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Vacuum tunneling and fluctuations around a most probable escape path

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We study vacuum tunneling in field theory directly in Minkowski space. We do this by extending the concept of "most probable escape path" (MPEP) first introduced by Banks, Bender, and Wu to the infinite-dimensional configuration space of fields and then constructing a wave functional, satisfying a Schrödinger-type equation, by a WKB expansion along this MPEP. The first-order results show that the tunneling process may indeed be described by one quantum variable tunneling in a one-dimensional potential barrier as proposed by us earlier for gauge theory. Corrections to this picture can now be calculated systematically. The first-higher-order corrections are shown to take the form of a "free energy" term that may be interpreted as a modification to the one-dimensional potential barrier obtained earlier.

I. INTRODUCTION

The existence of instanton solutions[1] appears to play an important role in the nature of the ground state of non-Abelian gauge theories. It is generally accepted that these Euclidean solutions are the realization of tunnelings between different vacuum states.[2] Thus the nature of these instanton solutions and their possible implications for hadron physics are at present among the important and exciting topics under consideration in particle physics.[3]

In a previous paper,[4] we studied the vacuum tunneling between two vacuums in a non-Abelian gauge theory from the Minkowski-space point of view. We consider the vacuum tunneling as the result of a continuous change of field configurations $A(x,t)=f(x,\lambda(t))$; the two vacuum states are described by $A^{(1)}=f(x,\lambda_1)$ and $A^{(2)}=f(x,\lambda_2)$, and as the parameter λ varies from λ_1 to λ_2, the system changes from one vacuum state to another. Assuming that the vacuum tunneling is described faithfully by this one-parameter family of field configurations, we can reduce the vacuum tunneling phenomena to a one-dimensional quantum-mechanical problem with λ as the dynamical variable. We then determine the optimal field configuration by choosing a function $f(x,\lambda)$ which maximizes the tunneling amplitude.

In this paper, we continue our analyses of the vacuum tunneling problem in Minkowski space. For notational simplicity, we restrict ourselves to a scalar field theory with the field variable $\phi(x)$. As we shall demonstrate in the last section of this paper, we can extend our results to non-Abelian gauge theories as well as to the inclusion of fermion fields.

We start with the full Hamiltonian $H(\phi,\dot{\phi})=H(\phi,(\hbar/i)\delta/\delta\phi)$, and write down a Schrödinger wave equation $H\psi(\phi)=E\psi(\phi)$ in the ϕ representation. We then apply the WKB approximation to the wave functional $\psi(\phi)$. By extending a method developed by Banks, Bender, and Wu,[5,6] we can study[7] $\psi(\phi)$ as a power series in $\hbar$. The lowest-order WKB approximation gives rise to the equation of the most probable escape path (MPEP). The equation which determines the MPEP is identical to the equation which determines the optimal field configuration as described in Ref. 4. The next-order WKB approximation includes the contribution due to the Gaussian fluctuations around the MPEP. We have obtained a set of equations which determines the effect of these fluctuations.

By a proper choice of the parameter $\lambda(t)$ which describes the field configurations along an MPEP, we can always express the equation of an MPEP in the same form as a Euclidean field equation. The parameter $\tau\equiv\lambda(t)$ now plays the role of a "Euclidean time". We wish to emphasize here that the parameter τ has nothing to do with the analytical continuation of t to the imaginary axis. In our approach, even though the equation of an MPEP has the same functional form as a Euclidean solution, it has a completely different physical interpretation.

The contribution associated with the Gaussian fluctuations around an MPEP also has an interesting physical interpretation. It has the same functional form as a partition function in statistical mechanics. Using an analog in statistical mechanics, we can introduce naturally a free energy $F(\lambda)$ along an MPEP. This free energy F gives rise to an additional contribution to the effective one-dimensional Hamiltonian $H(\lambda)$, and will modify the field equation associated with the MPEP. As we shall see in the text, the modified MPEP equation can be expressed as a Hartree field equation, and it has a simple mean-field interpretation.

The paper is organized as follows: In Sec. II, we formulate the vacuum tunneling problem as a Schrödinger equation with infinite degrees of freedom. In Sec. III, we apply the WKB approximation to this infinite system. We write down the equations which determine both the classical orbit and the MPEP. We show in Sec. IV how the Schrödinger equation near an MPEP reduces to the one-dimensional quantum equation as we have obtained in Ref. 4. In Secs. V and VI, we study the contribution due to the paths around a classical orbit or an MPEP. The Gaussian fluctuations provide a natural statistical-mechanical interpretation in the tunneling region. In the last section, we extend our results to non-Abelian gauge theories, and also discuss several related physics problems.

II. FORMULATION OF THE PROBLEM

A. Configuration space

We consider the infinite-dimensional space of fields $\phi(x)$ where x is one or more spatial continuous field variables. A point in this space is labeled by a parameter λ and has coordinates

$$\{\phi(x,\lambda)\}\equiv\{\phi(x_1,\lambda),\phi(x_2,\lambda),\ldots\}. \tag{2.1}$$

Let $\lambda=0$ designate the point whose coordinates are given by $\phi(x,0)$ and $\lambda=1$ designate the point with coordinates $\phi(x,1)$. If $\phi(x,\lambda)$ interpolates between these two points as λ varies continuously between $\lambda=0$ and $\lambda=1$ then $\phi(x,\lambda)$ $0\leq\lambda\leq 1$ defines a path in our space joining the two points in question. There are of course infinitely many such paths. Furthermore, each path may be parametrized by a variety of parameters λ.

The points $\lambda=0$ and $\lambda=1$ may describe two different states of a physical system if $\phi(x)$ is a quantum field. $\phi(x,\lambda)$ then corresponds to a specific quantum field fluctuation that contributes to the transition between these two different states. Indeed the functional integral formalism gives the amplitude for this transition as a certain sum over all such field fluctuations. Alternatively one may construct a wave functional in this space, which satisfies a Schrödinger-type equation, and whose magnitude at every point would give the relative probability of occurrence of the physical states these points correspond to. Thus the transition amplitude between the points $\lambda=0$ and $\lambda=1$ is proportional to the ratio of the magnitude of the wave functional at these points. In this section we lay the ground work for such an approach.

We define an increment of length dr along the path $\phi(x,\lambda)$ by

$$(dr)^2=\int dx\,[d\phi(x)]^2. \tag{2.2}$$

In other words we have

$$dr=\left[\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\right]^{1/2}d\lambda. \tag{2.3}$$

Thus if r, the length along the path, is used to parametrize this path we have $\lambda=r$ and

$$\int dx\left(\frac{\partial\phi}{\partial r}\right)^2=1. \tag{2.4}$$

We may then define a tangential vector along the path of length dr by

$$\delta\phi_{\tan}=dr\,\frac{\partial\phi}{\partial r}, \tag{2.5}$$

where $\partial\phi/\partial r$ is, from (2.4), a unit tangential vector. One may consequently define a vector orthogonal to the path $\phi(x,r)$ by

$$\delta\phi_\perp=\delta\phi-a\,dr\,\frac{\partial\phi}{\partial r}, \tag{2.6}$$

where $\delta\phi$ is an arbitrary variation and a is to be chosen such that

$$\int dx\,\delta\phi_\perp\left(\frac{\partial\phi}{\partial r}\right)=0. \tag{2.7}$$

This implies that

$$a\,dr=\int dx\,\delta\phi\left(\frac{\partial\phi}{\partial r}\right). \tag{2.8}$$

There are in general infinitely many such orthogonal vectors at every point $\phi(x,r)$ of the path. For every such direction $n(y)$ we may define a unit vector as

$$\hat{\phi}_{n(y)}(x,r)=\frac{\delta\phi_\perp}{[\int dx(\delta\phi_\perp)^2]^{1/2}} \tag{2.9}$$

where it is clear that $\hat{\phi}_{n(y)}(x,r)$ depends on r and the variable y denotes the coordinates of the direction $n(y)$. Let $n(y)$ also measure length along this orthogonal direction. Then we have in general

$$\delta\phi_\perp=\int dy\,\hat{\phi}_{n(y)}(x,r)\,n(y). \tag{2.10}$$

Consequently, if we consider a point $\phi(x)$ near the path defined by $\phi(x,r)$ then there is an r such that

$$\phi(x)=\phi(x,r)+\delta\phi_\perp(x,r), \tag{2.11}$$

where $\delta\phi_\perp(x,r)$ is given by the expansion of Eq. (2.10) above.

B. The Schrödinger equation

The Hamiltonian for a scalar field $\phi(x)$ is given by

$$H=\int dx\left[\tfrac{1}{2}\dot{\phi}^{2}+\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right], \qquad (2.12)$$

where $\dot{\phi}=\partial\phi/\partial t$. Since we have

$$[\dot{\phi}(x,t),\phi(x',t')]\,\delta(t-t')=i\hbar\,\delta^{4}(x-x'), \qquad (2.13)$$

we may in the configuration space of $\phi(x)$ represent ϕ by $(\hbar/i)\,\delta/\delta\phi$. Thus the Hamiltonian operator in this space reads

$$H=\int dx\left[-\frac{\hbar^{2}}{2}\left(\frac{\delta}{\delta\phi}\right)^{2}+\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]. \qquad (2.14)$$

The time-independent Schrödinger equation is then

$$H\psi(\phi)=E\,\psi(\phi), \qquad (2.15)$$

where $\psi(\phi)$ is an eigenfunctional of H corresponding to an eigenvalue E of the total energy of the system. Therefore we have

$$\left\{\int dx\left[-\frac{\hbar^{2}}{2}\left(\frac{\delta}{\delta\phi}\right)^{2}+\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]\right\}\psi(\phi)=E\,\psi(\phi). \qquad (2.16)$$

$\psi(\phi)$ has the usual probability interpretation, and hence its magnitude for various choices of $\phi(x)$ is a measure of the likelihood of their occurrence.

III. WKB APPROXIMATION

As a first step towards solving the eigenfunctional equation (2.16) above we consider a WKB-type solution and assume that $\psi(\phi)$ has the form

$$\psi(\phi)=A\,e^{(i/\hbar)S(\phi)}, \qquad (3.1)$$

where A is a constant. Equation (2.16) then reads

$$\left\{\int dx-\frac{\hbar^{2}}{2}\left[\frac{i}{\hbar}\,\frac{\delta^{2}S(\phi)}{\delta\phi^{2}}-\frac{1}{\hbar^{2}}\left(\frac{\delta S}{\delta\phi}\right)^{2}\right]+\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right\}e^{(i/\hbar)S(\phi)}=E\,e^{(i/\hbar)S(\phi)}. \qquad (3.2)$$

Proceeding as usual by expanding $S(\phi)$ in powers of $\hbar$ and then comparing equal powers of $\hbar$ we obtain

$$S(\phi)=S_{0}(\phi)+\hbar S_{1}(\phi)+\cdots, \qquad (3.3)$$

and the Schrödinger equation (3.2) leads to the following set of equations:

$$\int dx\left[\frac{1}{2}\left(\frac{\delta S_{0}}{\delta\phi}\right)^{2}+\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]=E, \qquad (3.4)$$

$$\int dx\left[-i\,\frac{\delta^{2}S_{0}}{\delta\phi^{2}}+2\,\frac{\delta S_{0}}{\delta\phi}\,\frac{\delta S_{1}}{\delta\phi}\right]=0, \qquad (3.5)$$

etc.

The solutions to Eqs. (3.4), (3.5), etc., are, however, nontrivial as the equations are highly nonlinear. It is, however, possible to reduce these equations, in first approximation, to one-dimensional equations. In doing so two cases must be considered. Define the potential

$$U(\phi)\equiv\int dx\left[\tfrac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]. \qquad (3.6)$$

Then there are two types of regions in configuration space. The first is where $E>U(\phi)$ which is the so-called classically allowed region, and the second is where $E<U(\phi)$ which is the so-called classically forbidden region and in which quantum-mechanical tunneling may occur. We are mainly interested in this region and we shall concentrate on it in the following.

A. Classically forbidden region: most probable escape path

If we define, in the classically forbidden region $E<U(\phi)$, R_0 and R_1 by

$$R_{0}=\frac{1}{i}S_{0}, \qquad (3.7)$$

$$R_{1}=\frac{1}{i}S_{1}, \qquad (3.8)$$

then the WKB equations (3.4) and (3.5) become

$$\int dx\,\frac{1}{2}\left(\frac{\delta R_{0}}{\delta\phi}\right)^{2}=U(\phi)-E, \qquad (3.9)$$

$$\int dx\left(\frac{\delta^{2}R_{0}}{\delta\phi^{2}}-2\,\frac{\delta R_{0}}{\delta\phi}\,\frac{\delta R_{1}}{\delta\phi}\right)=0. \qquad (3.10)$$

Now for an arbitrary path $\phi(x,\lambda)$ in this region one has, in general, nonvanishing values for $\delta R_0/\delta\phi$ for all variations $\delta\phi$, along the path and along all orthogonal directions to the path. We define the most probable escape path (MPEP) by $\phi_0(x,\lambda)$ such that $\delta R_0/\delta\phi$ is nonvanishing only along this path. This is a direct generalization of the concept of MPEP to our configuration space, as first introduced in two dimensions by Banks, Bender, and Wu.[4] In other words the MPEP satisfies

$$\left.\frac{\delta R_{0}}{\delta\phi_{\parallel}}\right|_{\phi_{0}(x,\lambda)}=C(\lambda)\,\frac{\delta\phi_{0}}{\delta\lambda},\quad\left.\frac{\delta R_{0}}{\delta\phi_{\perp}}\right|_{\phi_{0}(x,\lambda)}=0. \qquad (3.11)$$

For if such a path exists then the solutions to Eqs. (3.9) and (3.11) lead both to a determination of $R_0(\phi)$ along the MPEP and to an equation that determines the MPEP itself. We have along the MPEP,

$$\frac{\partial R_{0}}{\partial\lambda}=\int dx\left.\frac{\partial R_{0}}{\delta\phi_{\parallel}}\right|_{\phi_{0}(x,\lambda)}\frac{\delta\phi_{0}}{\delta\lambda}. \qquad (3.12)$$

When Eq. (3.12) is combined with Eq. (3.11) we are led to

$$C(\lambda)=\frac{\partial R_0}{\partial\lambda}\left[\int dx\left(\frac{\partial\phi_0}{\partial\lambda}\right)^2\right]^{-1}, \tag{3.13}$$

and hence the WKB equation (3.9) becomes

$$\frac{1}{2}\left[\int dx\left(\frac{\partial\phi_0}{\partial\lambda}\right)^2\right]^{-1}\left(\frac{\partial R_0}{\partial\lambda}\right)^2=U(\phi_0(x,\lambda))-E. \tag{3.14}$$

If we now use the length r along the MPEP for the parameter λ, this equation reduces further to

$$\frac{1}{2}\left(\frac{\partial R_0}{\partial r}\right)^2=U(r)-E, \tag{3.15}$$

where the potential $U(\phi_0(x,r))$ becomes, along the MPEP,

$$U(r)=\int dx\{\tfrac{1}{2}[\nabla\phi_0(x,r)]^2+V(\phi_0(x,r))\}. \tag{3.16}$$

Equation (3.15) is clearly soluble and one has then, along the MPEP,

$$R_0=\int_0^r dr'\{2[U(r')-E]\}^{1/2}. \tag{3.17}$$

Furthermore, Eq. (3.10) becomes, along the MPEP,

$$\frac{\partial^2 R_0}{\partial r^2}-2\frac{\partial R_0}{\partial r}\frac{\partial R_1}{\partial r}=0, \tag{3.18}$$

which with Eq. (3.17) gives

$$R_1=\tfrac{1}{2}\ln\{2[U(r)-E]\}^{1/2}, \tag{3.19}$$

which is of course the magnitude of R_1 along the MPEP. Thus if the MPEP is known then Eqs. (3.17) and (3.19) determine the WKB wave functional (to first order) at any point $\phi_0(x,r)$ along this path. We have

$$\psi(\phi_0(x,r))=A\frac{1}{\{2[U(r)-E]\}^{1/4}}\times\exp\left(-\frac{1}{\hbar}\int_0^r dr'\{2[U(r')-E]\}^{1/2}+\cdots\right). \tag{3.20}$$

Furthermore, Eq. (3.17) combined with Eq. (3.11) leads to an equation whose solution is the MPEP. To see how this comes about note that R_0 is such that

$$\delta R_0=\delta\int_0^r dr'\{2[U(r')-E]\}^{1/2}=0 \tag{3.21}$$

for arbitrary variations $\delta\phi_\perp$ about $\phi_0(x,r)$. The MPEP is the solution to the Euler-Lagrange equation that follows from (3.21). In order to derive this Euler-Lagrange equation we transform back to the arbitrary parameter λ which, in contrast to r, is not affected by the variations $\delta\phi$ to be applied to R_0. Thus we have

$$\frac{\delta}{\delta\phi}\left\{\int_{\lambda_1}^{\lambda_2}d\lambda'\left[\int dx\left(\frac{\partial\phi(x,\lambda')}{\partial\lambda'}\right)^2\right]^{1/2}\times\{2[U(\lambda')-E]\}^{1/2}\right\}_{\phi_0(x,\lambda)}=0, \tag{3.22}$$

where from Eq. (3.5) we have

$$U(\lambda')=\int dx\{\tfrac{1}{2}[\nabla\phi(x,\lambda')]^2+V(\phi(x,\lambda'))\}. \tag{3.23}$$

The Euler-Lagrange equation following from Eq. (3.22) is

$$\frac{\partial}{\partial\lambda}\left(\frac{\delta}{\delta(\partial\phi/\partial\lambda)}\left\{2\left[\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\right][U(\lambda)-E]\right\}^{1/2}\right)-\frac{\partial}{\partial\phi}\left\{2\left[\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\right][U(\lambda)-E]\right\}^{1/2}\Bigg|_{\phi=\phi_0(x,\lambda)}=0, \tag{3.24}$$

which leads to

$$\frac{\partial}{\partial\lambda}\left\{\frac{\partial\phi}{\partial\lambda}\frac{\{2[U(\lambda)-E]\}^{1/2}}{[\int dx(\partial\phi/\partial\lambda)^2]^{1/2}}\right\}_{\phi=\phi_0}=\left(-\nabla^2\phi+\frac{\partial V}{\partial\phi}\right)\left[\int dx(\partial\phi/\partial x)^2\right]^{1/2}\{2[U(\lambda)-E]\}^{-1}\Bigg|_{\phi=\phi_0}. \tag{3.25}$$

Note that in Eq. (3.25)

$$\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2=\left(\frac{dr}{d\lambda}\right)^2.$$

If we choose the parameter λ such that

$$\frac{dr}{d\lambda}=\{2[U(\lambda)-E]\}^{1/2}, \tag{3.26}$$

then the Euler-Lagrange equation (3.25) for the MPEP in the "tunneling region" becomes

$$\frac{\partial^2}{\partial\lambda^2}\phi_0(x,\lambda)+\nabla^2\phi_0(x,\lambda)-\frac{\partial V}{\partial\phi}\bigg|_{\phi_0}=0. \tag{3.27}$$

Note that $\lambda(r)$ as determined by Eq. (3.26) is real.

We remark now that Eq. (3.27) is identical in form to the Euclidean equation of motion for the field $\phi(x)$ with λ replacing Euclidean time. We emphasize here, however, that λ is *not* Euclidean time nor an analytic continuation thereof; for it must be recalled that our method involves constructing a (true) time-independent Schrödinger wave functional in ordinary Minkowski space. The similarity of Eq. (3.27), nevertheless, to the Euclidean equations of motion allows us to construct the MPEP in the tunneling region if real solutions to the Euclidean equations of motion are known. For if we formally replace Euclidean time in these solutions by λ we have a real solution to equation (3.27) which is the MPEP.

Before we turn our attention to the classically allowed region let us consider the following procedure for an arbitrary path $\phi(x,\lambda)$ that is not an MPEP.

Construct on $\phi(x,\lambda)$ the potential $U(\lambda)$ as given by Eq. (3.5). Now clearly $(\delta R_0/\delta\phi)^2$ along this path is not simply $[\int dx(\partial\phi/\partial\lambda)^2](\partial R_0/\partial\lambda)^2$ as on an MPEP but is in general

$$\left(\frac{\partial R_0}{\delta\phi}\right)^2=\left[\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\right]\left(\frac{\partial R_0}{\partial\lambda}\right)^2 + \text{positive-definite terms}\,. \tag{3.28}$$

Thus for this arbitrary path

$$\left[\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\right]\left(\frac{\partial R_0}{\partial\lambda}\right)^2\leq\{2[U(\lambda)-E]\}^{1/2}. \tag{3.29}$$

Now for $\lambda=\tau$, the length along this path, we have

$$\frac{\partial R_0}{\partial\tau}\leq\{2[U(\tau)-E]\}^{1/2}, \tag{3.30}$$

and hence if we evaluate

$$R_0'=\int_0^\tau d\tau'\{2[U(\tau')-E]\}^{1/2}, \tag{3.31}$$

we would be evaluating a quantity R_0' larger than the true R_0 along this path. Clearly the MPEP is that path for which R_0' as given in Eq. (3.31) is the true R_0 and hence is the path which minimizes the integral of (3.31). Thus if we can parametrize a family of paths $\phi(x,\tau)$ by a parameter ζ then evaluate $R_0'(\zeta)$ as given in Eq. (3.31) for a member of this family we may obtain the MPEP, if it is within this family, by finding ζ for which

$$\frac{\partial R_0'(\zeta)}{\partial\zeta}=0\,. \tag{3.32}$$

We mention this procedure here as it is equivalent to our discussion above and in particular cases may be a fast and direct way of finding the MPEP and the magnitude of R_0 along this MPEP.

B. The classically allowed region

In the classically allowed region $E>U(\phi)$ the WKB equations (3.4) and (3.5) may be transformed into one-dimensional equations if one defines a "classical orbit" by $\phi_c(x,\eta)$ such that

$$\left.\frac{\delta S_0}{\delta\phi_\parallel}\right|_{\phi_c(x,\eta)}=\frac{\partial S_0}{\partial\eta}\,\frac{\delta\phi_c}{\delta\eta},\quad\left.\frac{\delta S_0}{\delta\phi_\perp}\right|_{\phi_c(x,\eta)}=0\,. \tag{3.33}$$

For then one obtains along this "classical orbit" $\phi_c(x,\eta)$ from Eq. (3.4),

$$\frac{1}{2}\left[\int dx\left(\frac{\partial\phi_c}{\partial\eta}\right)^2\right]\left(\frac{\partial S_0}{\partial\eta}\right)^2=E-U(\eta), \tag{3.34}$$

which when η is replaced by the length s along this classical orbit leads to

$$S_0=\int_0^s ds'\{2[E-U(s')]\}^{1/2}. \tag{3.35}$$

Furthermore, along this classical orbit Eq. (3.5) becomes

$$-i\frac{\partial^2 S_0}{\partial s^2}+2\frac{\partial S_0}{\partial s}\frac{\partial S_1}{\partial s}=0, \tag{3.36}$$

leading to

$$S_1=\tfrac{1}{2}i\ln\{2[E-U(s)]\}^{1/2}. \tag{3.37}$$

Thus the wave functional along the classical orbit in this region is as expected,

$$\psi(\phi_c(x,s))=\frac{A}{\{2[E-U(s)]\}^{1/4}}\times\exp\left(\frac{i}{\hbar}\int_0^s ds'\{2[E-U(s')]\}^{1/2}+\cdots\right). \tag{3.38}$$

The classical orbit itself $\phi_c(x,s)$ is determined by the stationary phase condition

$$\delta S_0=\delta\int_0^s ds'\{2[E-U(s')]\}^{1/2}=0\,. \tag{3.39}$$

When Eq. (3.39) is expressed in terms of the parameter η, one obtains the Euler-Lagrange equation for the classical orbit as

$$\frac{\partial^2\phi_c(x,\eta)}{\partial\eta^2}-\nabla^2\phi_c(x,\eta)+\left.\frac{\partial V}{\partial\phi}\right|_{\phi_c(x,\eta)}=0, \tag{3.40}$$

where η is defined by

$$\frac{ds}{d\eta}=\{2[E-U(\eta)]\}^{1/2}. \tag{3.41}$$

As may be seen from Eq. (3.40) the equation determining the "classical orbit" $\phi_c(x,\eta)$ is similar in form to the equation of motion of the field $\phi(x)$ in Minkowski space. Thus one may obtain the classical orbit from a real solution to the Minkowski field equation by the *formal* replacement of the time variable by the parameter η.

We wish to turn our attention now to a better evaluation of the wave functional that takes into account the influence of points in the neighborhood of the MPEP in the classically forbidden regions of our configuration space. However, we digress a little to discuss in the following section the procedure, introduced by us in Ref. 4, of reducing the problem discussed above to a one-dimensional problem, and showing its equivalence to the above treatment.

IV. REDUCTION OF THE SCHRÖDINGER EQUATION NEAR AN MPEP

In Ref. 4, we obtain the MPEP by first reducing the field theory problem to an effective one-dimensional quantum-mechanical problem along a particular path,

$$\phi(x,t)=\phi(x,\lambda(t))\,. \tag{4.1}$$

The MPEP $\phi=\phi_0(x,\lambda(t))$ is the path which gives rise to the fastest decay rate. To obtain the effective one-dimensional quantum-mechanical system, we substitute (4.1) into the Hamiltonian H given in (2.12), and have

$$\begin{aligned} H_\lambda &= \int dx\,\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \dot{\lambda}^2+U(\lambda) \\ &= \tfrac{1}{2}m(\lambda)\dot{\lambda}^2+U(\lambda) \\ &= \frac{p_\lambda{}^2}{2m(\lambda)}+U(\lambda)\,, \end{aligned} \tag{4.2}$$

where

$$U(\lambda)\equiv\int dx\left[\tfrac{1}{2}(\nabla\phi)^2+V(\phi)\right], \tag{4.3}$$

$$m(\lambda)\equiv\int dx\,\tfrac{1}{2}\left(\frac{\partial\phi}{\partial\lambda}\right)^2>0\,, \tag{4.4}$$

and

$$p_\lambda=m(\lambda)\dot{\lambda}\,. \tag{4.5}$$

We now wish to interpret H_λ as a quantum-mechanical operator. Since p_λ and λ do not commute, there is no unique way of writing down the quantum-mechanical H_λ. We find that the most natural way of introducing the quantum Hamiltonian is to reparametrize $\lambda(t)$ and to obtain a Hamiltonian with a constant mass. Indeed, if we define an $r(t)$ through $r=r(\lambda(t))$ with

$$\frac{dr}{d\lambda}=[m(\lambda)]^{1/2}\,, \tag{4.6}$$

we obtain

$$H_r=\frac{p_r{}^2}{2}+U(r)\,, \tag{4.7}$$

where

$$U(r)=U(\lambda)\,, \tag{4.8}$$

$$p_r=\dot{r}=\frac{1}{[m(\lambda)]^{1/2}}\,p_\lambda\,. \tag{4.9}$$

The parametrization $r(t)$ is the same as r appearing in Sec. II. Using this familiar Hamiltonian (4.7), we obtain the quantum-mechanical Hamiltonian operator

$$H_r=-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial r}\right)^2+U(r)\,. \tag{4.10}$$

The Schrödinger equation associated with (4.10) is

$$\left[-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial r}\right)^2+U(r)\right]\psi(r)=E\,\psi(r)\,. \tag{4.11}$$

The WKB wave function in the tunneling region is

$$\psi(r)=\frac{1}{\{2[U(r)-E]\}^{1/4}}\times\exp\left(-\frac{1}{\hbar}\int_0^r dr'\{2[U(r')-E]\}^{1/2}\right), \tag{4.12}$$

which agrees with (3.20). Thus the field configuration (path) which gives rise to the maximal tunneling rate is the MPEP as discussed in Sec. III.

To obtain the corresponding quantum-mechanical Hamiltonian and wave function for an arbitrary parametrization $\lambda(t)$, we make a coordinate transformation and have

$$\psi'(\lambda)=[m(\lambda)]^{1/4}\psi(r(\lambda))\,, \tag{4.13}$$

and

$$\begin{aligned} H_\lambda &= [m(\lambda)]^{1/4}H_r(m(\lambda))^{-1/4} \\ &= [m(\lambda)]^{1/4}\left[-\frac{\hbar^2}{2}\left(\frac{1}{[m(\lambda)]^{1/2}}\frac{\partial}{\partial\lambda}\right)^2+U(\lambda)\right][m(\lambda)]^{-1/4} \\ &= -\frac{\hbar^2}{2}\,m(\lambda)^{-1/4}\frac{\partial}{\partial\lambda}\frac{1}{[m(\lambda)]^{1/2}}\frac{\partial}{\partial\lambda}[m(\lambda)]^{-1/4}+U(\lambda)\,. \end{aligned} \tag{4.14}$$

In the WKB approximation, the wave function in an arbitrary parametrization $\lambda(t)$ becomes

$$\psi'=(\lambda)\frac{1}{\{2m(\lambda)[U(\lambda)-E]\}^{1/4}}\times\exp\left(-\frac{1}{\hbar}\int_0^\lambda d\lambda'\{2m(\lambda')[U(\lambda')-E]\}^{1/2}\right), \tag{4.15}$$

as expected.

In Sec. II, we have written down the full quantum-mechanical Hamiltonian (2.14) and the Schrödinger equation (2.16) as

$$H=\int dx\left[-\frac{\hbar^2}{2}\left(\frac{\delta}{\delta\phi}\right)^2+\tfrac{1}{2}(\nabla\phi)^2+V(\phi)\right] \tag{4.16}$$

and

$$H\,\psi(\phi)=E\,\psi(\phi)\,. \tag{4.17}$$

It would be far more desirable if we could obtain the effective one-dimensional Hamiltonian H_λ and the corresponding Schrödinger equation near the MPEP directly from (4.16) and (4.17) without going through the detailed WKB expansion. We shall achieve this in the remaining part of this section. First, we summarize the MPEP formulation as follows:

(1) The wave functional $\psi(\phi)$ is dominated by a family of field configurations $\phi_0(x,\lambda)$ which varies continuously from one local minimum to another as λ varies from some boundary values λ_1 to λ_2.

(2) Near $\phi=\phi_0$, the wave functional $\psi(\phi)$ varies smoothly along the tangential direction

$$\delta\phi_{\tan}=\frac{\partial\phi_0}{\partial\lambda}\,\delta\lambda\,,$$

and damps rapidly to zero in the perpendicular direction. Hence, it is reasonable to assume that, acting on the wave functional $\psi(\phi)$ near $\phi=\phi_0$, we have approximately

$$\frac{\delta}{\delta\phi(x)}=\frac{1}{m(\lambda)}\,\frac{\partial\phi_0(x,\lambda)}{\partial\lambda}\,\frac{\partial}{\partial\lambda}, \tag{4.18}$$

where $m(\lambda)$ is a proportionality constant which will be identified as the mass given in (4.4). To verify that $m(\lambda)$ is indeed the mass, we note that near $\phi=\phi_0$,

$$\frac{\partial}{\partial\lambda}=\int dx\,\frac{\partial\phi_0}{\partial\lambda}\,\frac{\delta}{\partial\phi(x)}\,. \tag{4.19}$$

Substituting (4.18) into the right-hand side of (4.19), and identifying the coefficients of $\partial/\partial\lambda$, we obtain

$$1=\frac{1}{m(\lambda)}\int dx\left(\frac{\partial\phi_0}{\partial\lambda}\right)^2,$$

or

$$m(\lambda)=\int dx\left(\frac{\partial\phi_0(x,\lambda)}{\partial\lambda}\right)^2 \tag{4.20}$$

as promised. Now, we consider the Schrödinger equation (4.17) in the neighborhood of $\phi=\phi_0(x,\lambda)$. We denote, to within a normalization factor $A(\lambda)$ to be determined later,

$$\psi(\phi)\big|_{\phi=\phi_0}=A(\lambda)\,\psi(\lambda)\,, \tag{4.21a}$$

or

$$\psi(\lambda)=A(\lambda)^{-1}\psi(\phi_0)\,. \tag{4.21b}$$

According to our approximation, we have

$$\begin{aligned}\int dx\left[-\frac{\hbar^2}{2}\left(\frac{\partial}{\delta\phi}\right)^2\psi(\phi)\right]_{\phi=\phi_0}&=-\frac{\hbar^2}{2}\int dx\,\frac{1}{m(\lambda)}\left(\frac{\partial\phi_0}{\partial\lambda}\right)\frac{\partial}{\partial\lambda}\left(\frac{1}{m(\lambda)}\,\frac{\partial\phi_0}{\partial\lambda}\,\frac{\partial}{\partial\lambda}A\,\psi(\lambda)\right)\\&=-\frac{\hbar^2}{2}\left[\int dx\left(\frac{\partial\phi_0}{\partial\lambda}\right)^2\frac{1}{m(\lambda)}\,\frac{\partial}{\partial\lambda}\,\frac{1}{m(\lambda)}\,\frac{\partial}{\partial\lambda}+\int dx\,\frac{\partial\phi_0}{\partial\lambda}\,\frac{\partial^2\phi_0}{\partial\lambda^2}\,\frac{1}{m(\lambda)^2}\,\frac{\partial}{\partial\lambda}\right]A(\lambda)\psi(\lambda)\\&=-\frac{\hbar^2}{2}\left[\frac{\partial}{\partial\lambda}\,\frac{1}{m(\lambda)}\,\frac{\partial}{\partial\lambda}+\frac{1}{2}\,\frac{1}{m(\lambda)^2}\,\frac{\partial m(\lambda)}{\partial\lambda}\,\frac{\partial}{\partial\lambda}\right]A(\lambda)\,\psi(\lambda)\\&=-\frac{\hbar^2}{2}\left[\frac{1}{[m(\lambda)]^{1/2}}\,\frac{\partial}{\partial\lambda}\,\frac{1}{[m(\lambda)]^{1/2}}\,\frac{\partial}{\partial\lambda}\right]A(\lambda)\,\psi(\lambda)\,,\end{aligned} \tag{4.22}$$

and

$$\int dx\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+V(\phi)\right]\psi(\phi)\,\bigg|_{\phi=\phi_0}=U(\lambda)A(\lambda)\,\psi(\lambda)\,, \tag{4.23}$$

where

$$U(\lambda)=\int dx\left[\frac{1}{2}\left(\frac{\partial\phi_0}{\partial x}\right)^2+V(\phi_0)\right]. \tag{4.24}$$

Thus the Schrödinger equation (4.17) leads, in the neighborhood of $\phi=\phi_0$, to

$$H_\lambda\,\psi=\left\{A^{-1}(r)\left[-\frac{\hbar^2}{2}\left(\frac{1}{[m(\lambda)]^{1/2}}\,\frac{\partial}{\partial\lambda}\,\frac{1}{[m(\lambda)]^{1/2}}\,\frac{\partial}{\partial\lambda}\right)+U(\lambda)\right]A(r)\right\}\psi(r)=E\,\psi(r)\,. \tag{4.25}$$

Choosing

$$A(r)=m(\lambda)^{-1/4}, \tag{4.26}$$

we reproduce the Hamiltonian H_λ as given in (4.14).

V. CONTRIBUTION FROM POINTS NEAR THE MPEP TO THE WAVE FUNCTIONAL

Consider a point $\phi(x)$ near the MPEP defined by $\phi_0(x,r)$. Then there is an r such that

$$\phi(x)=\phi_0(x,r)+\int dy\, n(y)\,\hat{\phi}_{n(y)}(x,r)\,. \tag{5.1}$$

Our aim now is to solve the WKB equation (3.4) for S_0 when it is not strictly one-dimensional as on the MPEP but nearly so. Thus we must first express the operation

$$\int dx\left(\frac{\delta R_0}{\delta\phi}\right)^2$$

for points on and near the MPEP. For this purpose we consider a local orthogonal coordinate system[6] at every point r of the MPEP $\phi_0(x,r)$. The unit vectors of this local orthogonal system are $\partial\phi_0(x,r)/\partial r$ along the MPEP and $\hat{\phi}_{n(y)}(x,r)$ orthogonal to it. Our first step is to find the metric tensor in this system. For this purpose consider an arbitrary variation of $\phi(x)$ with r at constant $n(y)$. Then

$$\delta\phi\,\big|_n=\frac{\partial\phi_0(x,r)}{\partial r}\,\delta r+\int dy\, n(y)\,\frac{\partial\hat{\phi}_{n(y)}(x,r)}{\partial r}\,\delta r\,. \tag{5.2}$$

It must be clear, however, that as long as we consider variations at constant n the point $\phi(x)$ is moving parallel to the MPEP $\hat{\phi}_0(x,r)$ and hence all variations in the unit vectors $\hat{\phi}_{n(y)}(x,r)$ are zero except for $n(y)=n_1$ which is the unit vector lying in the plane of the MPEP. Thus Eq. (4.2) reduces to

$$\frac{\partial\phi(x)}{\partial r}=\frac{\partial\phi_0(x,r)}{\partial r}+\frac{\partial\hat{\phi}_{n_1}(x,r)}{\partial r}\,n_1\,, \tag{5.3}$$

where $\hat{\phi}_{n_1}(x,r)$ is the unit vector orthogonal to $\partial\phi_0(x,r)/\partial r$ in the plane of the MPEP.

If we define the curvature of the MPEP in its plane by

$$\frac{\partial\hat{\phi}_{n_1}(x,r)}{\partial r}=\rho\,\frac{\partial\phi_0(x,r)}{\partial r}\,, \tag{5.4}$$

then we have, from Eq. (5.3),

$$\int dx\left(\frac{\partial\phi(x)}{\partial r}\right)^2=(1+n_1\rho)^2\,. \tag{5.5}$$

For variations of $\phi(x)$ at constant r and along any unit vector $\hat{\phi}_{n(y)}(x,r)$ we simply have

$$\delta\phi(x)\,\big|_r=\hat{\phi}_{n(y)}(x,r)\,\delta n(y)\,. \tag{5.6}$$

Thus

$$\int dx\left(\frac{\partial\phi(x)}{\partial n(y)}\right)^2=1\,. \tag{5.7}$$

Equations (5.5) and (5.7), along with the orthogonal nature of the local system under consideration, allow us to determine the metric tensor in this new system. For by definition

$$\begin{bmatrix} g_{rr} & g_{rn_1} & g_{rn_2} & \cdots \\ g_{n_1r} & g_{n_1n_1} & g_{n_1n_2} & \cdots \\ g_{n_2r} & g_{n_2n_1} & g_{n_2n_2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}=\begin{bmatrix} \int dx\left(\frac{\partial\phi}{\partial r}\right)^2 & \int\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n_1}dx & \int\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n_2}dx & \cdots \\ \int dx\,\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n_1} & \int\left(\frac{\partial\phi}{\partial n_1}\right)^2dx & \int\frac{\partial\phi}{\partial n_1}\frac{\partial\phi}{\partial n_2}dx & \cdots \\ \int dx\,\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n_2} & \int\frac{\partial\phi}{\partial n_2}\frac{\partial\phi}{\partial n_1}dx & \int\left(\frac{\partial\phi}{\partial n_2}\right)^2dx & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{5.8}$$

which leads to

$$\begin{aligned} g_{rr}&=(1+n_1\rho)^2\,,\\ g_{nn'}&=\delta_{nn'}\,,\\ g_{rn}&=0\,. \end{aligned} \tag{5.9}$$

The reciprocal metric tensor is then

$$\begin{aligned} g^{rr}&=(1+n_1\rho)^{-2}\,,\\ g^{nn'}&=\delta^{nn'}\,,\\ g^{rn}&=0\,. \end{aligned} \tag{5.10}$$

Using this metric we can effect a change of variable from configuration space $\phi(x)$ to this locally orthogonal system of coordinates along the MPEP. We get then

$$\int dx \left(\frac{\delta}{\delta\phi}\right)^2 - g^{rr}\left(\frac{\partial}{\partial r}\right)^2 + \int dy \left(\frac{\partial}{\partial n(y)}\right)^2, \qquad (5.11)$$

which is what we need in the WKB equation (3.4) above.

For points $\phi(x)$ near the MPEP which are given by Eq. (5.1) we can then express all relevant quantities in this local orthogonal system by expanding around the MPEP. We obtain for $R_0(\phi(x))$,

$$R_0(\phi) = R_0(\phi_0(x,r)) + \int dx_1 \frac{\delta R_0}{\delta\phi_\perp(x_1)}\bigg|_{\phi_0(x,r)} \delta\phi_\perp(x_1) + \frac{1}{2}\int dx_1 \int dx_2 \frac{\delta^2 R_0}{\delta\phi_\perp(x_1)\delta\phi_\perp(x_2)}\bigg|_{\phi_0(x,r)} \times \delta\phi_\perp(x_1)\delta\phi_\perp(x_2) + \cdots; \qquad (5.12)$$

and for $U(\phi)$ defined by

$$U(\phi) = \int [\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)]\,dx, \qquad (5.13)$$

we obtain

$$U(\phi) = U(r) + \int dx_1 \frac{\delta U}{\delta\phi_\perp(x_1)}\bigg|_{\phi_0(x,r)} \delta\phi_\perp(x_1) + \frac{1}{2}\int dx_1 \int dx_2 \frac{\delta^2 U}{\delta\phi_\perp(x_1)\delta\phi_\perp(x_2)}\bigg|_{\phi_0(x,r)} \times \delta\phi_\perp(x_1)\delta\phi_\perp(x_2) + \cdots, \qquad (5.14)$$

with $U(r)$ given by Eq. (3.14) for $\phi = \phi_0(x,r)$.

Now by construction of the MPEP,

$$\frac{\delta R_0}{\delta\phi_\perp}\bigg|_{\phi_0(x,r)} = 0, \qquad (5.15)$$

so there is no linear term in Eq. (5.12). Furthermore, since $\delta\phi_\perp$ is a variation orthogonal to $\phi_0(x,r)$ we have

$$\delta\phi_\perp(x,r) = \int dy\, \hat{\phi}_{n(y)}\delta n(y), \qquad (5.16)$$

where δn is considered small. Thus substituting these expansions into the WKB equation (3.9) and making use of Eq. (5.11) we would obtain a sequence of approximate equations away from the MPEP valid for arbitrary powers of δn.

Define

$$\kappa(y,y';r) \equiv \int dx_1 \int dx_2 \frac{\delta^2 R_0(\phi)}{\delta\phi_\perp(x_1)\delta\phi_\perp(x_2)}\bigg|_{\phi_0(x,r)} \times \hat{\phi}_{n(y)}(x_1,r)\hat{\phi}_{n(y')}(x_2,r), \qquad (5.17)$$

$$U_{n(y)}(r) \equiv \int dx_1 \frac{\delta U(\phi)}{\delta\phi_\perp(x_1)}\bigg|_{\phi_0(x,r)} \hat{\phi}_{n(y)}(x_1), \qquad (5.18)$$

and

$$U_{n(y),n(y')} \equiv \int dx_1 \int dx_2 \frac{\delta^2 U(\phi)}{\delta\phi_\perp(x_1)\delta\phi_\perp(x_2)}\bigg|_{\phi_0(x,r)} \times \hat{\phi}_{n(y)}(x_1)\hat{\phi}_{n(y')}(x_2). \qquad (5.19)$$

Note that $\kappa(y,y';r)$ is positive semidefinite since $\phi_0(x,r)$ is a minimum of $R_0(\phi)$.

The WKB equation (3.9) then becomes, using Eq. (5.11),

$$\frac{1}{2}\left[(1+n_1\rho)^{-2}\left(\frac{\partial R_0(\phi)}{\partial r}\right)^2 + \int dy \left(\frac{\partial R_0(\phi)}{\partial n(y)}\right)^2\right] = U(\phi) - E. \qquad (5.20)$$

Using the expansions of Eqs. (5.12) and (5.14) with definitions (5.17), (5.18), and (5.19) we have

$$R_0(\phi) = R_0(\phi_0(x,r)) + \tfrac{1}{2}\int dy \int dy'\, \kappa(y,y';r)\,\delta n(y)\,\delta n(y') + \cdots, \qquad (5.21)$$

thus

$$\frac{\partial R_0(\phi)}{\partial r} = \frac{\partial R_0(\phi_0(x,r))}{\partial r} + \tfrac{1}{2}\int dy \int dy' \frac{\partial\kappa(y,y';r)}{\partial r}\,\delta n(y)\,\delta n(y') + \cdots$$

and

$$\frac{\partial R_0(\phi)}{\partial n(y)} = \int dy'\, \kappa(y,y';r)\,\delta n(y') + \cdots. \qquad (5.22)$$

Equation (5.20) then becomes

$$(1+n_1\rho)^{-2}\left[\left(\frac{\partial R_0}{\partial r}\right)^2 + \left(\frac{\partial R_0}{\partial r}\right)\left(\int dy \int dy' \frac{\partial\kappa(y,y';r)}{\partial r}\,\delta n(y)\,\delta n(y')\right) + \cdots\right] + \int dy \int dy'\,\kappa(y,y';r) \int dy''\,\kappa(y,y'';r)\,\delta n(y')\,\delta n(y'') + \cdots$$
$$= 2\left[-E + U(r) + \int dy\, U_{n(y)}(r)\,\delta n(y) + \frac{1}{2}\int dy \int dy'\, U_{n(y),n(y')}(r)\,\delta n(y)\,\delta n(y') + \cdots\right]. \qquad (5.23)$$

Finally by comparing similar powers of $\delta n(y)$ we obtain the following sequence of equations:

$$\left(\frac{\partial R_0}{\partial r}\right)^2 = 2[U(r) - E], \qquad (5.24)$$

$$\rho\left(\frac{\partial R_0}{\partial r}\right)^2=-U_{n_1}(r), \tag{5.25}$$

$$6\left(\frac{\partial R_0}{\partial r}\right)^2\rho^2+\left(\frac{\partial R_0}{\partial r}\right)\frac{\partial\kappa(n_1,n_1;r)}{\partial r}+\int dy''\,\kappa(n_1,y'';r)\,\kappa(n_1,y'';r)=U_{n_1n_1}(r), \tag{5.26}$$

$$U_{n(y)}(r)=0\,,\qquad n(y)\neq n_1 \tag{5.27}$$

$$\frac{\partial R_0}{\partial r}\,\frac{\partial\kappa(y,y';r)}{\partial r}+\int dy''\,\kappa(y'',y;r)\,\kappa(y'',y';r)=U_{n(y)n(y')}\,,\qquad n(y),n(y')\neq n_1, \tag{5.28}$$

etc.

Now Eq. (5.24) is the zeroth-order equation we treated in the preceding section of this paper. It leads to the MPEP and to the evaluation of R_0 along the MPEP. Furthermore, the quantities $U_{n(y)}, U_{n(y),n(y')}$ also become known as they are evaluated at the MPEP. From Eqs. (5.24) and (5.25) we may first calculate ρ. We have

$$\rho=\frac{-U_{n_1}(r)}{2[U(r)-E]}\quad. \tag{5.29}$$

Then Eqs. (5.26) and (5.28) determine $\kappa(y,y';r)$ and read

$$\frac{3[U_{n_1}(r)]^2}{U(r)-E}+\{2[U(r)-E]\}^{1/2}\frac{\partial\kappa_{n_1n_1}}{\partial r}+\kappa_{n_1n_1}{}^2=U_{n_1n_1}(r), \tag{5.30}$$

$$\{2[U(r)-E]\}^{1/2}\,\frac{\partial\kappa_{yy'}}{\partial r}+\kappa_{yy'}{}^2=U_{n(y)n(y')}\,. \tag{5.31}$$

These are of course still nontrivial but in certain circumstances manageable.[5]

Finally Eq. (5.27) is a consistency equation on the MPEP. Note that in this tunneling region where $E \leq U(r)$ we see that $\kappa(y,y';r)$ is real and as noted earlier positive semidefinite and hence the wave function in general picks up a further exponentially damping factor.

Collecting terms together we find from Eqs. (3.11), (3.13), and (5.12) that the wave function at a point $\phi(x,r)$ is

$$\psi(\phi(x,r))=\frac{A}{\{2[U(r)-E]\}^{1/4}}\exp\left[-\frac{1}{\hbar}\left(\int_0^r dr'\{2[U(r')-E]\}^{1/2}+\frac{1}{2}\int dy\int dy'\,\kappa(y,y',r)\delta n(y)\delta n(y')+\cdots\right)\right]. \tag{5.32}$$

This is to be compared with corresponding wave functional in the classically allowed region around the classical orbit which would read

$$\psi(\phi(x,s))=\frac{A}{\{2[E-U(s)]\}^{1/2}}\exp\left[\frac{i}{\hbar}\left(\int_0^s ds'\{2[E-U(s')]\}^{1/2}+\frac{1}{2}\int dy\int dy'\,K(y,y';s)\delta n(y)\delta n(y')+\cdots\right)\right], \tag{5.33}$$

where K is now defined as

$$K(y,y';s)=\int dx_1\int dx_2\,\frac{\delta^2 S_0}{\delta\phi_\perp(x_1)\delta\phi_\perp(x_2)}\bigg|_{\phi_c(x,s)}\hat{\phi}_{n(y)}(x_1)\hat{\phi}_{n(y')}(x_2) \tag{5.34}$$

in analogy with the above.

We see that in the tunneling region, paths near the MPEP give exponentially damped contributions to the wave functional, thus justifying the leading character of the MPEP.

VI. THE EFFECT OF NEIGHBORING PATHS ON THE TUNNELING AMPLITUDE

A. Matching the WKB solutions

In Sec. V, we worked out the WKB wave functional in the neighborhood of both a classical solution (in a classically allowed region), and an MPEP (in a classically forbidden region). Now, we consider the tunneling of field configurations between two classically allowed regions I and III, separated by a classically forbidden region II as indicated in Fig. 1. In Fig. 1, λ is a parameter which characterizes the change of field configurations, and $U(\lambda)$ is the effective potential defined in Eq. (3.5). The system is originally in region I. The WKB wave functional in region I can be chosen as

$$\psi^{(\mathrm{I})}(\lambda,\delta n)=\frac{1}{\{2m(\lambda)[E-U(\lambda)]\}^{1/4}}\times\exp\left[\frac{i}{\hbar}\left(\int_{\lambda_1}^{\lambda}d\lambda'\{2m(\lambda')[E-U(\lambda')]\}^{1/2}+\frac{1}{2}\int dy\,dy'\,K(y,y',\lambda)\delta n(y)\delta n(y')+\cdots\right)\right]+\text{reflected wave}. \tag{6.1}$$

To within constant phase factors, the WKB wave functionals in regions II and III are

$$\psi^{(\mathrm{II})}(\lambda,\delta n)=\frac{1}{\{2m(\lambda)[U(\lambda)-E]\}^{1/4}}\times\exp\left[-\frac{1}{\hbar}\left(\int_{\lambda_1}^{\lambda}d\lambda'\{2m(\lambda')[U(\lambda')-E]\}^{1/2}+\frac{1}{2}\int dy\,dy'\,\kappa(y,y',\lambda)\delta n(y)\delta n(y')+\cdots\right)\right] \tag{6.2}$$

and

$$\psi^{(\mathrm{III})}(\lambda,\delta n)=\frac{A}{\{2m(\lambda)[E-U(\lambda)]\}^{1/4}}\times\exp\left[\frac{i}{\hbar}\left(\int_{\lambda_2}^{\lambda}d\lambda'\{2m(\lambda')[E-U(\lambda')]\}^{1/2}+\frac{1}{2}\int dy\,dy'\,K(y,y',\lambda)\delta n(y)\delta n(y')+\cdots\right)\right], \tag{6.3}$$

respectively. The definitions of $K(y,y',\lambda)$, $\kappa(y,y',\lambda)$, and $\delta n(\lambda)$ are given in the preceding section; and $m(\lambda)$, $U(\lambda)$ are defined in Secs. III and IV. The relative coefficients in $\psi^{(\mathrm{I})}$ and $\psi^{(\mathrm{II})}$ are fixed by matching the boundary condition at $\lambda=\lambda_1$. From the boundary condition at $\lambda=\lambda_2$, we can determine the tunneling amplitude A as

$$A=\exp\left[-\frac{1}{\hbar}\left(\int_{\lambda_1}^{\lambda_2}d\lambda'\{2m(\lambda')[U(\lambda')-E]\}^{1/2}+\frac{1}{2}\int dy\,dy'[\kappa(y,y',\lambda_2)-\kappa(y,y',\lambda_1)]\delta n(y)\delta n(y')\right)\right]. \tag{6.4}$$

Note that A depends not only on the value of $m(\lambda)$ and $U(\lambda)$ along a MPEP, but also on the deviation of a given path from the MPEP as denoted by δn. Since A is always real and positive, the contributions from paths described by different sets of δn will add coherently. Hence the total tunneling amplitude is given by the sum of A over all these paths,

$$\begin{aligned}P&=\int\mathfrak{D}(\delta n)J\exp\left[-\frac{1}{\hbar}\left(\int_{\lambda_1}^{\lambda_2}d\lambda'\{2m(\lambda')[U(\lambda')-E]\}^{1/2}+\frac{1}{2}\int dy\,dy'[\kappa(y,y',\lambda_2)-\kappa(y,y',\lambda_1)]\delta n(y)\delta n(y')+\cdots\right)\right]\\&=\exp\left[-\frac{1}{\hbar}\int_{\lambda_1}^{\lambda_2}d\lambda'\{2m(\lambda')[U(\lambda')-E]\}^{1/2}\right]\\&\quad\times\int\mathfrak{D}(\delta n)J\exp\left(-\frac{1}{2\hbar}\int dy\,dy'[\kappa(y,y',\lambda_2)-\kappa(y,y',\lambda_1)]\delta n(y)\delta n(y')+\cdots\right),\end{aligned} \tag{6.5}$$

where J is the Jacobian associated with the change of integration variables $\mathfrak{D}\delta\phi$ to $\mathfrak{D}\lambda(t)\,\mathfrak{D}\delta n$. In (6.5), the first factor denotes the lowest-order WKB tunneling amplitude; and the second factor denotes the additional contribution due to the neighboring paths.

B. Contribution due to the neighboring paths

The contribution due to the integration over all nearby paths around an MPEP is

$$Z\equiv\int\mathfrak{D}(\delta n)J\times\exp\left(-\frac{1}{2\hbar}\int dy\,dy'[\kappa(y,y',\lambda_1)-\kappa(y,y',\lambda_2)]\times\delta n(y)\delta n(y')\right). \tag{6.6}$$

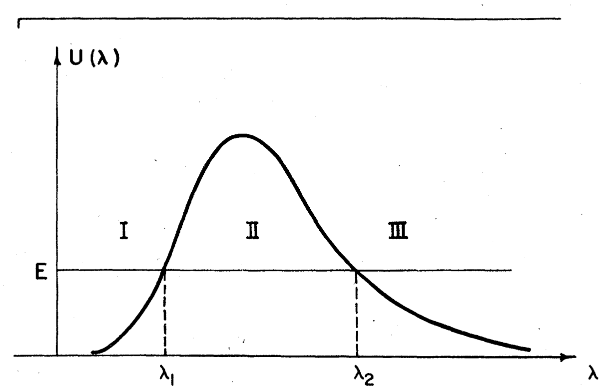

FIG. 1. The potential $U(\lambda)$ for a tunneling configuration.

In principle, we can determine $\kappa(y,y',\lambda)$ by solving the coupled equations (5.26)–(5.28). In practice, however, this is too involved. We shall make here a further simplification. We assume that $R_0(\phi)$ near an MPEP is given by

$$R_0(\phi)=\int_{\lambda_1}^{\lambda} d\lambda' \{2m(\lambda')[U(\lambda')-E]\}^{1/2}. \tag{6.7}$$

Note that Eq. (6.7) is exact along an MPEP. Here, we assume that we can use the *same* functional form for the neighboring paths as well.

Now, we want to express our result in terms of $\delta\phi_\perp(x,\lambda)$ directly. Define

$$\tilde{\kappa}(x',\lambda';x'',\lambda'')\equiv \left.\frac{\delta^2 R_0(\phi)}{\delta\phi(x',\lambda')\delta\phi(x'',\lambda'')}\right|_{\phi=\phi_0(x,\lambda)} \tag{6.8}$$

as the second functional derivative of R_0 with respect to an arbitrary variation $\delta\phi(x)$. Then, we have

$$\kappa(y',y'',\lambda)=\int dx'\,d\lambda' \int dx''\,d\lambda''\, \hat{\phi}_{n(y')}(x',\lambda') \times \hat{\phi}_{n(y'')}(x'',\lambda'') \times \tilde{\kappa}(x',\lambda';x'',\lambda''), \tag{6.9}$$

and

$$\int dy\,dy'\,\delta n(y)\delta n(y')\,\kappa(y,y',\lambda) =\int dx'\,d\lambda' \int dx''\,d\lambda''\,\delta\phi_\perp(x',\lambda') \times\tilde{\kappa}(x',\lambda';x'',\lambda'')\delta\phi_\perp(x'',\lambda''). \tag{6.10}$$

It is straightforward to compute $\tilde{\kappa}$ in the neighborhood of an MPEP. Varying Eq. (6.7) with respect to $\delta\phi(x,\lambda)$ we have

$$\begin{aligned}\delta R_0 &= \int d\lambda\,\delta\phi\left[\frac{\delta}{\delta\phi}[2m(U-E)]^{1/2} - \frac{\partial}{\partial\lambda}\left(\frac{\delta}{\delta(\partial\phi/\partial\lambda)}[2m(U-E)]^{1/2}\right)\right]\\ &= \int d\lambda\,\delta\phi(x,\lambda)\left\{\left(\frac{2m}{U-E}\right)^{1/2}\left(-\tfrac{1}{2}\nabla^2\phi+\tfrac{1}{2}\frac{\partial V}{\partial\phi}\right) - \frac{\partial}{\partial\lambda}\left[\left(\frac{2(U-E)}{m}\right)^{1/2}\frac{\partial\phi}{\partial\lambda}\right]\right\}.\end{aligned} \tag{6.11}$$

Introducing a new parametrization $\tau=\tau(\lambda)$ via

$$\frac{d\tau}{d\lambda}=\left(\frac{m(\lambda)}{2[U(\lambda)-E]}\right)^{1/2}, \tag{6.12}$$

we obtain

$$\delta R_0=\int d\tau\,\delta\phi(x,\tau)\left(-\frac{\partial^2\phi}{\partial\tau^2}-\nabla^2\phi+\frac{\partial V}{\partial\phi}\right), \tag{6.13}$$

and consequently

$$\frac{\delta R_0}{\delta\phi(x,\tau)}=\left(-\frac{\partial^2\phi}{\partial\tau^2}-\nabla^2\phi+\frac{\partial V}{\partial\phi}\right). \tag{6.14}$$

Differentiating (6.14) once more, we have

$$\kappa(x,\tau;x',\tau')=\frac{\delta^2 R_0}{\delta\phi(x,\tau)\delta\phi(x',\tau')} =\left(-\frac{\partial^2}{\partial\tau^2}-\nabla^2+\frac{\partial^2 V}{\partial\phi^2}\right)\delta(x-x')\delta(\tau-\tau'). \tag{6.15}$$

Substituting (6.15) into (6.10) and (6.6), we have

$$Z=\int \mathfrak{D}\delta\phi_\perp J' \exp\left[-\frac{1}{2\hbar}\int_{\tau_1}^{\tau_2} d\tau \int dx\,\delta\phi_\perp(x)\left(-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2 V}{\partial\phi^2}\right)\delta\phi_\perp(x)\right], \tag{6.16}$$

with

$$\tau_1=\tau(\lambda_1),\quad \tau_2=\tau(\lambda_2), \tag{6.17}$$

and J' is the Jacobian associated with the transformation $\mathfrak{D}\delta\phi\to\mathfrak{D}\delta\phi_\perp\mathfrak{D}\lambda(t)$. We can write Z and P symbolically as

$$Z=\left[\det{}'\left(-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2 V}{\partial\phi^2}\right)\right]^{-1/2}, \tag{6.18}$$

and

$$P=e^{-(1/\hbar)R_0(\phi_0)}\left[\det{}'\left(-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2 V}{\partial\phi^2}\right)\right]^{-1/2}. \tag{6.19}$$

This is the same expression (unrenormalized) as obtained by Callan and Coleman.[8,9] Even though the above expression is identical in form to the Euclidean action, it is *not* the analytical continuation of the Minkowski action into the Euclidean region. The parameter $\tau=\tau(\lambda)$ describes the time dependence of the tunneling field configuration in the Minkowski space. It has nothing to do with the Euclidean time variable.

Several remarks are in order: (1) The kernel

$$-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2 V}{\partial\phi^2}$$

is a positive semidefinite operator. It implies that the contribution from any individual path is always smaller than (or, in exceptional cases, equal to) the contribution due to the MPEP. (2) It is known that the vacuum tunneling can happen at any space-time region. Thus one expects an explicit VT factor in computing the vacuum tunneling amplitude. This VT factor comes from the contributions due to integrations over the zero modes associated with the space and time translations. For a detailed analysis of these zero-energy translational modes, we refer the readers to the paper by Gervais and Sakita in Ref. 7. (3) Depending on the interaction, the integration over $D(\delta\phi_\perp)$ may lead to ultraviolet as well as infrared divergences. It is expected that[8] the ultraviolet divergences will be removed by standard mass and wave-function renormalizations.

C. Free energy

The contribution due to the neighboring paths has the form

$$Z(\tau)=\int \mathfrak{D}\delta\phi_\perp J' \exp\left[-\frac{1}{\hbar}\int_{\tau_1}^{\tau} d\tau' H(\delta\phi,\tau')\right], \tag{6.20}$$

with

$$H(\delta\phi,\tau)=\frac{1}{2}\int dx\,\delta\phi_\perp\left(-\frac{\partial^2}{\partial\tau^2}-\nabla^2+\frac{\partial^2 V}{\partial\phi^2}\right)\delta\phi_\perp . \tag{6.21}$$

It is important to note that (6.20) has the same functional form as the partition function in statistical mechanics, where τ is identified as the inverse temperature. The interaction energy has an explicit τ dependence. It is convenient to define, to within a renormalization factor,

$$\begin{aligned}Z(\tau)&\equiv\int \mathfrak{D}\delta\phi_\perp J'\exp\left(-\frac{1}{\hbar}\int_{\tau_1}^{\tau} d\tau' H\right)\\ &\equiv\exp\left[-\frac{1}{\hbar}\int_{\tau_1}^{\tau} d\tau' F(\tau')\right]\\ &\equiv\exp\left[-\frac{1}{\hbar}B(\tau)\right],\end{aligned} \tag{6.22}$$

where $F(\tau)$ represents the free energy, and B the Boltzmann factor. We can now write the total tunneling amplitude as

$$P=\exp\left[-\frac{1}{\hbar}\int_{\lambda_1}^{\lambda_2} d\lambda\{2m(\lambda)[U(\lambda)-E]\}^{1/2} -\frac{1}{\hbar}\int_{\tau_1}^{\tau_2} d\tau F(\tau)\right]. \tag{6.23}$$

For small F, we can absorb the second term into the first term by modifying U to an effective $U_{\rm eff}$. Indeed, if we define

$$U_{\rm eff}\equiv U+F, \tag{6.24}$$

we have

$$\begin{aligned}\int_{\lambda_1}^{\lambda} d\lambda'[2m(\lambda')(U_{\rm eff}-E)]^{1/2}&=\int_{\lambda_1}^{\lambda} d\lambda'[2m(U-E)]^{1/2}+\int_{\lambda_1}^{\lambda} d\lambda'\left(\frac{2m}{U-E}\right)^{1/2}\tfrac{1}{2}F+O(F^2)\\ &=\int_{\lambda_1}^{\lambda} d\lambda'[2m(U-E)]^{1/2}+\int_{\tau_1}^{\tau} d\tau' F(\tau')+O(F^2),\end{aligned} \tag{6.25}$$

and

$$P=\exp\left(-\frac{1}{\hbar}\int_{\lambda_1}^{\lambda} d\lambda\,[2m(\lambda)(U_{\rm eff}-E)]^{1/2}\right) \tag{6.26}$$

as desired. Thus, the overall contribution of the neighboring paths is to modify the classical energy U associated with the path to the total energy $U_{\rm eff}$. The above finding suggests that an improved method of obtaining the MPEP with corrections from neighboring paths is as follows: We choose a path $\phi=\phi(x,\lambda)$, and compute both the classical energy U and the renormalized free energy F due to the neighboring paths. Then, we determine the MPEP by requiring that the tunneling amplitude P in (6.26) be maximal. This improved method is closely related to the self-consistent Hartree approximation applied to the tunneling problem.

To make the connection to the Hartree approximation, we first derive the equation for the improved MPEP through the variation of (6.26), giving

$$\delta \int d\lambda\,[2m(\lambda)(U_{\rm eff}-E)]^{1/2} = \delta\left(\int d\lambda\,[2m(\lambda)(U-E)]^{1/2} + \int d\tau\, F\right)$$

$$= \int d\lambda\left(\delta\phi(x,\lambda)\left\{\left(\frac{2m}{U-E}\right)^{1/2}\left(-\tfrac{1}{2}\nabla^2\phi + \tfrac{1}{2}\frac{\partial V}{\partial\phi}\right) - \frac{\partial}{\partial\lambda}\left[\left(\frac{2(U-E)}{m}\right)^{1/2}\frac{\partial\phi}{\partial\lambda}\right]\right\} + \left(\frac{2m}{U-E}\right)^{1/2}\delta F\right) = 0\,. \qquad (6.27)$$

We express our result in terms of the parametrization τ defined in (6.12), and obtain the equation for the MPEP $\phi_0(x,\tau)$ as

$$-\frac{\partial^2\phi_0}{\partial\tau^2} - \frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial V(\phi_0)}{\partial\phi} + \frac{\delta B(\phi_0)}{\delta\phi} = 0\,. \qquad (6.28)$$

Using the definition of B in (6.22), we differentiate B with respect to ϕ and obtain

$$\frac{\delta B}{\delta\phi(x,\tau)} = \frac{1}{Z(\tau_2)}\int \mathfrak{D}\delta\phi_\perp J'\left(\tfrac{1}{2}\delta\phi_\perp^2\,\frac{\partial^3 V(\phi_0)}{\partial\phi^3}\right) \times \exp\left(-\frac{1}{\hbar}\int_{\tau_1}^{\tau_2} d\tau\, H\right)$$

$$= \frac{1}{2}\,\frac{\partial^3 V(\phi_0)}{\partial\phi^3}\langle\delta\phi_\perp^2\rangle$$

$$= \frac{1}{2}\,\frac{\partial^3 V(\phi_0)}{\partial\phi^3}\langle(\phi-\phi_0)^2\rangle \qquad (6.29)$$

where

$$\langle(\phi-\phi_0)^2\rangle \equiv \frac{1}{Z}\int \mathfrak{D}\delta\phi_\perp J'(\delta\phi_\perp)^2 \exp\left(-\frac{1}{\hbar}\int_{\tau_1}^{\tau_2} d\tau\, H\right). \qquad (6.30)$$

Now, we can relate the last two terms in Eq. (6.30) to the expectation value of $\partial V/\partial\phi$ via

$$\left\langle\frac{\partial V(\phi)}{\partial\phi}\right\rangle = \left\langle\frac{\partial V(\phi_0)}{\partial\phi} + \frac{\partial^2 V(\phi_0)}{\partial\phi^2}(\phi-\phi_0) + \frac{1}{2}\,\frac{\partial^3 V(\phi_0)}{\partial\phi^3}(\phi-\phi_0)^2 + \cdots\right\rangle$$

$$= \frac{\partial V(\phi_0)}{\partial\phi} + \frac{1}{2}\,\frac{\partial^3 V(\phi_0)}{\partial\phi^3}\langle(\phi-\phi_0)^2\rangle$$

$$= \frac{\partial V(\phi_0)}{\partial\phi} + \frac{\delta B}{\delta\phi}\,, \qquad (6.31)$$

where we have used the fact that $\langle\phi-\phi_0\rangle = 0$. Thus we can interpret (6.28) as the expectation value of the field equation

$$-\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{\partial V(\phi)}{\partial\phi} = 0 \qquad (6.32)$$

under the Gaussian fluctuation. It is well known that the use of Gaussian fluctuations as trial functions is equivalent to the Hartree approximation. Equations (6.31) and (6.32) are the realization of the Hartree approximation in the familiar field operator forms.

We can extend the above interpretations to study the contribution from the neighboring paths of a classical orbit (in the classically allowed region) as well. We can write the equation of a classical orbit $\phi_c(x,\eta)$ as [see Eq. (3.40)]

$$\frac{\partial^2\phi_c}{\partial\eta^2} - \frac{\partial^2\phi_c}{\partial x^2} + \left\langle\frac{\partial V}{\partial\phi}\right\rangle_{\phi_c} = \frac{\partial^2\phi_c}{\partial\eta^2} - \frac{\partial^2\phi_c}{\partial x^2} + \frac{\partial V(\phi_c)}{\partial\phi} + \frac{1}{2}\,\frac{\partial^3 V(\phi_c)}{\partial\phi^3}\langle(\phi-\phi_c)^2\rangle = 0\,. \qquad (6.33)$$

For a classical orbit, the quantity which is the analog to the free energy F in the tunneling case is the zero-point energy, $E_0(\phi_c)$, in the presence of ϕ_c. We can express E_0 symbolically as

$$E_0 = \sum_{\text{all modes}} \tfrac{1}{2}\hbar\omega(\phi_c) - \sum \tfrac{1}{2}\hbar\omega(\text{vacuum})\,. \qquad (6.34)$$

VII. DISCUSSION

A. Non-Abelian gauge theories

We can generalize the method developed in the previous sections easily to the non-Abelian gauge theory described by

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}\,(F_{\mu\nu})^2\,, \qquad (7.1)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{i}[A_\mu, A_\nu]\,. \qquad (7.2)$$

In the (temporal) gauge

$$A^0 = 0\,, \qquad (7.3)$$

we have

$$E_k \equiv F^0{}_k = \partial^0 A_k \qquad (7.4)$$

and

$$B_m \equiv F_{kl} = \partial_k A_l - \partial_l A_k + \frac{1}{i}[A_k, A_l]\,, \quad (k,l,m \text{ cyclic})\,. \qquad (7.5)$$

A path in the function space is defined by the field configuration

$$\vec{A} = \vec{A}(x, \lambda(t)). \tag{7.6}$$

Most of the results that we have obtained in Secs. II–VI are still valid after we make the identification

$$m(\lambda) = \frac{2}{g^2}\int d^3x\, \mathrm{Tr}\left(\frac{\partial \vec{A}}{\partial\lambda}\right)^2 \tag{7.7}$$

and

$$U(\lambda) = \frac{1}{g^2}\int d^3x\, \mathrm{Tr}\, \vec{B}^2. \tag{7.8}$$

In terms of the new parametrization $\tau(t)$ defined by

$$\frac{d\tau}{d\lambda} = \left(\frac{m(\lambda)}{2[U(\lambda)-E]}\right)^{1/2}, \tag{7.9}$$

we find that the field configuration associated with the tunneling region obeys the equation

$$\frac{\partial^2}{\partial\tau^2}A_l + D_k F_{kl} = 0, \tag{7.10}$$

where D_k is the gauge-covariant derivative.

Define the field

$$\mathscr{E}_l \equiv F_{\tau l} \equiv \frac{\partial A_l}{\partial\tau}, \tag{7.11}$$

which is related to the electric field by

$$\mathscr{E}_l = \left(\frac{\partial\tau}{\partial\lambda}\right)^{-1} E_l; \tag{7.12}$$

we have

$$D_\tau F_{\tau l} + D_k F_{kl} = 0, \tag{7.13}$$

where $D_\tau = \partial/\partial\tau$ in the $A_0 = 0$ gauge. One recognizes that (7.13) is the Euclidean field equation for a non-Abelian gauge field. In Ref. 4, we showed that $\vec{E}$ and $\vec{B}$ fields associated with the MPEP obey [see Eq. (7.1) in Ref. 4 and note that the energy $E = 0$]

$$\vec{E} = \vec{B}\left(\frac{m(\lambda)}{2[U(\lambda)-E]}\right)^{1/2}\dot{\lambda}, \tag{7.14}$$

which implies that

$$\vec{B} = \vec{E}\left(\frac{d\tau}{dt}\right)^{-1} = \vec{\mathscr{E}}. \tag{7.15}$$

Equations (7.13) and (7.15) imply that the field tensor $F_{\tau l} = \mathscr{E}_l$, $F_{kl} = B_m$, expressed in terms of the τ variable, obeys both the Euclidean-type non-Abelian gauge equation and the self-dual conditon.[10] This explains why a formal replacement of $\tau \to \lambda(t)$ in a self-dual Euclidean instanton solution ($A_0 = 0$ gauge) gives rise to a maximal tunneling field configuration (MPEP) in the Minkowski space, as demonstrated in Ref. 4.

We can evaluate the contribution for the nearby paths in an analogous way. The total tunneling amplitude after including the contribution from the neighboring paths is

$$P = Z\exp\left(-\frac{1}{\hbar}R_0\right),$$

where

$$R_0 = \int_{\lambda_1}^{\lambda_2} d\lambda\, \{2m(\lambda)[U(\lambda)-E]\}^{1/2} \tag{7.16}$$

is the tunneling exponential factor associated with the MPEP, and

$$Z = \int \mathfrak{D}'A\, J\exp\left\{-\tfrac{1}{2}\int d\tau\, dx\, \delta A_k\left[-\frac{\delta^2}{\delta\tau^2}\delta_{kl} - \frac{\partial}{\partial A_k}\left(D_m F_{ml}\right)\right]\delta A_l\right\}. \tag{7.17}$$

In Eq. (7.17), $\mathfrak{D}'A$ contains a gauge-fixing term which we have omitted for notational simplicity, and J is the Jacobian associated with the transformation $\mathfrak{D}A$ to $\mathfrak{D}'A\,\mathfrak{D}\lambda(t)$. As we have done in the scalar case, we introduce a Boltzmann factor B and a free energy F by

$$Z = e^{-B(\tau)} = \exp\left[-\int^{\tau} d\tau'\, F(\tau')\right]. \tag{7.18}$$

We can express the total tunneling amplitude compactly as

$$P = \exp\left(-\int_{\lambda_1}^{\lambda_2} d\lambda\, [2m(\lambda)(U+F-E)]^{1/2}\right). \tag{7.19}$$

Just as in the scalar case, we can obtain an improved MPEP by applying the variational principle to the total amplitude P. The resultant field equation for the MPEP obeys

$$\frac{\partial^2 A_l}{\partial\tau^2} + \langle D_k F_{kl}\rangle = \frac{\partial^2 A_l}{\partial\tau^2} + D_k F_{kl} + \frac{1}{2}\frac{\partial^2}{\partial A_m \partial A_n}\left(D_k F_{kl}\right)\langle\delta A_m \delta A_n\rangle = 0, \tag{7.20}$$

which has a simple mean-field interpretation.

B. Coupling to a fermion field

If the scalar (or gauge) field is coupled to a fermion field ψ, both the tunneling amplitude and the MPEP will be affected by the presence of this field. Consider a simple system with the Lagrange density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \bar{\psi}\left(\frac{i}{2}\overleftrightarrow{\partial\!\!\!/} - m\right)\psi - g\bar{\psi}\psi\phi, \tag{7.21}$$

where ψ is the fermion field. To find the MPEP in ϕ and ψ, consider the field configuration $\phi(x,\lambda(t))$, $\psi(x,\lambda(t))$ which depends on the single parameter $\lambda(t)$. Then as a function of $\lambda(t)$ and $\dot\lambda(t)$ the Lagrangian becomes

$$L=\int dx\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\lambda}\right)^2\dot\lambda^2-(\nabla\phi)^2-V(\phi)+\tfrac{1}{2}\left(i\psi^\dagger\frac{\overleftrightarrow{\partial}}{\partial\lambda}\psi\right)\dot\lambda - i\bar\psi\gamma_i\partial_i\psi-\bar\psi\psi m-g\bar\psi\psi\phi\right]. \tag{7.22}$$

Treating λ as a canonical variable we then get for the canonical conjugate momentum

$$P_\lambda=\frac{\partial L}{\partial\dot\lambda}=\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2\dot\lambda+\tfrac{1}{2}\int dx\left(i\psi^\dagger\frac{\overleftrightarrow{\partial}}{\partial\lambda}\psi\right), \tag{7.23}$$

and the Hamiltonian then reads

$$H=P_\lambda\dot\lambda-L=\int dx\,\frac{1}{2}\left(\frac{\partial\phi}{\partial\lambda}\right)^2\dot\lambda^2+U_\phi(\lambda)+U_\psi(\lambda), \tag{7.24}$$

or

$$H=\frac{1}{2m(\lambda)}(P_\lambda-Q)^2+U_\phi(\lambda)+U_\psi(\lambda) \tag{7.25}$$

where we have defined

$$U_\phi(\lambda)=\int dx\{[\nabla\phi(x,\lambda)]^2+V(\phi(x,\lambda))\}, \tag{7.26}$$

$$U_\psi(\lambda)=\int dx[i\bar\psi\gamma_i\nabla_i\psi(x,\lambda)+m\bar\psi\psi(x,\lambda)+g\phi\bar\psi\psi], \tag{7.27}$$

$$Q=\int dx\,\tfrac{1}{2}i\left(\psi^\dagger\frac{\partial}{\partial\lambda}\psi-\frac{\partial\psi^\dagger}{\partial\lambda}\psi\right), \tag{7.28}$$

$$m(\lambda)=\int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2. \tag{7.29}$$

One could obtain H in Eq. (7.25) by first constructing $H(\psi,\phi)$ from $\mathcal{L}$ of Eq. (7.21) and then substituting $\phi(x,\lambda)$ and $\psi(x,\lambda)$ in that expression. To get the correct momentum canonical to $\lambda(t)$, however, one must work with L of Eq. (7.22) directly to obtain Eq. (7.23) and hence Eq. (7.25).

For a tunneling transition at constant energy E we have now

$$P_\lambda-Q=i\{2m(\lambda)[U_\phi(\lambda)+U_\psi(\lambda)-E]\}^{1/2}, \tag{7.30}$$

so that the WKB tunneling amplitude becomes to first order e^{-R_0} with

$$R_0=\int_{\lambda_1}^{\lambda_2}d\lambda\{2m(\lambda)[U_\phi(\lambda)+U_\psi(\lambda)-E]\}^{1/2}+\frac{1}{i}\int_{\lambda_1}^{\lambda_2}d\lambda\,Q. \tag{7.31}$$

Note that R_0 is now complex. It indicates that a single $\lambda(t)$ parametrization for ϕ, ψ, and $\psi^\dagger$ may not be adequate. We shall assume tentatively that the equations determining the MPEP now in ϕ,ψ space may be obtained by varying R_0 with respect to ϕ and ψ. As before the first term gives us the Euclidean-type equation for $\phi(\lambda)$ that we discussed before except that $U_\psi(\lambda)$ gives an extra contribution due to the coupling of ψ with ϕ.

Furthermore, $U_\psi(\lambda)$ and Q when varied with respect to ψ give us a Euclidean-type equation for ψ coupled to ϕ. Thus the MPEP in $\phi-\psi$ space must satisfy the following Euclidean-type coupled equations [after changing to the parameter given by Eq. (6.12) with $U(\lambda)=U_\phi+U_\psi$]:

$$\frac{\partial^2}{\partial\tau^2}\phi(x,\tau)+\nabla^2\phi(x,\tau)-V'(\phi)=g\bar\psi\psi, \tag{7.32}$$

$$\left[\gamma_0\frac{\partial}{\partial\tau}+i\gamma_i\nabla_i+m+g\phi(x,\tau)\right]\psi=0, \tag{7.33a}$$

$$\bar\psi\left[-\gamma_0\frac{\overleftarrow{\partial}}{\partial\tau}-i\gamma_i\overleftarrow{\nabla}_i+m+g\phi(x,\tau)\right]=0. \tag{7.33b}$$

It is important to note that (7.33b) is not equivalent to the Hermitian adjoint of (7.33a). The latter can only be obtained by varying R_0^*. The only solutions which are consistent with both (7.33a) and (7.33b) are $\psi=0$. This leads to the usual MPEP solution. One may now take as the MPEP the solution $\phi=\phi_c(x,\lambda)$, $\psi=0$, where ϕ_c satisfies

$$\frac{\partial^2}{\partial\tau^2}\phi_c(x,\tau)+\nabla^2\phi_c(x,\tau)-V'(\phi_c)=0, \tag{7.34}$$

and treat the whole fermion field ψ as a fluctuation around ϕ_c in the same manner one treated $\delta\phi$ in Secs. V and VI. Doing so one must introduce the quantity $\tilde K_\psi$ corresponding to $\tilde K$ introduced in Sec. VI [see Eq. (6.8)],

$$\tilde\kappa_\psi=\frac{\delta^2R_0}{\delta\psi\delta\bar\psi}. \tag{7.35}$$

Using Eq. (7.31) as an approximation to R_0 around the MPEP and performing the differentiation with respect to the fermion fields one obtains

$$\tilde\kappa_\psi=\gamma_0\frac{\partial}{\partial\tau}+i\gamma_i\nabla_i+g\phi_c+m, \tag{7.36}$$

so that the correct wave functional has now the extra contribution

$$\exp\left[-\int dx\,\bar\psi\left(\gamma_0\frac{\partial}{\partial\tau}+i\gamma_i\partial_i+g\phi_c+m\right)\psi\right]. \tag{7.37}$$

Upon matching wave functionals one must integrate out the fermion fluctuations to obtain the Fermi contribution to the tunneling amplitude in the form

$$\exp\left(-\int_{\tau_1}^{\tau_2} d\tau F_\psi\right) = \exp\left[-\mathrm{Tr}\ln\left(\gamma^0 \frac{\partial}{\partial\tau} + i\gamma_i\partial_i + m + g\phi_c\right)\right] \Big/ \exp\left[-\mathrm{Tr}\ln\left(\gamma^0\frac{\partial}{\partial\tau} + i\gamma_i\partial_i + m\right)\right], \tag{7.38}$$

where Tr indicates a trace in spin as well as $(\tau, \vec{x})$ space, and

$$F_\psi = \sum_{\substack{\text{negative}\\ \text{energy states}}} \omega_k(\phi_c) - \sum_{\substack{\text{negative}\\ \text{energy states}}} \omega_k(0).$$

The full amplitude then reads

$$P = e^{-(1/\hbar)R}, \tag{7.39}$$

where R is

$$R = \int d\lambda \{2m(\lambda)[U_{\text{eff}}(\lambda) - E]\}^{1/2}, \tag{7.40}$$

with

$$U_{\text{eff}}(\lambda) = U(\lambda) + F_\phi(\lambda) + F_\psi(\lambda). \tag{7.41}$$

$U(\lambda)$ is the classical potential energy defined in (4.3), $F_\phi(\lambda)$ is the free energy defined in (4.22), and F_ψ is given above.

Thus the fermions contribute an additional energy to the tunneling potential which describes the change of negative energy sea in the presence of ϕ_c. In the case of a massless fermion field the contribution due to F_ψ can be extremely important and it may modify the MPEP significantly. In fact in gauge theory where the tunneling involves a change in winding number it is known that F_ψ for massless fermions is logarithmically divergent due to the necessary presence of zero modes of the Euclidean Dirac operator that appears in F_ψ.[11]

In general one can then obtain a modified MPEP by applying the variational principle to R as given in (7.40). This improved variational calculation leads to a modified field equation

$$\frac{\partial^2\phi_c}{\partial\tau^2} + \left\langle \frac{\partial V}{\partial\phi_c}\right\rangle + g\langle\bar{\psi}\psi\rangle = 0 \tag{7.42}$$

as given in a Hartree approximation.

It is interesting to note that to the one-fermion-loop approximation, we may replace the Lagrange function (7.21) by

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \psi'^\dagger\gamma_5(i\partial\!\!\!/ - m)\psi' - g\psi'^\dagger\gamma_5\psi'\phi, \tag{7.43}$$

for they both lead to the same fermion contribution F_ψ in (7.38). Then, under the same approximation by considering the field configurations $\phi(x,\lambda(t))$, $\psi(x,\lambda(t))$, we arrive at a slightly different effective Hamiltonian

$$H' = \frac{1}{2m(\lambda)}(p_\lambda - Q')^2 + U_\phi(\lambda) + U_\psi'(\lambda), \tag{7.44}$$

where U_ϕ and $m(\lambda)$ are the same as in (7.26) and (7.29), and

$$U_{\psi'} = \int dx (i\psi'^\dagger\gamma_5\gamma_i\nabla_i\psi' + m\psi'^\dagger\gamma_5\psi' + g\phi\psi'^\dagger\gamma_5\psi'), \tag{7.45}$$

$$Q' = \int dx \frac{i}{2}\left(\psi'^\dagger\gamma_5\gamma_0\frac{\partial}{\partial\lambda}\psi' - \frac{\partial\psi'^\dagger}{\partial\lambda}\gamma_5\gamma_0\psi'\right). \tag{7.46}$$

Now, Q' is purely imaginary. The new WKB tunneling amplitude becomes $e^{-R_0'}$ with

$$R_0' = \int_{\lambda_1}^{\lambda_2} d\lambda\{2m(\lambda)[U_\phi(\lambda) + U_\psi'(\lambda) - E]\}^{1/2} + \frac{1}{i}\int_{\lambda_1}^{\lambda_2} d\lambda\, Q'. \tag{7.47}$$

R_0' is now real. The equation determining the MPEP in ϕ, ψ' space can indeed be obtained by varying R_0' with respect to ϕ and ψ', giving

$$\frac{\partial^2}{\partial\tau^2}\phi(x,\tau) + \nabla^2\phi(x,\tau) - V'(\phi) = g\psi'^\dagger\gamma_5\psi', \tag{7.48}$$

$$\left[\gamma_0\frac{\partial}{\partial\tau} + i\gamma_i\nabla_i + m + g\phi(x,\tau)\right]\psi' = 0, \tag{7.49}$$

$$\psi^{\dagger\prime}\gamma_5\left[-\gamma_0\frac{\overleftarrow{\partial}}{\partial\tau} - i\gamma_i\overleftarrow{\nabla}_i + m + g\phi(x,\tau)\right] = 0. \tag{7.50}$$

Equations (7.48)–(7.50) are identical to the Euclidean equations of motion of the system.

We can apply the same procedure to gauge theory. In particular, the MPEP is determined by the following set of Euclidean-type equations:

$$D_\mu F_{\mu\nu} = \psi'^\dagger\gamma_5\gamma_\nu\psi', \quad \mu,\nu = (\tau,1,2,3) \tag{7.51}$$

$$(\gamma_0 D_\tau + i\gamma_i D_i)\psi' = 0. \tag{7.52}$$

It is well known[11] that Eq. (7.52) always has solutions if the gauge field carries a nonzero winding number. Furthermore, these solutions are eigenstates of γ_5 with an eigenvalue equal to the sign of the winding number. This property means, however, that the source term on the right-hand side of Eq. (7.51) always vanishes for these solutions. Thus there is a *family* of MPEP's with the same action carried by a gauge field component that satisfies a sourceless Euclidean-type equation and a fermion component satisfying Eq. (7.52) which contributes zero action. This degeneracy of MPEP's reflects the existence of the fermion zero mode inasmuch as the degeneracy of pseudoparticle action with respect to position or size reflect the gauge field zero modes. The significance of this MPEP degeneracy and its relation to tunneling in gauge theory will be discussed in a forthcoming publication.

C. Lorentz transformation

We can work out the Lorentz transformation property of the wave functional $\psi(\phi)$ defined in Sec. II*B formally*. Consider the system

$$\mathcal{L} = \tfrac{1}{2}(\partial_0\phi)^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi) . \tag{7.53}$$

The stress tensor of the system is

$$T^{\mu\nu} = \partial^\mu\phi\, \partial^\nu\phi - g^{\mu\nu}\mathcal{L} . \tag{7.54}$$

In particular, we have

$$T^{00} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) , \tag{7.55}$$

$$T^{0k} = \dot{\phi}\, \partial^k\phi . \tag{7.56}$$

In the Schrödinger representation, we have

$$T^{00} = -\frac{\hbar^2}{2}\left(\frac{\delta}{\delta\phi}\right)^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) , \tag{7.57}$$

$$T^{0k} = \frac{\hbar}{i}\frac{\delta}{\delta\phi}\partial^k\phi . \tag{7.58}$$

The energy-momentum, and the angular momentum operators are

$$P^\mu = \int dx\, T^{0\mu} , \tag{7.59}$$

$$J^{\mu\nu} = \int dx (x^\mu T^{0\nu} - x^\nu T^{0\mu}) . \tag{7.60}$$

The time-independent wave functional $\psi(\phi)$ obeys

$$\int dx \left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right]\psi(\phi) = E\,\psi(\phi) , \tag{7.61}$$

$$\int dx \frac{\hbar}{i}\frac{\delta}{\delta\phi}\vec{\nabla}\phi\,\psi(\phi) = \vec{P}\,\psi(\phi) . \tag{7.62}$$

Under a Lorentz transformation, $\psi(\phi)$ transforms to

$$\psi'(\phi) = e^{(i/2)\alpha_{\mu\nu}J^{\mu\nu}}\psi(\phi) . \tag{7.63}$$

It is easy to see that ψ' is also an eigenstate of P^μ. The new eigenvalues $P'^\mu \equiv (E', \vec{P}')$ are the Lorentz transform of the old eigenvalues $P^\mu \equiv (E, \vec{P})$. For the vacuum states, we have $P^\mu = 0$ which implies $P'^\mu = 0$.

We can obtain the vacuum tunneling amplitude from the WKB wave functional as outlined in Sec. III. For gauge theory one finds that both the instanton number and the tunneling amplitude are proportional to the total phase space VT. The transition amplitude per unit phase space (or, per unit instanton) is then Lorentz invariant.

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[10]Under appropriate boundary conditions Eqs. (7.13) and (7.15) imply

$$D_\mu F_{\mu\nu} = 0, \quad \mu,\nu = (\tau, 1, 2, 3).$$

We can obtain the above equation directly from the variational principle $\delta\int d\lambda\,[U(\lambda) - E]^{1/2} = 0$ by considering the field configuration in a general gauge:

$$A^0 = a(x, \lambda(t))\,\dot{\lambda}, \quad \vec{A} = \vec{A}(x, \lambda(t)).$$

The variation $\delta\vec{A}$ leads to Eq. (7.13) and the variation of δa leads to $\vec{D}\cdot\vec{\epsilon} = 0$ as desired.

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