^2 + M_2{}^2) , \tag{3.11}$$

where $\eta_{\mu\nu}$ is the Lorentz metric. This strongly suggests that we *define* a metric tensor by

$$g_{rs} = a^A{}_r g_{\hat{A}B} a^B{}_s \equiv \frac{1}{\lambda} R_{(rs)} , \tag{3.12}$$

where $R_{(rs)}$ is the symmetric part of R_{rs}, and with

$$\lambda = 4\,\mathrm{tr}(M_1{}^2 + M_2{}^2) . \tag{3.13}$$

Then from Eq. (3.9) we have

$$D_t g_{rs} = 0 \tag{3.14}$$

and hence the covariant derivative of g_{rs} vanishes. If we denote by $g^0{}_{\hat{A}B}$ the "vacuum metric," then all components of $g^0{}_{\hat{A}B}$ vanish except for

$$g^0{}_{\mu\nu} = \eta_{\mu\nu} . \tag{3.15}$$

Thus while we can define a metric $g_{\hat{A}B}$, its inverse g^{AB} does *not* exist (in the whole superspace). Hence a Riemannian geometry cannot be developed on the superspace under these circumstances and we must continue to work with a purely affine theory. We note that the internal-symmetry group has been left totally *arbitrary* up to this point.

We now wish to start considering the equations of motion obtained from Eq. (3.9) for the dynamical fields. We here restrict our attention to the simplest case, which is the free gravitational field. Thus we introduce the usual vierbein field $e^a{}_\mu(x)$, where a is the local Lorentz index and μ is the world index. It is related to $g_{\mu\nu}(x)$ by

$$e^a{}_\mu(x) e^b{}_\nu(x) \eta_{ab} = g_{\mu\nu}(x) . \tag{3.16}$$

From our previous considerations we also have

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) , \tag{3.17}$$

where $h_{\mu\nu}(x)$ is the dynamical gravitational field. We now modify Eq. (3.10a) so as to include the gravitational field via $e^a{}_\mu(x)$. Thus the only nonvanishing components of the connection are (for this free gravitational case)

$$\Gamma^{(\beta n)}_{(\alpha m)\mu} = \Gamma^{(\beta n)}_{\mu(\alpha m)} = (\gamma_a)^\beta{}_\alpha e^a{}_\mu(x)(M_1)^n{}_m + (\gamma_5\gamma_a)^\beta{}_\alpha e^a{}_\mu(x)(M_2)^n{}_m , \tag{3.18a}$$

and

$$\Gamma^{\mu}_{\nu\lambda}(x) = \Gamma^{\mu}_{\lambda\nu}(x) , \tag{3.18b}$$

with $\Gamma^\mu_{\nu\lambda}(x)$ to be determined. In fact, from Eq. (3.14) and Eqs. (3.18), we obtain

$$g_{\mu\nu}\Gamma^\nu_{\lambda\sigma} = \tfrac{1}{2}(g_{\mu\lambda,\sigma} + g_{\mu\sigma,\lambda} - g_{\lambda\sigma,\mu}) . \tag{3.19}$$

This equation only involves Bose indices $[g_{\mu\lambda,\sigma} \equiv (\partial/\partial x^\sigma) g_{\mu\lambda}(x)]$. From Eq. (3.17) it is clear that the inverse of $g_{\mu\nu}(x)$ exists in the Bose subspace. That is, we may define $g^{\lambda\sigma}(x)$ by

$$g^{\lambda\sigma} g_{\sigma\mu} = \delta^\lambda_\mu . \tag{3.20}$$

Hence Eq. (3.19) becomes

$$\Gamma^\nu_{\lambda\sigma} = \tfrac{1}{2} g^{\nu\mu}(g_{\mu\lambda,\sigma} + g_{\mu\sigma,\lambda} - g_{\lambda\sigma,\mu}) , \tag{3.21}$$

which gives $\Gamma^\nu_{\lambda\sigma}$ as the usual Christoffel symbol of the second kind in terms of the metric $g_{\mu\nu}(x)$ in the Bose subspace only.

We now insert Eq. (3.18a) and Eq. (3.21) into Eq. (3.12) which may be written as

$$R_{\underline{\hat{A}B}} = \lambda g_{\hat{A}B}\,, \tag{3.22}$$

where

$$R_{\underline{\hat{A}B}} \equiv \tfrac{1}{2}[R_{\hat{A}B} + (-1)^{a+b+ab} R_{\hat{B}A}]\,. \tag{3.23}$$

$R_{\hat{A}B}$ is given by Eq. (3.8), and λ by Eq. (3.13). Letting $\hat{A} = \mu$, $B = \nu$, and using Eq. (3.16) we find that the $\lambda g_{\hat{A}B}$ term on the right-hand side of Eq. (3.22) is exactly canceled and we obtain

$$R^{E}{}_{\mu\nu}(g_{\sigma\lambda}) = 0\,, \tag{3.24}$$

where $R^{E}{}_{\mu\nu}(g_{\sigma\lambda})$ is the contracted curvature tensor, formed with respect to only the Bose components of the metric. We have thus recovered Einstein's equation of motion for the free gravitational field in spite of the fact that we do not have a Riemannian geometry on the full superspace.

Similar results were obtained in Ref. 10. However, there the internal-symmetry group had to be U(1) in contrast to the present situation, where the internal-symmetry group is left totally arbitrary.

IV. FLAT SUPERSPACE

The procedure for obtaining the Lorentz group from the general linear group GL(4,R) is well known. A constant symmetric "metric" $g_{\mu\nu}$ is used to lower the indices of the group generators $G_\lambda{}^\sigma$. The generators $h_{\mu\nu}$ of the Lorentz group are then given by the antisymmetric combination

$$h_{\mu\nu} = G_\mu{}^\lambda g_{\lambda\nu} - G_\nu{}^\lambda g_{\lambda\mu}\,.$$

We can carry over this procedure with only minor modifications to obtain the analog of the Lorentz group in superspace.

A constant metric $g_{\hat{A}B}$ needs to be found in superspace which plays the role of $g_{\mu\nu}$ used above. The symmetry requirement is

$$g_{\hat{A}B} = (-1)^{a+b+ab} g_{\hat{B}A}\,. \tag{4.1}$$

Then, the combination

$$h_{AB} \equiv G_A{}^C g_{\hat{C}B} - (-1)^{ab} G_B{}^C g_{\hat{C}A} \tag{4.2}$$

satisfies the algebra

$$\begin{aligned} h_{AB}h_{CD} - (-1)^{(a+b)(c+d)} h_{CD}h_{AB} &= (-1)^{(b+c)(c+d)} h_{AD} g_{\hat{C}B} \\ &- (-1)^{(c+a)(a+b)} h_{CB} g_{\hat{A}D} \\ &- (-1)^{ab+(a+c)(c+d)} h_{BD} g_{\hat{C}A} \\ &+ (-1)^{ab+(b+c)(a+b)} h_{CA} g_{\hat{B}D}\,. \end{aligned} \tag{4.3}$$

This is our generalization of the (homogeneous) Lorentz group to superspace. Before we discuss physical implications, a few words about $g_{\hat{A}B}$ are in order. If we do not wish to entertain extraneous parameters, we are limited to choosing

$$g_{\mu\alpha} = 0\,. \tag{4.4}$$

Even though $g_{\mu\alpha}$ has been chosen to be zero, the Bose-Fermi operators $h_{\mu\alpha}$ are *not* zero. (Thus, the Bose and Fermi spaces are not disjoint.) Using Eq. (4.4), we can write Eq. (4.3) in terms of its tensor components explicitly as

$$\begin{aligned} [h_{\mu\nu}, h_{\lambda\sigma}] &= h_{\mu\sigma}g_{\nu\lambda} - h_{\lambda\nu}g_{\mu\sigma} \\ &\quad - h_{\nu\sigma}g_{\lambda\mu} + h_{\lambda\mu}g_{\nu\sigma}\,, \end{aligned} \tag{4.5a}$$

$$[h_{\mu\nu}, h_{\lambda\alpha}] = h_{\mu\alpha}g_{\lambda\nu} - h_{\nu\alpha}g_{\lambda\mu}\,, \tag{4.5b}$$

$$[h_{\mu\nu}, h_{\alpha\beta}] = 0\,, \tag{4.5c}$$

$$\{h_{\mu\alpha}, h_{\nu\beta}\} = -h_{\alpha\beta}g_{\mu\nu} + h_{\mu\nu}g_{\alpha\beta}\,, \tag{4.5d}$$

$$[h_{\mu\alpha}, h_{\beta\gamma}] = -h_{\mu\gamma}g_{\alpha\beta} - h_{\mu\beta}g_{\alpha\gamma}\,, \tag{4.5e}$$

$$\begin{aligned} [h_{\alpha\beta}, h_{\gamma\delta}] &= h_{\alpha\delta}g_{\gamma\beta} - h_{\gamma\beta}g_{\alpha\delta} \\ &\quad + h_{\beta\delta}g_{\gamma\alpha} - h_{\gamma\alpha}g_{\beta\delta}\,. \end{aligned} \tag{4.5f}$$

An interpretation of the set of operators h_{AB} can be attempted along the same lines as the Bose operator $h_{\mu\nu}$, which generates Lorentz transformations. Just as an arbitrary Lorentz transformation can be written as

$$U_L = e^{-(i/2)h_{\mu\nu}\omega^{\nu\mu}}\,,$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the six parameters necessary to characterize the infinitesimal Lorentz transformation

$$x'^{\mu} \simeq (\delta^\mu{}_\nu + \omega^{\mu\lambda}g_{\lambda\nu})x^\nu\,,$$

we can say that the transformation

$$U = e^{-(1/2)h_{AB}\omega^{BA}}$$

with (constant) parameters. $\omega^{BA} = -(-1)^{ab}\omega^{AB}$, is related to the infinitesimal transformation of the coordinates

$$z'^{A} \simeq (\delta^A{}_B + \omega^{AC}g_{\hat{C}B})z^B\,. \tag{4.6}$$

Thus, scalar products of the type $V\cdot W = V^A g_{\hat{A}B} W^B$, where V^A and W^B are vectors, are invariants under U.

Killing vectors for flat superspace have been obtained and discussed by Kannenberg.[14]

To form the analog of the Poincaré (or inhomogeneous Lorentz) group for superspace, we need the "translation operators" $\hat{P}^A = (\hat{P}^\mu, \hat{P}^\alpha)$. In the next section, we discuss in some detail the appropriate contraction scheme of a de Sitter superspace with one extra Bose dimension to obtain these momentum operators $\hat{P}^A$. For now, we borrow some results obtained there, which show that

$$[\hat{P}^\mu,\hat{P}^\nu]=0\,, \tag{4.7a}$$

$$[\hat{P}^\mu,\hat{P}^\alpha]=0\,, \tag{4.7b}$$

$$[\hat{P}^\mu,h_{\nu\lambda}]=(\delta^\mu{}_\nu\hat{P}_\lambda-\delta^\mu{}_\lambda\hat{P}_\nu)\,, \tag{4.7c}$$

$$[\hat{P}^\mu,h_{\nu\alpha}]=\delta^\mu{}_\nu\hat{P}_\alpha\,, \tag{4.7d}$$

$$[\hat{P}^\mu,h_{\alpha\beta}]=0\,, \tag{4.7e}$$

$$[\hat{P}^\alpha,h_{\nu\lambda}]=0\,, \tag{4.7f}$$

$$\{\hat{P}^\alpha,h_{\beta\mu}\}=\delta^\alpha{}_\beta\hat{P}_\mu\,, \tag{4.7g}$$

$$[\hat{P}^\alpha,h_{\beta\gamma}]=\delta^\alpha{}_\beta\hat{P}_\gamma+\delta^\alpha{}_\gamma\hat{P}_\beta\,. \tag{4.7h}$$

Equations (4.5) and (4.7) constitute the totality of generalized commutation relations for the generators of inhomogeneous tranformations in FS. These transform the coordinates according to

$$z'^A=\Lambda^A{}_Bz^B+C^A\,, \tag{4.8}$$

where $\Lambda^A{}_B$ and C^A are constant parameters.

The Fermi (Majorana) coordinates $\theta^{\alpha m}$ carry the internal-symmetry index m along with the Dirac index α. We can define the "generalized charges" on the *curved* superspace as

$$Q_k=\sum_r\bar{a}^r{}_{\alpha m}(\mathfrak{M}_k)^{\alpha m}{}_{\beta n}a_r{}^{\beta n}\,, \tag{4.9}$$

where the matrices $\mathfrak{M}_k$ are the (direct) product matrices formed out of the 16 Dirac matrices with the internal-symmetry matrices T_l of the internal-symmetry group GL(N), where $m,n=1,2,\ldots,N$.

For FS, however, we have a smaller allowed set of charges:

$$\hat{Q}_k\equiv-\tfrac{1}{2}h_{(\alpha m)(\beta n)}(\mathfrak{M}_kg^{-1})^{(\alpha m)(\beta n)}\,, \tag{4.10}$$

where the symmetry $h_{(\alpha m)(\beta n)}=h_{(\beta n)(\alpha m)}$ tells us that only those matrices $\mathfrak{M}_k$ are allowed for which

$$\tilde{\mathfrak{M}}_k\equiv g^{-1}\mathfrak{M}_k{}^Tg=-\mathfrak{M}_k\,. \tag{4.11}$$

Thus, for the Majorana case, i.e., *no* internal symmetry, only two sets of operators survive:

$$\hat{Q}_\mu\equiv2\Gamma_\mu=-\tfrac{1}{2}h_{\alpha\beta}(\gamma_\mu g^{-1})^{\alpha\beta}\,, \tag{4.12a}$$

and

$$\hat{Q}_{\mu\nu}\equiv+2\Sigma_{\mu\nu}=-\tfrac{1}{2}h_{\alpha\beta}(\sigma_{\mu\nu}g^{-1})^{\alpha\beta}\,. \tag{4.12b}$$

It can easily be shown, using Eq. (4.5f), that

$$[\Gamma_\mu,\Gamma_\nu]=\Sigma_{\mu\nu}\,. \tag{4.13}$$

$\Sigma_{\mu\nu}$ is clearly the spin part of the total angular momentum operator $M_{\mu\nu}$,

$$M_{\mu\nu}=h_{\mu\nu}+\Sigma_{\mu\nu}\,. \tag{4.14}$$

On the other hand, if we consider purely internal-symmetry charges, formed out of the identity element of the Dirac (Clifford) algebra and some internal-symmetry matrices T_l, then the condition (4.11) tells us that T_l's must be *antisymmetric*.

V. SUPERSYMMETRY AND OTHER ALGEBRAS AS CONTRACTIONS OF GRADED DE SITTER GROUPS

Let us consider a flat superspace (FS) with one extra Bose dimension, i.e., Bose coordinates ξ^a, $a=0,1,2,3,5$ and Fermi coordinates $\xi^{\alpha m}$ with the same Fermi indices (αm) as before.[11] This possesses a four-dimensional Majorana representation of the γ matrices of the form given by Eq. (2.7),

$$\{\gamma_a,\gamma_b\}=2\eta_{ab}\,, \tag{5.1}$$

with all γ's real, $\gamma_5{}^2=-1$ and $\eta_{55}=-1$. Thus, homogeneous tranformations in the Bose sector form the group O(3,2) since $\eta_{ab}=(-1,1,1,1,-1)$. If we let μ run over the index 5 as well, Eqs. (4.5) remain valid. Let us choose for the extended space, $g_{a\alpha}=0, g_{\alpha\beta}=ik\eta_{\alpha\beta}$ with k fixed independent of the coordinates, and $g_{ab}=\eta_{ab}$.

Consider the hypersurface defined by

$$\begin{aligned}\xi^5&=R\varphi_1(z)\,,\\ \xi^\mu&=x^\mu\varphi_2(z)\,,\\ \xi^\alpha&=\lambda\theta^\alpha\varphi_2(z)\,,\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}\varphi_1(z)&=\frac{1+(1/4R^2)(x^2+i\lambda^2k\bar\theta\theta)}{1-(1/4R^2)(x^2+i\lambda^2k\bar\theta\theta)}\,,\\ \varphi_2(z)&=\frac{1}{1-(1/4R^2)(x^2+i\lambda^2k\bar\theta\theta)}\,,\end{aligned} \tag{5.3}$$

and R and λ are fixed, independent of coordinates. Clearly, then

$$\begin{aligned}\xi^Ag_{\hat{A}B}\xi^B&=-R^2\varphi_1{}^2+(x^2+i\lambda^2k\bar\theta\theta)\varphi_2{}^2\\ &=-R^2\,,\end{aligned} \tag{5.4a}$$

by virtue of Eq. (5.3), defines a hypersphere of radius R. This induces the following line element:

$$\begin{aligned}ds^2&=d\xi^Ag_{\hat{A}B}d\xi^B\\ &=\varphi_2{}^2(z)(dx^\mu\eta_{\mu\nu}dx^\nu+ik\lambda^2d\theta^\alpha\eta_{\alpha\beta}d\theta^\beta)\,.\end{aligned} \tag{5.4b}$$

In Eqs. (5.4a) and (5.4b), the Bose values of A are now those of a.

Next consider the tranformations $U(\xi)$ given by

$$U(\xi)=e^{-(1/2)h_{AB}\omega^{BA}}\,.$$

Equation (4.6) tells us that

$$\xi'^A\simeq(\delta^A{}_B+\omega^{AC}g_{\hat{C}B})\xi^B\,. \tag{5.5}$$

Let us first consider the transformations with

$$\omega^{5\mu} = c^\mu / R\,,$$
$$\omega^{5\alpha} = \lambda\epsilon^\alpha / R\,, \qquad (5.6)$$
$$\omega^{\mu\alpha} = \frac{1}{\lambda k}{}^\alpha(\Gamma^\mu \epsilon)\,,$$

where as in Ref. 5, $\Gamma^\mu = \gamma^\mu M + \gamma^\mu\gamma_5 N$, with M and N internal-symmetry matrices possessing the symmetries, $M = M^T$ and $N = -N^T$, such that $(\eta\Gamma^\mu)^T = (\eta\Gamma^\mu)$. ϵ^α is a constant Majorana spinor. Then, Eqs. (5.2)–(5.6) can be used to show that for *large* R,

$$x'^\mu \simeq x^\mu + c^\mu - i(\bar\epsilon\Gamma^\mu\theta) + O(1/R)\,, \qquad (5.7a)$$

$$\theta'^\alpha \simeq \theta^\alpha + \epsilon^\alpha - \frac{1}{k\lambda^2} x^\mu \eta_{\mu\nu}{}^\alpha(\Gamma^\nu\epsilon) + O(1/R)\,. \qquad (5.7b)$$

Thus, for large R, the operator $\hat P_\mu = [(1/R)h_{5\mu}]$ generates translations in x^μ—a well-known result. However, the same extra Bose dimension also provides a Fermi generator, $[(1/R)h_{5\alpha}]$, which for large R generates translations in θ^α. Thus, as alluded to in Sec. II, we have indeed obtained the momentum operators $\hat P^A$ as large-R limits of a graded O(3,2). If we extend Eqs. (4.5) so as to include the index $\mu = 5$, one easily derives Eqs. (4.7) in the limit $R \to \infty$.

But where is supersymmetry? Equation (5.7a) is acceptable as is, Eq. (5.7b) is not. We need the further contraction $k\lambda^2 \to \infty$. Only then, are the supersymmetry transformations

$$x'^\mu \simeq x^\mu - i\bar\epsilon\Gamma^\mu\theta$$

and (5.8)

$$\theta'^\alpha \simeq \theta^\alpha + \epsilon^\alpha$$

retrieved.

More can be learned from the generators. Consider the spinor operator

$$V_\alpha \equiv \frac{1}{k\lambda} h_{\mu\beta}(\Gamma^\mu)^\beta{}_\alpha + \left(\frac{\lambda}{R}\right) h_{5\alpha}\,. \qquad (5.9)$$

Using Eqs. (4.5), one finds that

$$\{V_\alpha, V_\beta\} = 2i\left(\frac{h_{5\mu}}{R}\right)(\eta\Gamma^\mu)_{\alpha\beta} = 2i\hat P_\mu(\eta\Gamma^\mu)_{\alpha\beta}\,, \qquad (5.10)$$

only when the following limits are taken:

$$R \to \infty\,,$$
$$\lambda^2 k \to \infty\,, \qquad (5.11)$$

but

$$\frac{\lambda^2 k}{R^2} \to 0\,.$$

This is the set of contractions which have to be made to obtain supersymmetry.

This procedure has the added advantage of indicating how to obtain algebras *other* than those of supersymmetries. The crucial point is to notice that the fermion operators

$$l_{\mu\alpha} \equiv \frac{1}{k\lambda} h_{\mu\alpha}\,,$$
$$n_\alpha \equiv \frac{\lambda}{R} h_{5\alpha} \qquad (5.12)$$

have the following asymptotic (anti) commutation relations:

$$\{l_{\mu\alpha}, l_{\nu\beta}\} \doteq 0\,, \qquad (5.13a)$$

$$\{n_\alpha, n_\beta\} \doteq 0\,, \qquad (5.13b)$$

and

$$\{l_{\mu\alpha}, n_\beta\} \doteq -i\hat P_\mu \eta_{\alpha\beta}\,, \qquad (5.13c)$$

where $\doteq$ denotes that the particular contractions of Eq. (5.11) have been performed. This immediately suggests that spinor-vector operators $W_{\mu\alpha}$ can be constructed using $l_{\mu\alpha}$, n_β, and γ matrices, such that they anticommute asymptotically to the momentum operator $\hat P_\nu$ multiplied by some γ matrices. We show two examples of such spinor-vector algebras.

A. Consider

$$W_{\mu\alpha} \equiv l_{\mu\alpha} + n_\beta(\gamma_\mu)^\beta{}_\alpha\,. \qquad (5.14)$$

It is easily seen using Eqs. (5.13) that

$$\{W_{\mu\alpha}, W_{\nu\beta}\} \doteq (-i)[\hat P_\mu(\eta\gamma_\nu)_{\alpha\beta} + \hat P_\nu(\eta\gamma_\mu)_{\alpha\beta}]\,. \qquad (5.15)$$

It is also obvious that under the Lorentz group generators $h_{\mu\nu}$, $W_{\mu\alpha}$ behaves as a four-vector and under $h_{\alpha\beta}$, it behaves as a four-spinor. Also,

$$\{W_{\mu\alpha}, \hat P_\nu\} \doteq 0\,.$$

This example can be straightforwardly generalized to include internal-symmetry matrices.

B. As another example consider

$$S_{\mu\alpha} \equiv l_{\nu\beta}(\sigma^\nu{}_\mu)^\beta{}_\alpha + n_\beta(\gamma_\mu)^\beta{}_\alpha\,. \qquad (5.16)$$

For this case, we find

$$\{S_{\mu\alpha}, S_{\nu\beta}\} \doteq (-i)\hat P^\lambda[\eta_{\mu\lambda}(\eta\gamma_\nu)_{\alpha\beta} + \eta_{\nu\lambda}(\eta\gamma_\mu)_{\alpha\beta} - 2\eta_{\mu\nu}(\eta\gamma_\lambda)_{\alpha\beta} - 4\varepsilon_{\mu\nu\lambda\sigma}(\eta\gamma^\sigma\gamma_5)_{\alpha\beta}]\,. \qquad (5.17)$$

An algebra similar to the above has been presented in Ref. 12. It contains only the last term on the right-hand side of Eq. (5.17) but misses the first three. With an appropriate subtraction [see Eq. (5.15)], the first two terms on the right-hand side of Eq. (5.17) may be eliminated, but *not* the third term. Thus, that algebra does not follow, at least from our proposed contraction scheme.

From our point of view, the basic algebraic structure is given by Eqs. (5.13). If spin-$\frac{1}{2}$ constant spinor parameters ϵ'^{α}, are introduced via $\omega^{\mu\alpha}$ and $\omega^{5\alpha}$ then generalized supersymmetry follows.[15] On the other hand, if one entertains constant vector-spinor parameters $\epsilon^{\mu\alpha}$, then spin-$\frac{3}{2}$ algebras emerge.

VI. MASS AND SPIN OF FS MULTIPLETS

We now consider very briefly the physically interesting and important question of defining the "mass" operator on FS. It can be shown that the following operator, $\mathfrak{M}^2$, is a Casimir operator on FS:

$$\mathfrak{M}^2 \equiv -\frac{(-1)^c}{2R^2} h_{AB}(g^{-1})^{BC} h_{CD}(g^{-1})^{DA} . \tag{6.1}$$

For large R, it reduces to

$$\mathfrak{M}^2 \underset{R\text{ large}}{\simeq} -(\hat{P}_\mu \hat{P}^\mu) - \frac{1}{(\lambda^2 k)}[n_\alpha (\eta^{-1})^{\alpha\beta} n_\beta] - \left(\frac{\lambda^2 k}{R^2}\right)[l_{\mu\alpha}(\eta^{-1})^{\alpha\beta} l_{\nu\beta}\eta^{\mu\nu}] . \tag{6.2}$$

Equation (6.2) shows very clearly that if one performs the necessary limits given by Eq. (5.11) to obtain supersymmetry, both the second *and* third terms go to zero giving us the standard result

$$\mathfrak{M}^2 \Rightarrow -\hat{P}_\mu \hat{P}^\mu .$$

But, if $\lambda^2 k$ remains *finite* then only the third term is negligible and we have a left-over "spin contribution" to the mass operator. In general, therefore, FS multiplets should be labeled by the eigenvalues of the $\mathfrak{M}^2$ operator which does include the second term. This should be contrasted with the mass degeneracy present in supersymmetric multiplets, where mass is defined completely through the momentum operator $\hat{P}_\mu$. Presently, we are following this exciting possiblity to obtain allowed patterns of "mass shifts" in an FS multiplet.

The Pauli-Lubanski operator

$$W_\mu = \frac{1}{8R} \mathcal{E}_{\mu\nu\lambda\sigma\rho} M^{\nu\lambda} M^{\sigma\lambda}$$

(where the indices μ, etc., take on five values) allows us to form

$$I_2 = W_\mu W_\nu \eta^{\mu\nu} ,$$

which, however, is not an invariant on FS. This is, of course, the underlying reason for a given supermultiplet containing particles of different spin.[13]

VII. FORM INVARIANCE IN THE FERMI SECTOR

As discussed in Sec. II, the general formalism envisions expanding the connection (and/or the metric tensor) in a power series in $\theta^{\alpha m}$, whose coefficients are the dynamical fields. In examples with realistic internal symmetries, the resulting number of terms is very large. For instance, calculations of every Taylor coefficient of the Ricci tensor $R_{\hat{A}B}$, in terms of the fields introduced in the metric tensor $g_{\hat{A}B}$, becomes prohibitively tedious. In fact, no complete result for such cases exists so far.

In this section we show that a judicious use of invariance under *global* Fermi displacements *allows us to calculate all the coefficients of* θ *in* $R_{\hat{A}B}$ *provided the zeroth order in* θ *has been calculated in an arbitrary gauge.*

The method starts out from the following identity. Consider the θ expansion of a tensor $g_{\hat{A}B}(x,\theta)$:

$$g_{\hat{A}B}(x,\theta) = g_{\hat{A}B[0]}(x) + \sum_{n=1}^{N} \frac{1}{n!} g_{\hat{A}B[\alpha_1\cdots\alpha_n]}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_n} . \tag{7.1}$$

Clearly then, the following functional identity holds:

$$g_{\hat{A}B}(x,\theta) = \exp\left(\sum_{n=0}^{N-1} \frac{(-1)^n}{n!} g_{\hat{C}D[\alpha_1\cdots\alpha_n]}(x)\theta^\alpha \frac{\delta}{\delta g_{\hat{C}D[\alpha_1\cdots\alpha_n]}(x)}\right) g_{\hat{A}B[0]}(x) \exp\left(-\sum_{n=0}^{N-1}\cdots\right) , \tag{7.2}$$

where in Eq. (7.2) no symmetry is assumed for $g_{\hat{C}D}$ with respect to the indices $\hat{C}$ and D.

Now, consider a global Fermi displacement

$$\theta^\alpha \to \theta'^\alpha = \theta^\alpha + \epsilon^\alpha , \tag{7.3}$$

where ϵ^α is a constant spinor.

We now require form invariance of the tensor $g_{\hat{A}B}$ under the transformation (7.3). That is to say,

$$g_{\hat{A}B}(\{\Omega_n'(x)\}, \theta') = g_{\hat{A}B}(\{\Omega_n(x)\}, \theta) , \tag{7.4}$$

where $\Omega_n'(x)$ is the transformed field corresponding to Ω_n. So, the change

$$\delta g_{\hat{A}B[\alpha_1\alpha_2\ldots\alpha_n]}(x) \equiv J_{\hat{A}B[\alpha_1\ldots\alpha_n\alpha]}(x)\epsilon^\alpha$$

$$= (-1)^{n+1} g_{\hat{A}B[\alpha_1\ldots\alpha_n\alpha]}(x)\epsilon^\alpha . \tag{7.5}$$

Thus, for form-invariant tensors, we may substitute for Eq. (7.2)

$$g_{\hat{A}B}(x,\theta) = \exp\left(-\sum_{n=0}^{N-1}\frac{1}{n!}J_{\hat{C}D[\alpha_1\ldots\alpha_n\alpha]}(x)\theta^\alpha \frac{\delta}{\delta g_{\hat{C}D[\alpha_1\ldots\alpha_n]}(x)}\right) g_{\hat{A}B[0]}(x)\exp\left(\sum_{n=0}^{N-1}\cdots\right) . \tag{7.6}$$

Defining the operator

$$Q_\alpha \equiv \sum_{n=0}^{N-1}\frac{1}{n!}J_{\hat{C}D[\alpha_1\ldots\alpha_n\alpha]}(x)(-)^{c+d+n+1}\frac{\delta}{\delta g_{\hat{C}D[\alpha_1\ldots\alpha_n]}(x)} \tag{7.7}$$

we may equivalently write

$$g_{\hat{A}B}(x,\theta) = e^{Q_\alpha\theta^\alpha} g_{\hat{A}B[0]}(x) e^{-Q_\alpha\theta^\alpha} . \tag{7.8}$$

Now consider the connection $\Gamma_{\hat{B}}{}^A{}_C(x,\theta)$. In a metric theory, $\Gamma_{\hat{B}}{}^A{}_C(x,\theta)$ is a known functional of $g_{\hat{A}B}$, $\partial_\mu g_{\hat{A}B}$, and $(\partial/\partial\theta^\alpha)g_{\hat{A}B}$. Noting that for any field ψ

$$\partial_\mu\delta\psi = \delta(\partial_\mu\psi) ,$$

one easily derives that

$$\Gamma_{\hat{B}}{}^A{}_C(x,\theta) = e^{Q_\alpha\theta^\alpha}\Gamma_{\hat{B}}{}^A{}_{C[0]}(x)e^{-Q_\alpha\theta^\alpha} , \tag{7.9}$$

where $\Gamma_{\hat{B}}{}^A{}_{C[0]}(x)$ is the coefficient of the zeroth term in θ of $\Gamma_{\hat{B}}{}^A{}_C(x,\theta)$. The construction of $\Gamma_{\hat{B}}{}^A{}_{C[0]}(x)$ requires *only* $g^{AB}{}_{[0]}(x)$, $g_{\hat{A}B[0]}(x)$, and $g_{\hat{A}B[\alpha]}(x)$. However $g^{AB}{}_{[0]}(x)$ is completely determined from $g_{\hat{A}B[0]}(x)$ and thus a considerable amount of labor involved in inverting the full metric is avoided. Finally, Eq. (7.9) allows us to obtain the full $\Gamma_{\hat{B}}{}^A{}_C(x,\theta)$. Of course, we do not need to carry it out completely, because the desired quantity $R_{\hat{A}B[0]}$ only requires knowledge of $\Gamma_{\hat{B}}{}^A{}_C(x,\theta)$ to first order in θ. $R_{\hat{A}B}(x,\theta)$ can be obtained from $R_{\hat{A}B[0]}(x)$ through an equation analogous to Eq. (7.9). Clearly, similar procedures can be used to obtain the Lagrangian density $L(x,\theta)$ provided $L_{[0]}(x)$ is known. Anyone who has ever attempted to generate a superspace Lagrangian or all the equations of motion will appreciate the great saving of labor.

In the Appendix, we illustrate this technique via an example which is nonphysical but computationally involved.

The spontaneous breaking of a local SS to a global one implies the choice of a gauge and hence for such theories the above procedure is not directly applicable.

VIII. CONCLUDING REMARKS

In this paper, we have reviewed a theory of interacting fields,[4] formulated on superspace. We have shown how this theory contains Einstein's free gravitational equation of motion. We have also examined the relationship between the general superspace theory and its flat limit. This study shows that the "antisymmetric" part of $GGL(N_B, N_F, R)$ contains generalized supersymmetry upon certain group contractions. It also permitted us to construct the displacement operators for both the Bose and the Fermi coordinates via a graded de Sitter group. Form invariance under Fermi displacements was utilized in developing a procedure for calculating all the coefficients of θ of any tensor provided the zeroth order in θ is known in an arbitrary gauge.

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APPENDIX: AN ILLUSTRATION OF THE FERMI DISPLACEMENT METHOD

Here we present, in some detail, an example of the Fermi displacement method developed in Sec. VII. The model we consider has, $N_B=1$ and $N_F=2$, i.e., one Bose (x^0) and two Fermi dimensions (θ_m, $m=1,2$). On the Fermi (internal) space, we have $GL(2,R)$ invariance, for which we choose to write the generators G_a ($a=0,1,2,3$) as

$$G_0=\frac{i}{2} ,\quad G_1=\frac{i}{2}\tau_1,\quad G_2=\tfrac{1}{2}\tau_2,\quad G_3=\frac{i}{2}\tau_3 . \tag{A1}$$

The metric tensor $g_{\hat{A}B}(z)$ can be decomposed in terms of fields as

$$g_{00}(z) = \Lambda(x) + i\Psi(x)\theta + F(x)(\theta G_2\theta) , \tag{A2}$$

$$g_{0m}(z) = -g_{m0}(z)$$
$$= i\chi_m(x) + H_a(x)_m(G_a\theta) + iz_m(x)(\theta G_2\theta)\,, \quad \text{(A2b)}$$
$$g_{mn}(z) = -g_{nm}(z)$$
$$= (G_2)_{mn}\{2s(x) - 2K(x)(\theta G_2\theta) - 4[\omega(x)G_2\theta]\}. \quad \text{(A2c)}$$

Form invariance, under a global Fermi displacement,

$$x^{0\prime} = x^0\,, \quad \text{(A3)}$$
$$\theta' = \theta + \epsilon$$

(with ϵ a constant spinor) gives, through Eqs. (7.4) and (7.5), the following rules for variations of the fields:

$$\delta\Lambda(x) = -i\Lambda(x)(\psi_1(x)\epsilon)\,, \quad \text{(A4a)}$$
$$\delta\psi_1(x) = -2i[f(x) + \psi_1(x)G_2\psi_1(x)](G_2\epsilon)\,, \quad \text{(A4b)}$$
$$\delta f(x) = i(\psi_1(x)\epsilon)f(x)\,, \quad \text{(A4c)}$$
$$\delta\chi(x) = iH_a(x)(G_a\epsilon)\,, \quad \text{(A4d)}$$
$$\delta H_a(x) = -4i(z(x)G_aG_2\epsilon)\,, \quad \text{(A4e)}$$
$$\delta z(x) = 0\,, \quad \text{(A4f)}$$
$$\delta s(x) = 2[\epsilon G_2\omega(x)]\,, \quad \text{(A4g)}$$
$$\delta\omega(x) = -K(x)\epsilon\,, \quad \text{(A4h)}$$
$$\delta K(x) = 0\,. \quad \text{(A4i)}$$

In the above, we have redefined some fields:

$$\psi_1(x) = \frac{1}{\Lambda(x)}\psi(x) \quad \text{(A5a)}$$

and

$$f(x) = \frac{1}{\Lambda(x)}F(x)\,. \quad \text{(A5b)}$$

The connection $\Gamma_{\hat{B}}{}^{A}{}_{C}(z)$ is given by

$$\Gamma_{\hat{E}}{}^{D}{}_{C}(z) = (-1)^{cd}\tfrac{1}{2}[(-1)^{ac}g_{\hat{E}A,C}(z) + (-1)^{e+ae+ec+c}g_{\hat{C}A,E}(z) - g_{\hat{E}C,A}]g^{AD}(z)\,. \quad \text{(A6)}$$

Thus, to calculate "zeroth Γ", we need $g^{AD}{}_{[0]}(x)$. Let

$$g^{00}{}_{[0]}(x) = \hat{\Lambda}(x)\,, \quad \text{(A7a)}$$
$$g^{0m}{}_{[0]}(x) = \hat{\chi}_m(x)\,, \quad \text{(A7b)}$$
$$g^{mn}{}_{[0]}(x) = 2\hat{s}(x)(G_2)_{mn}\,. \quad \text{(A7c)}$$

The zeroth-order (in θ) version of the equation,

$$g^{AB}(z)g_{\hat{B}C}(z) = \delta^A{}_C\,,$$

allows us to solve for the fields with carets in terms of $g_{\hat{A}B[0]}(x)$:

$$\hat{\Lambda}(x) = \frac{1}{\Lambda(x)} + \frac{2[\chi(x)G_2\chi(x)]}{\Lambda^2(x)s(x)}\,, \quad \text{(A8a)}$$

$$\hat{\chi}(x) = \frac{2i}{\Lambda(x)s(x)}[G_2\chi(x)]\,, \quad \text{(A8b)}$$

$$\hat{s}(x) = \frac{1}{s(x)} - \frac{[\chi(x)G_2\chi(x)]}{s^2(x)\Lambda(x)}\,. \quad \text{(A8c)}$$

Now, Eqs. (A2), (A6), and (A8) give for $\Gamma_{\hat{B}}{}^{A}{}_{C[0]}(x)$

$$\Gamma_0{}^0{}_{0[0]}(x) = \tfrac{1}{2}\partial_0\ln\Lambda(x) - \frac{1}{s(x)}\partial_0\left[\frac{\chi(x)G_2\chi(x)}{\Lambda(x)}\right] + \frac{\psi_1(x)G_2\chi(x)}{s(x)}\,, \quad \text{(A9a)}$$

$$\Gamma_m{}^0{}_{0[0]}(x) = -\frac{i}{2}\psi_{1m}(x) - i\left[\frac{\chi(x)G_2\chi(x)}{\Lambda(x)s(x)}\right]\psi_{1m}(x) + \frac{i}{2\Lambda(x)}\left[\partial_0\ln s(x) + \frac{H_2(x)}{s(x)}\right]\chi_m(x) + \frac{2i}{\Lambda(x)s(x)}H_a(x)[G_a{}^TG_2\chi(x)]\,, \quad \text{(A9b)}$$

$$\Gamma_m{}^p{}_{0[0]}(x) = \frac{\psi_{1m}(x)_p(G_2\chi(x))}{s(x)} + \left[\tfrac{1}{2}[\partial_0\ln s(x)]\delta_{mp} + \frac{2H_{a'}(x)}{s}(G_{a'}G_2)_{mp}\right]\left[1 - \frac{\chi(x)G_2\chi(x)}{\Lambda(x)s(x)}\right]\,, \quad \text{(A9c)}$$

$$\Gamma_m{}^0{}_{n[0]}(x) = (G_2)_{mn}\left\{-\frac{s(x)}{\Lambda(x)}\left[\partial_0\ln s(x) + \frac{H_2(x)}{s(x)}\right]\left[1 + \frac{2}{\Lambda(x)s(x)}\chi(x)G_2\chi(x)\right] + \frac{2i}{\Lambda(x)s(x)}[\omega(x)\chi(x)]\right\}\,, \quad \text{(A9d)}$$

$$\Gamma_m{}^l{}_{n[0]}(x) = (G_2)_{mn}\left\{\frac{2i}{\Lambda(x)}{}_l[G_2\chi(x)]\left[\partial_0\ln s(x) + \frac{H_2(x)}{s(x)}\right] - \frac{2\omega_l(x)}{s(x)}\left[1 - \frac{\chi(x)G_2\chi(x)}{\Lambda(x)s(x)}\right]\right\}\,, \quad \text{(A9e)}$$

$$\Gamma_0{}^m{}_{0[0]}(x) = \frac{i\Lambda(x)}{s(x)}{}_m[G_2\psi_1(x)] - \frac{2i}{s(x)}(\partial_0 - \tfrac{1}{2}\partial_0\ln\Lambda)\,{}_m[G_2\chi(x)] - \frac{i\chi(x)G_2\chi(x)}{s^2(x)}\left[{}_m(G_2\psi_1) - \frac{2}{\Lambda(x)}(G_2\partial_0\chi(x))\right]. \quad \text{(A9f)}$$

[The notation $H_{a'}(x)G_{a'}$ implies a sum over all $a \neq 2$.]

To obtain $R_{\hat{A}B[0]}(x)$, we need all the $\Gamma_{\hat{B}}{}^{A}{}_{C}(z)$ up to first order (in θ), but $\Gamma_m{}^0{}_n(z)$ to zeroth order only. We apply the method developed in Sec. VII to boost Γ's to first order in θ through Eqs. (A4) and (A9). Thus

$$\Gamma_0{}^0{}_{0[m]}(x)=\frac{i}{2}\left[\partial_0\psi_1(x)-\frac{2H_a(x)}{s(x)}\psi_1(x)G_2G_a\right]_m+\frac{i}{s(x)}\partial_0\left[\frac{\chi(x)G_2\chi(x)}{\Lambda(x)}\psi_{1m}(x)\right]$$
$$+\frac{2i}{s(x)}\partial_0\left[\frac{H_a(x)}{\Lambda}\chi(x)G_2G_a\right]_m+\frac{i}{2s(x)}[f(x)+\psi_1(x)G_2\psi_1(x)]\chi_m(x)$$
$$+\frac{2}{s^2(x)}[\psi_1(x)G_2\chi(x)][\omega(x)G_2]_m-\frac{2}{s^2(x)}\partial_0\left[\frac{\chi(x)G_2\chi(x)}{\Lambda(x)}\right](\omega G_2)_m\,, \qquad \text{(A10a)}$$

$$\Gamma_m{}^0{}_{0[n]}(x)=\left[f(x)-\frac{1}{2s(x)}\mathcal{H}(x)+(\psi_1(x)G_2\psi_1(x))\right](G_2)_{mn}$$
$$+\frac{1}{2\Lambda(x)}\left[\partial_0\ln s(x)+\frac{H_2(x)}{s(x)}\right]H_a(x)(G_a)_{mn}-\frac{2i\psi_{1m}}{\Lambda(x)s^2(x)}[\chi(x)G_2\chi(x)](\omega G_2)_n$$
$$-\frac{2}{\Lambda(x)s(x)}\psi_{1m}(x)H_a(x)(\chi(x)G_2G_a)_n+\frac{2}{\Lambda(x)s(x)}[\chi(x)G_2\chi(x)][f(x)+2\psi_1(x)G_2\psi_1(x)](G_2)_{mn}$$
$$+\frac{i\chi_m(x)}{\Lambda(x)s(x)}\left[\partial_0\ln s+\frac{H_2(x)}{s(x)}\right](\omega G_2)_n+\frac{\chi_m}{2\Lambda}\left(\partial_0\ln s+\frac{H_2}{s}\right)\psi_{1n}-\frac{\chi_m(x)}{2\Lambda(x)s(x)}z_n(x)$$
$$-\frac{4}{\Lambda(x)s(x)}[z(x)G_2\chi(x)](G_2)_{mn}-\frac{i\chi_m(x)}{\Lambda(x)s(x)}(\partial_0\omega(x)G_2)_n$$
$$+\frac{4i}{\Lambda(x)s^2(x)}H_a(x)\,{}_m(G_a{}^TG_2\chi(x))(\omega(x)G_2)_n+\frac{2}{\Lambda(x)s(x)}H_a(x)\,{}_m[G_A{}^TG_2\chi(x)]\psi_{1n}(x)\,, \qquad \text{(A10b)}$$

$$\Gamma_m{}^p{}_{0[l]}(x)=-i\psi_{1m}(x)\left[\frac{H_a(x)}{s(x)}\right](G_2G_a)_{pl}+\frac{2\psi_{1m}(x)}{s^2(x)}\,{}_p[G_2\chi(x)](\omega(x)G_2)_l-\frac{2i}{s(x)}[f(x)+\psi_1(x)G_2\psi_1(x)]_p(G_2\chi)(G_2)_{ml}$$
$$+\left[1-\frac{\chi(x)G_2\chi(x)}{\Lambda(x)s(x)}\right]\left(-\frac{1}{s(x)}(\partial_0\omega(x)G_2)_l\delta_{mp}-\frac{2i}{s(x)}\sum_{a\neq 0}(G_a)_{mp}(zG_a)_l\right)$$
$$+\left[(\partial_0\ln s(x))\delta_{mp}+4\frac{H_{a'}}{s(x)}(G_{a'}G_2)_{mp}\right]\left[\frac{(\omega G_2)_l}{s(x)}+\frac{iH_a}{\Lambda s}(\chi G_2G_a)_l+\frac{1}{2\Lambda s}(\chi G_2\chi)\left(i\psi_{1l}-\frac{4}{s}(\omega G_2)_l\right)\right], \qquad \text{(A10c)}$$

$$\Gamma_m{}^l{}_{n[p]}(x)=(G_2)_{mn}\left\{\frac{2}{\Lambda}\,{}_l(G_2\chi)\left[\partial_0\ln s+\frac{H_2}{s}\right]\left[\psi_{1p}+\frac{2i}{s}(\omega G_2)_p\right]\right.$$
$$+\frac{2}{\Lambda}\left[\partial_0\ln s+\frac{H_2}{s}\right]H_a(G_2G_a)_{lp}-\frac{4i}{\Lambda s}\,{}_l(G_2\chi)\left[(\partial_0\omega G_2)_p-\frac{i}{2}z_p\right]-\frac{2K}{s}\delta_{lp}\left[1-\frac{\chi G_2\chi}{\Lambda s}\right]+\frac{2}{s^2}\delta_{lp}(\omega G_2\omega)$$
$$\left.-\frac{2i\omega_l}{s^2}\left[\frac{2H_a}{\Lambda}(\chi G_2G_a)_p+\frac{(\chi G_2\chi)}{\Lambda}\left(\psi_{1p}+\frac{4i}{s}(\omega G_2)_p\right)\right]\right\}, \qquad \text{(A10d)}$$

$$\Gamma_0{}^m{}_{0[n]}(x)=\frac{2i\Lambda}{s^2}\,{}_m(G_2\psi_1)(\omega G_2)_n-\frac{\Lambda f}{2s}\delta_{mn}-\frac{2}{s}(\partial_0-\tfrac{1}{2}\partial_0\ln\Lambda)H_a(G_2G_a)_{mn}$$
$$+\frac{4i}{s^2}(\omega G_2)_n[(\partial_0-\tfrac{1}{2}\partial_0\ln\Lambda)_m(G_2\chi)]-\frac{1}{s}\,{}_m(G_2\chi)(\partial_0\psi_{1n})$$
$$+\frac{4H_a}{\Lambda s^2}\,{}_m(G_2\partial_0\chi)(\chi G_2G_a)_n+\frac{2}{\Lambda s^2}(\chi G_2\chi)(\partial_0H_a)(G_2G_a)_{mn}$$
$$-\frac{2}{s^2}\,{}_m(G_2\psi_1)H_a(\chi G_2G_a)_n-\frac{4i}{s^3}\,{}_m(G_2\psi_1)(\chi G_2\chi)(\omega G_2)_n$$
$$+\frac{(\chi G_2\chi)}{2s^2}[f+(\psi_1G_2\psi_1)]\delta_{mn}+\frac{2}{\Lambda s^2}(\chi G_2\chi)\,{}_m(G_2\partial_0\chi)\left[\psi_{1n}+\frac{4i}{s}(\omega G_2)_n\right]. \qquad \text{(A10e)}$$

We have checked the validity of Eqs. (A10) by recomputing it through the standard method. Needless to say, our proposed method is much more convenient.

Given Eqs. (A9) and (A10) we are in a position to compute $R_{\hat{A}B[0]}(x)$. The answer is less awkward in terms of the tilded fields defined below.

Let

$$
\begin{aligned}
&\gamma_0{}^0{}_0=\tfrac{1}{2}(\partial_0 \ln\Lambda)\,,\\
&h_a=\frac{H_a}{s}\,,\\
&\mathcal{H}=\frac{1}{\Lambda}(H_0{}^2+H_2{}^2-H_1{}^2-H_3{}^2)\,,\\
&A_0=(\partial_0 \ln s+h_2)\,,\\
&\tilde f=\frac{1}{s}\left(f-\frac{\mathcal{H}}{2s}\right),\\
&\tilde\psi_1=\frac{\psi_1}{s^{1/2}},\\
&\tilde\chi=\frac{\chi}{s^{1/2}}\,,\\
&\tilde\omega=\frac{(G_2\omega)}{s^{3/2}}\,,\\
&\tilde\omega^\dagger=\frac{(\omega G_2)}{s^{3/2}}\,,\\
&\tilde\psi=\tilde\psi_1-\frac{4h_a}{\Lambda}(G_a^T G_2\tilde\chi)\,,\\
&\tilde K=\frac{K}{s^2}\,,\\
&\tilde z=\frac{z}{s^{3/2}}+4ih_a(G_aG_2\tilde\omega)\,.
\end{aligned}
\tag{A11}
$$

The tilded fields have been defined such that under the local (internal-symmetry) gauge transformation

$$
\begin{aligned}
&x^{0\prime}=x\,,\\
&\theta'=[e^{2\lambda_a(x)G_2G_a\theta}]\,,
\end{aligned}
\tag{A12}
$$

they transform simply:

$$
\tilde\psi'=e^{2\lambda_{a'}G_{a'}G_2}\tilde\psi\quad(a'\neq 2)\,. \tag{A13}
$$

$\tilde\chi$, $\tilde\omega$, and $\tilde z$ transform as $\tilde\psi$ does. Also,

$$
\begin{aligned}
&\Lambda'=\Lambda\,,\\
&A_0'=A_0\,,\\
&\tilde K'=\tilde K\,,\\
&h_a'=h_a+f_{ab'c}\lambda_{b'}h_c+\partial_0\lambda_a\quad(b'\neq 2)\,,\\
&s'=e^{-\lambda_2}s\,,
\end{aligned}
\tag{A14}
$$

where $f_{ab'c}=-4\,\mathrm{tr}[(G_aG_{b'}G_2G_c)-(G_aG_cG_2G_{b'})]$.

We also define some "covariant" derivatives:

$$
\begin{aligned}
&D_0A_0=(\partial_0-\gamma_0{}^0{}_0)A\,,\\
&D_0\tilde\omega+(\partial_0-2h_{a'}G_{a'}G_2)\tilde\omega\quad(a'\neq 2)\,,\\
&D_0\tilde\chi=(\partial_0-\gamma_0{}^0{}_0-2h_{a'}G_{a'}G_2)\tilde\chi\,.
\end{aligned}
\tag{A15}
$$

Armed with these definitions then, we can write $R_{\hat A B[0]}(x)$:

$$
\begin{aligned}
R_{00[0]}(x)={}&\frac{\Lambda}{2}(\tilde\psi G_2\tilde\psi)\left[1+\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right]+\Lambda\tilde f\left[1-\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right]+2A_0\left[\frac{1}{\Lambda}(\tilde\chi G_2D_0\tilde\chi)-(\tilde\chi G_2\tilde\psi)\right]+\frac{A_0{}^2}{2}\left[1+2\,\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right]\\
&+(D_0A_0)\left[1-\frac{(\tilde\chi G_2\tilde\chi)}{\Lambda}\right]-4i\Lambda(\tilde\psi G_2\tilde\omega)\left[1-\frac{2\tilde\chi G_2\tilde\chi}{\Lambda}\right]-4iA_0(\tilde\omega^\dagger G_2\tilde\chi)-8i(\tilde\omega^\dagger G_2D_0\tilde\chi)\left[1-2\,\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right],
\end{aligned}
\tag{A16a}
$$

$$
\begin{aligned}
R_{0m[0]}(x)={}&-\frac{4s^{1/2}}{\Lambda}(\tilde\chi G_2D_0\tilde\chi)\omega_m+s^{1/2}\{6\tilde\psi_m(\tilde\chi G_2\tilde\omega)-2(\tilde\psi G_2\tilde\omega)\tilde\chi_m-\frac{4s^{1/2}}{\Lambda}(\tilde\chi G_2\tilde\omega)_m(D_0\tilde\chi)+3s^{1/2}{}_m(D_0\tilde\omega)\left(1-\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)\\
&+\frac{3s^{1/2}}{2}A_0\tilde\omega_m\left(1+\tfrac{1}{3}\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)+\frac{3is^{1/2}}{\Lambda}\tilde z_m\left(1-\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)+\frac{is^{1/2}}{2\Lambda}A_0{}^2\tilde\chi_m-\frac{i}{2\Lambda}s^{1/2}(D_0A_0)\tilde\chi_m\\
&+\frac{is^{1/2}}{2\Lambda}A_{0\,m}(D_0\tilde\chi)\frac{(\tilde\chi G_2\tilde\chi)}{\Lambda}-\frac{3i}{4}s^{1/2}A_0\tilde\psi_m\left(1+\frac{5}{3}\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)-\frac{is^{1/2}}{4}(\tilde\psi G_2\tilde\psi)\tilde\chi_m-\frac{is^{1/2}}{2}\tilde f\,\tilde\chi_m\,,
\end{aligned}
\tag{A16b}
$$

$R_{mn[0]}(x)=s(G_2)_{mn}X$, where

$$
\begin{aligned}
X={}&4i(\tilde\psi G_2\tilde\omega)\left[1+\frac{4(\tilde\chi G_2\tilde\chi)}{\Lambda}\right]+\frac{6}{\Lambda}(\tilde\chi G_2\tilde z)+\frac{8i}{\Lambda}(\tilde\omega^\dagger G_2D_0\tilde\chi)\left(1+\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)-\frac{12i}{\Lambda}(\tilde\chi G_2D_0\tilde\omega)+\frac{14i}{\Lambda}A_0(\tilde\chi\,G_2\tilde\omega)\\
&-\tilde f\left(1+2\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)-\tfrac{1}{2}(\tilde\psi G_2\tilde\psi)\left(1+4\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)+48(\tilde\omega^\dagger G_2\tilde\omega)\left(1-2\frac{\tilde\chi\,G_2\tilde\chi}{\Lambda}\right)-6\tilde K\left(1-\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)\\
&-\frac{(D_0A_0)}{\Lambda}\left(1+2\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)-\frac{2A_0}{\Lambda^2}(\tilde\chi G_2D_0\tilde\chi)-\frac{A_0}{\Lambda}(\tilde\psi G_2\tilde\chi)+\frac{A_0{}^2}{\Lambda}\left(1+2\frac{\tilde\chi G_2\tilde\chi}{\Lambda}\right)\,.
\end{aligned}
\tag{A16c}
$$

Following the same procedure utilized to boost Γ's, we can construct $R_{\hat A B[m]}(x)$ and $R_{\hat A B[m,n]}(x)$, using Eqs. (A4) and (A16). Since we do not intend to pursue the resulting equations of motion, we do not bother to write them here.

We hope that the above discussion will convince the reader of the virtues of the proposed method of calculation.

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