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Pomeranchuk-type theorems for total and elastic cross sections

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In the framework of local field theory the asymptotic properties of the crossing-odd forward scattering amplitude $F(E)$ are investigated, E being the laboratory energy. Confronting the analyticity properties of the logarithm of the amplitude with the Froissart-Martin bound, we obtain a series of sufficient conditions for a fast asymptotic vanishing of $\mathrm{Im}F(E)/E$ and $\mathrm{Re}F(E)/E$ in the mean. Analogous conditions for the vanishing of the total-cross-section difference $\Delta\sigma(E)$ follow via the optical theorem. We also find conditions for the existence of Meiman's generalized high-energy limit of $\Delta\sigma(E)$. Finally, two independent asymptotic bounds on $\Delta\sigma(E)$ are derived. The method is extended to nonforward scattering to give asymptotic bounds on differential cross sections.

I. INTRODUCTION

Since Pomeranchuk[1] proved from dispersion relations and some additional assumptions that the total cross sections for particle-particle and antiparticle-particle scattering become equal at infinite energy, considerable progress has been made in generalizing the theorem and relaxing the assumptions.

It is well known that the asymptotic vanishing of the total-cross-section difference $\Delta\sigma(E)$ has not been proved from the principles of local field theory. The ratio $\Delta\sigma(E)/\sigma_s(E)$, however, should tend to zero with E tending to infinity (E being the laboratory energy) if the crossing-even total cross section $\sigma_s(E)$ rises unboundedly. This was proved by using unitarity several times in the past,[2-4] by making assumptions of various degrees of rigor.

On the other hand, defining a generalized high-energy limit $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$ in which a point set of zero asymptotic density can be omitted, Meiman[5] has recently obtained a condition of the existence and vanishing of this limit. Assuming that $\Delta\sigma(E)$ does not change sign indefinitely and that the real part of the crossing-odd forward scattering amplitude $F(E)$ does not grow too fast,

$$\mathrm{Re}F(E)=o(E\ln E)\,, \tag{1.1}$$

Meiman has shown that $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$ exists and is equal to zero. This is a generalization of earlier results obtained by other authors.[4,6,7]

Experimental data strongly suggest that $\Delta\sigma(E)$ does tend to zero with increasing energy. Moreover, the data indicate that a rather fast asymptotic vanishing of $\Delta\sigma(E)$ takes place. It is therefore worthwhile to search for sufficient conditions of a *fast* vanishing of $\Delta\sigma(E)$. The present paper is devoted to problems related to this subject.

Starting from the basic results of axiomatic field theory as usually adopted in the S-matrix framework, we obtain high-energy asymptotic correlations for the crossing-odd forward scattering amplitude $F(E)$ (Sec. II, theorem 1). Then we establish necessary and sufficient conditions of the unbounded rise, of the boundedness and of the asymptotic vanishing of $\Delta\sigma(E)$ (Sec. II). (These results are nothing but a slight generalization of Theorem 6 and Corollary 2 of Ref. 8.) Then, in Sec. III, Theorem 1 is used for finding various sufficient conditions of a fast asymptotic vanishing of the total-cross-section difference $\Delta\sigma(E)$ and of the whole crossing-odd forward scattering amplitude $F(E)$. To this end, we introduce the integrals

$$\Sigma_{\alpha,\beta}(E)=\int_{E_0}^{E}\Delta\sigma(E')E'^{\alpha}(\ln E')^{\beta}dE' \tag{1.2}$$

and

$$R_{\gamma,\delta}(E)=\int_{E_0}^{E}\frac{\mathrm{Re}F(E')}{E'}E'^{\gamma}(\ln E')^{\delta}dE'\,, \tag{1.3}$$

with E_0 being a positive constant. The asymptotic behavior of $F(E)$ can conveniently be investigated by considering the convergence properties of $\Sigma_{\alpha,\beta}(E)$ and $R_{\gamma,\delta}(E)$ with $E\to\infty$. The highest values of α, β and γ, δ for which $\lim_{E\to\infty}\Sigma_{\alpha,\beta}(E)$ and $\lim_{E\to\infty}R_{\gamma,\delta}(E)$ are finite indicate how fast $\Delta\sigma(E)$ and $\mathrm{Re}F(E)/E$, respectively, tend to zero. We find the general form of constraints to be imposed on the phase of $F(E)$ in order to ensure the convergence of the limits $\lim_{E\to\infty}\Sigma_{\alpha,\beta}(E)$ and $\lim_{E\to\infty}R_{\gamma,\delta}(E)$ for given α,β and γ,δ, respectively. Moreover, the same constraints are shown to imply that $F(E)$ obeys certain sum rules [see, e.g., (3.2) and (3.9)].

Among the cases considered, the integral

$\Sigma_{-1,0}(E)$ is of particular interest. We obtain in theorem 2 a new sufficient condition of its convergence with $E\to\infty$. Then, in the Appendix, we prove two theorems which demonstrate that the convergence properties of this integral with E tending to infinity are relevant for the existence of Meiman's generalized limit $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$ mentioned above.

The reader who is interested mainly in applications of Sec. III can focus on Sec. IV, where the results obtained are discussed under simplifying assumptions related to the existence of certain high-energy limits. It turns out, for example, that if the value of $\lim_{E\to\infty}\ln E\,\mathrm{Im}F(E)/\mathrm{Re}F(E)$ lies outside the narrow interval $\langle-\pi,-\pi/2\rangle$ then $\liminf_{E\to\infty}|\Delta\sigma(E)|$ always equals zero. Reversing this statement we infer that if the Pomeranchuk theorem were violated [in the sense that $\lim_{E\to\infty}\Delta\sigma(E)\neq0$], then $\xi(E)=\mathrm{Im}F(E)/\mathrm{Re}F(E)$ should obey severe restrictions in the asymptotic region, being confined, at least on an infinite sequence of energies, between the curves $-\pi/\ln E$ and $-\pi/(2\ln E)$. Thus, the assumption that $\lim_{E\to\infty}\Delta\sigma(E)$ is not zero has very restrictive consequences for the asymptotic behavior of the phase, even without assuming any specific high-energy model.

In Sec. V, several consequences are drawn from the results obtained. It is shown, in particular, that $\Delta\sigma(E)$ is asymptotically bounded either by (5.1) or by (5.3). Besides this, $\Delta\sigma(E)$ must satisfy either (5.5) or (5.6). This means that each of these two pairs of bounds amounts to one unconditional bound on $\Delta\sigma(E)$. Similar pairs of bounds are also obtained for nonforward scattering [see relations (5.9), (5.10) and (5.11), (5.12)].

II. ASYMPTOTIC PHASE-MODULUS CORRELATIONS

We shall assume the following general properties of the crossing-odd forward scattering amplitude $F(E)$:

(A1) $F(E)$ is an analytic function of complex E in the region $D_r=\{E;\ \mathrm{Im}E>0,|E|>r,r>0\}$ and is continuous on the closure of D_r.

(A2) For every $E\in D_r$ we have $F(E)=-F^*(-E^*)$.

(A3) $F(E)$ is bounded by a polynomial of degree N in E for large $|E|$ (N being independent of the direction).

(A4) $F(E)$ satisfies the Froissart-Martin bound on the real axis

$$\left|\frac{F(E)}{E\ln^2E}\right|<\mathrm{const}$$

for large enough energies.

The connection of these assumptions with the general properties of axiomatic quantum field theory is discussed, for instance, in Refs. 7 and 8.

The amplitude $F(E)$ is normalized in such a way that the optical theorem has asymptotically the following form:

$$\Delta\sigma(E)=\mathrm{Im}F(E)/E\,. \tag{2.1}$$

Unless otherwise stated, E is supposed to be real and positive.

Definition. Using the principal branch of $\tan^{-1}x$, we shall denote by $\arctan_c x$ the function defined by the following relations:

$$\arctan_c x=\tan^{-1}x\quad\text{for } x>-\cot(\pi c/2)$$
$$\arctan_c x=\tan^{-1}x+\pi\quad\text{for } x<-\cot(\pi c/2) \tag{2.2}$$

for positive c,

$$\arctan_c x=\tan^{-1}x\quad\text{for } x<-\cot(\pi c/2)$$
$$\arctan_c x=\tan^{-1}x-\pi\quad\text{for } x>-\cot(\pi c/2) \tag{2.3}$$

for negative c and $\arctan_0 x=\tan^{-1}x$ for all real x. The function $\arctan_c x$ satisfies the following inequality for $|c|<2$:

$$-\tfrac12+c/2\leq\frac{1}{\pi}\arctan_c x\leq\tfrac12+c/2\,. \tag{2.4}$$

Note that a dependence of c on x is not excluded by this definition. We shall work with $x=\xi(E)$ and $c=a+b/\ln E$, where a and b are arbitrary real numbers (a being subject to the constraint $-1\leq a\leq1$).

Further, for the sake of brevity, we introduce the symbol $f(E)$ for the function

$$f(E)=i(-iE)^a[\ln(-iE)]^bF(E) \tag{2.5}$$

defined for every $E\in D_r$, $r>1$.

The subsequent Theorem 1 represents a simple generalization of Theorem 2 of Ref. 8.

Theorem 1. Assume that $F(E)$ possesses properties (A1) to (A4) and that the following conditions are satisfied for large enough energies and for some real a and b, $|a|\leq1$:

$$\lim_{E\to\infty}F(E)\ln^bE\,E^{a-2}=0\,, \tag{2.6}$$

$$\mathrm{Im}f(E)\ \text{does not change sign}\,, \tag{2.7}$$

$$\int_{E_0}^{\infty}\mathrm{Im}f(E)\,\frac{dE}{E}\ \text{diverges}\,. \tag{2.8}$$

Then there exists an infinite sequence $\{E_k\}$ of points, $\lim_{k\to\infty}E_k=\infty$, on which

$$D_1E_k\exp\left[-\frac{2}{\pi}\int_{E_0}^{E_k}\arctan_c\xi(E)\,\frac{dE}{E}\right]\leq|F(E_k)|\leq D_2E_k\exp\left[-\frac{2}{\pi}\int_{E_0}^{E_k}\arctan_c\xi(E)\,\frac{dE}{E}\right], \tag{2.9}$$

where D_1, D_2 are constants independent of E_k and $c=a+b/\ln E$.

Remark 1. Theorem 1 remains unchanged if condition (2.8) is replaced by the weaker condition

$$\eta \int_{E_0}^{\infty} \text{Im} f(E)\, \frac{dE}{E} > \tfrac{1}{2}\eta \int_0^{\pi} \text{Re} f(E_0 e^{i\varphi})\, d\varphi\,, \quad E_0 > r \tag{2.10}$$

where η equals 1 or -1, the sign being determined by the requirement $\eta\,\text{Im} f(E) \geq 0$ at sufficiently high energies. This result follows immediately from Theorem A of Ref. 8. It is easy to express the conditions (2.8) and (2.10) in terms of the amplitude $F(E)$ by using relation (2.5). It will be shown below that the condition (2.10) leads to a sum rule for $F(E)$ which, however, can be of practical use only if some information on the amplitude for complex E is available allowing us to evaluate the right-hand side of (2.10).

The following Corollary 1 yields necessary and sufficient conditions for the various asymptotic behaviors of $\Delta\sigma(E)$.

Corollary 1. Let $F(E)$ satisfy (A1)–(A4) and let $\text{Im} f(E) = \text{Im} F(E) - b\pi\, \text{Re} F(E)/2\ln E$ not change sign for some $b < -2$, beyond some energy. If

$$\left| \lim_{E\to\infty} \left[\Sigma_{1,b}(E) - \frac{b\pi}{2} R_{1,b-1}(E) \right] \right| = \infty$$

and the limits $\lim_{E\to\infty} \Delta\sigma(E)$, $\lim_{E\to\infty} \tau_c(E)$ exist, $\tau_c(E)$ being defined as

$$\tau_c(E) = \frac{2}{\pi} \int_{E_0}^{E} \arctan_c \xi(E')\, \frac{dE'}{E'} + \tfrac{1}{2}\ln[1 + \xi^{-2}(E)] \tag{2.11}$$

with $c = 1 + b/\ln E$, then a necessary and sufficient condition for

$$\begin{aligned} &\lim_{E\to\infty} |\Delta\sigma(E)| = +\infty\,, \\ &0 < \lim_{E\to\infty} |\Delta\sigma(E)| < +\infty\,, \\ &\lim_{E\to\infty} \Delta\sigma(E) = 0 \end{aligned} \tag{2.12}$$

is

$$\begin{aligned} &\lim_{E\to\infty} \tau_c(E) = -\infty\,, \\ &-\infty < \lim_{E\to\infty} \tau_c(E) < +\infty\,, \\ &\lim_{E\to\infty} \tau_c(E) = +\infty\,, \end{aligned} \tag{2.13}$$

respectively.

It is perhaps useful to illustrate the conditions (2.13) by considering some special asymptotic behavior of $\xi(E)$. If, for instance, $\xi(E)$ tends to any nonvanishing limit then, in view of (2.2) (see Fig. 1), $\tau_c(E)$ tends to plus infinity and, consequently, $\lim_{E\to\infty} \Delta\sigma(E)$ necessarily vanishes. If $\xi(E)$ behaves so that $\lim_{E\to\infty} \xi(E)\ln E$ is in the interval $\langle -\pi, -\pi/2)$, then $|\Delta\sigma(E)|$ tends to infinity.

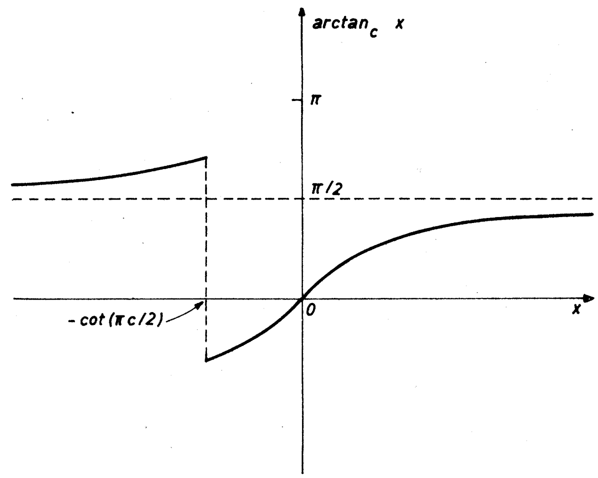

FIG. 1. Plot of the function $\arctan_c x$ for a value of c between 0 and 1. The discontinuity occurs at the point $x = -\cot \pi c/2$.

III. SUFFICIENT CONDITONS FOR A FAST VANISHING OF THE TOTAL-CROSS-SECTION DIFFERENCE AND OF THE REAL PART

The asymptotic behavior of the total-cross-section difference $\Delta\sigma(E)$ as well as of the real part $\text{Re}F(E)$ can be analyzed by considering the convergence properties of the integrals $\Sigma_{\alpha,\beta}(E)$ and $R_{\gamma,\delta}(E)$ defined by relations (1.2) and (1.3).

The integral $\Sigma_{-1,0}(E)$ was investigated by a number of authors and useful criteria of its convergence were found. Recently, Meiman[5] has found a condition for the convergence of $\Sigma_{-1,-1}(E)$ too.

In this section we present theorems giving sufficient conditions for the convergence of $\Sigma_{\alpha,\beta}(E)$ and $R_{\gamma,\delta}(E)$ in the following cases:

(i) $\alpha = -1$, $\beta = 0$;
(ii) $\alpha = 0$, $\beta = -1 - \epsilon$, $\gamma = 0$, $\delta = 0$;
(iii) α is any nonzero real number between -1 and 1, $\beta = 0$, $\gamma = \alpha$, $\delta = 0$.

The convergence is proved by showing that the divergence would contradict the Froissart-Martin bound.

Theorem 2. Assume that $F(E)$ fulfills (A1)–(A4), and that $\Delta\sigma(E)$ does not change sign for large enough energies, $E > E_0$. If

$$\int_{E_0}^{\infty} [\arctan_{-1} \xi(E) + \pi/\ln E]\, \frac{dE}{E} = -\infty \tag{3.1}$$

then $\Sigma_{-1,0}(E)$ converges for $E \to \infty$, and, moreover, the following sum rule holds:

$$\eta \int_{E_0}^{\infty} \Delta\sigma(E)\, \frac{dE}{E} < \tfrac{1}{2}\eta \int_0^{\pi} [\text{Re}F(E_0 e^{i\varphi}) \cos\varphi + \text{Im}F(E_0 e^{i\varphi}) \sin\varphi]\, \frac{d\varphi}{E_0} \tag{3.2}$$

where $E_0 > r$ and $\eta = \mathrm{sign}\Delta\sigma(E)$.

Proof. Setting $a=-1$, $b=0$ we see that assumptions (2.6) and (2.7) of theorem 1 are obviously fulfilled. Suppose that $\lim_{E\to\infty}\Sigma_{-1,0}(E)$ diverges. Then (2.8) holds and, combining (2.9) with the Froissart-Martin bound, we get a contradiction with (3.1):

$$\exp\left\{-\frac{2}{\pi}\int_{E_0}^{E_k}[\arctan_{-1}\xi(E)+\pi/\ln E]\,\frac{dE}{E}\right\}\leq \mathrm{const}. \tag{3.3}$$

Thus, $\Sigma_{-1,0}(E)$ cannot diverge as $E\to\infty$.

The sum rule (3.2) follows from relation (2.10) of remark 1.

Remark 2. Let us point out two cases in which condition (3.1) is satisified:

(i) The signs of $\Delta\sigma(E)$ and $\mathrm{Re}F(E)$ are asymptotically equal;

(ii) $\xi(E)\leq -\pi(1+\epsilon)/\ln E$, for some $\epsilon>0$, and all $E>E_0$.

In case (i), the real part $\mathrm{Re}F(E)$ can be shown, by using the relation (2.9), to obey the asymptotic bound

$$|\mathrm{Re}F(E_k)/E_k|\leq \mathrm{const}. \tag{3.4}$$

[Cf. an analogous relation[8] for the crossing-even amplitude $F_s(E)$, which shows that the unbounded rise of $\sigma_s(E)$ is incompatible with the negative sign of $\mathrm{Re}F_s(E)$]. In case (ii), (3.4) holds too, but, moreover, it can be proved that the integral $R_{0,0}(E)$ is convergent. This is stated below in Theorem 3.

Theorem 2 and its consequences deserve a closer discussion. It is well known that the Pomeranchuk theorem does not tell the rate by which $\Delta\sigma(E)$ should approach zero with $E\to\infty$. On the other hand, Theorem 2 establishes conditions under which the integral

$$\lim_{E\to\infty}\Sigma_{-1,0}(E)=\int_{E_0}^{\infty}\Delta\sigma(E)\frac{dE}{E} \tag{3.5}$$

is convergent and satisfies the inequality (3.2). Now the convergence of the integral (3.5) means that $\Delta\sigma(E)$ must vanish sufficiently quickly in the mean. If, for instance, $\Delta\sigma(E)$ does not oscillate too much it must satisfy the inequality

$$|\Delta\sigma(E)|\leq K/(\ln E\ln\ln E\cdots\ln\cdots\ln E) \tag{3.6}$$

for all E above some E_0, for some positive constant K and for any finite number of factors indicated in the denominator. The convergence properties of the integral (3.5) have been used since 1960 by various authors[2,6,9] to characterize the vanishing rate of $\Delta\sigma(E)$ and a number of different sufficient conditions of its convergence have been derived. Theorem 2 gives a new one.

We would like to stress in this connection that the convergence of the integral (3.5) is closely related to the existence and the vanishing of the generalized high-energy limit $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$, which was introduced by Meiman.[5] We recall its definition in the beginning of the Appendix. Then we show (see Theorem A) that if $\Sigma_{-1,0}(E)$ converges for $E\to\infty$ and $\Delta\sigma$ has a fixed sign above some energy, then the generalized limit $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$ exists and is equal to zero. This implies, in particular, that $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$ equals zero under the assumptions of theorem 2.

Being a sufficient condition for the vanishing of Meiman's limit, the convergence of $\Sigma_{-1,0}(E)$ with $E\to\infty$ is not a necessary one, but is not far from being so, as shown in Theorem B of the Appendix. This theorem states that the convergence of (3.5) does *not* imply that $\lim_{E\to\infty}\Delta\sigma(E)l(E)=0$, however slow the unbounded rise of the function $l(E)$ may be.

It is interesting to discuss the relation between Meiman's condition (1.1) and our condition (3.1). Both of them are sufficient conditions for the vanishing of $\mathrm{Lim}_{E\to\infty}\Delta\sigma(E)$, but are very different in form and by no means equivalent. It is not difficult to find amplitudes satisfying (1.1) and violating (3.1); one example is

$$F(E)=E[\ln(-iE)]^{\alpha} \tag{3.7}$$

with $\alpha<1$ and $E\in D_r$. On the other hand, functions obeying (3.1) but disobeying (1.1) are not so trivial to construct. One example is

$$F(E)=E\left\{\frac{1}{\ln(-iE)}+\sum_{n=1}^{\infty}\frac{2a_nb_n\ln(b_n)}{[1-ia_n(E-b_n)][1-ia_n(E+b_n)]}\right\} \tag{3.8}$$

with $a_n=1000\exp(n^3)$ and $b_n=\exp(n^2)$. It can be proved by a rather lengthy reasoning that this function satisfies both the requirements (A1) to (A4) and the conditions of Theorem 2, but violates (1.1). We shall be glad to supply interested readers with the proof.

This result indicates, on the other hand, that there is only a very narrow class of functions by which our theorem 2 extends the class of amplitudes defined by Meiman's general condition (1.1). It is worth mentioning in this connection that according to (3.4) and Theorem 3 (see below) every nonoscillating amplitude satisfying (3.1) satisfies (1.1) too.

Let us mention that the integral (3.5) has a very simple geometrical interpretation. Denoting by $\sigma_+(E)$ and $\sigma_-(E)$ the particle-particle and the antiparticle-particle total cross section, respectively, we can easily see that (3.5) represents the area between these two cross sections plotted in logarithmic scale. Thus, to illustrate Theorem 2 by an example, we conclude that if the signs of $\Delta\sigma(E)$ and Re$F(E)$ are asymptotically equal [see point (i) of Remark 2], then $\sigma_+(E)$ and $\sigma_-(E)$ must approach each other so quickly that the area confined by them is finite in logarithmic scale. Apart from this example, the general statement is that the area is always finite when the conditions of theorem 2 are satisfied.

The following Theorems 3 and 4 point out sufficient conditions of the convergence of $\lim_{E\to\infty}\Sigma_{\alpha,\beta}(E)$ and $\lim_{E\to\infty}R_{\gamma,\delta}(E)$ with $(\alpha,\beta)\neq(-1,0)$. While $\Sigma_{-1,0}(E)$ is interpreted as the area between the two total cross sections in logarithmic scale, $\Sigma_{\alpha,\beta}(E)$ for some other values of α and β represents the area weighted by some weight function in the integrand.

Theorem 3. Let $F(E)$ satisfy (A1)–(A4) and let Re$F(E)$ and Im$F(E)$ have fixed opposite signs above some E_0. If (3.1) holds then both $\Sigma_{0,-1-\epsilon}(E)$ and $R_{0,0}(E)$ converge for $E\to\infty$ and any $\epsilon>0$. Moreover,

$$\eta\int_{E_0}^{\infty}\frac{\mathrm{Re}F(E)}{E}\,dE<-\frac{\eta}{2}\int_0^{\pi}\mathrm{Im}F(E_0e^{i\varphi})d\varphi\,, \tag{3.9}$$

where $\eta=\mathrm{sign\,Re}F(E)$ for $E>E_0>r$.

The proof is entirely analogous to that of Theorem 2. The convergence of $\lim_{E\to\infty}R_{0,0}(E)$ and relation (3.9) are proved by setting $a=0$ and $b=0$, whereas the convergence of $\lim_{E\to\infty}\Sigma_{0,-1-\epsilon}(E)$ follows from the case $a=0$, $b=-\epsilon$.

Theorem 4. Let $F(E)$ possess the properties (A1) to (A4) and let $\Delta\sigma(E)$ not change its sign above some energy. If $\xi(E)$ satisfies one of the following constraints:

$$\lim_{E\to\infty}\sup\,[\xi(E)\ln E]<-\pi \tag{3.10}$$

and

$$\lim_{E\to\infty}\inf\xi(E)>-\cot(\tfrac{1}{2}\pi\alpha)$$

with $0<\alpha<1$, or

$$\lim_{E\to\infty}\inf\xi(E)>-\cot(\tfrac{1}{2}\pi\alpha) \tag{3.11}$$

with $-1<\alpha<0$, then the integrals $\Sigma_{\alpha,0}(E)$ and $R_{\alpha,0}(E)$ converge for $E\to\infty$.

The theorem can be proved analogously as Theorems 2 and 3 by using Theorem 1 and setting $a=\alpha$ and $b=0$. Note that the symbols "inf" and "sup" can be omitted if the corresponding limits exist.

There are two regions for which the fast vanishing of $\Delta\sigma(E)$ was not proved. The first one is defined by the requirement that $\xi(E)$ is negative for large enough energies and satisfies the inequality

$$\lim_{E\to\infty}\inf[\xi(E)\ln E]>-\pi/2\,; \tag{3.12}$$

the second one is defined by

$$\lim_{E\to\infty}\sup[\xi(E)\ln E]\leqslant-\pi/2\,,\qquad \lim_{E\to\infty}\inf[\xi(E)\ln E]\geqslant-\pi\,. \tag{3.13}$$

If $\xi(E)$ satisfies (3.12), $\Delta\sigma(E)$ can be shown to tend to zero but the rate may be extraordinarily slow. This is shown in the subsequent Theorem 5 and Remark 4. The second case (3.13), in which $\lim_{E\to\infty}\Delta\sigma(E)$ may even be nonvanishing, is discussed in Sec. IV.

Theorem 5. Let $F(E)$ fulfill (A1) to (A4). If $\xi(E)$ is negative for every finite, sufficiently high energy and

$$\lim_{E\to\infty}\inf[\xi(E)\ln E]>-\pi/2\,, \tag{3.14}$$

then

$$\lim_{E\to\infty}\inf|\Delta\sigma(E)|=0\,. \tag{3.15}$$

Proof. We set $a=-1$, $b=0$ in Theorem 1; then Im$f(E)=-\Delta\sigma(E)$. If $\Delta\sigma(E)$ changes sign indefinitely then (3.15) follows. Thus we can assume that $\Delta\sigma(E)$ does not change sign above some energy and therefore condition (2.7) is satisfied. Further, if $\Sigma_{-1,0}(E)$ converges then (3.15) follows. Thus, we can assume that (2.8) is satisfied and according to (2.9) we have

$$|\Delta\sigma(E_k)|\leqslant\frac{D_2\exp[-(2/\pi)\int_{E_0}^{E_k}\arctan_{-1}\xi(E)dE/E]}{[1+\xi^{-2}(E)]^{1/2}}\,. \tag{3.16}$$

Since $\arctan_{-1}\xi(E)=\tan^{-1}\xi(E)\sim\xi(E)$ the theorem follows now by using relation (3.14).

Remark 3. To give an example of a slowly vanishing $\Delta\sigma(E)$ let us consider the function of $E\in D_r$,

$$F(E)=E(\ln E-i\pi/2)^{1-\epsilon}\,,$$

with $0<\epsilon<1$. Since

$$\lim_{E\to\infty}[\xi(E)\ln E]=-\frac{\pi}{2}(1-\epsilon)\,,$$

we obtain an agreement with (3.12) and, besides, $\Delta\sigma(E)$ behaves like $-(\pi/2)(1-\epsilon)(\ln E)^{-\epsilon}$.

Details about the correlation between $\Delta\sigma(E)$ and $\xi(E)$ follow from relation (3.16).

IV. DISCUSSION

To give a better insight into the results obtained, we shall discuss several special cases after making some simplifying assumptions, which are specified below. The general properties (A1) to (A4) of the amplitude $F(E)$ are assumed throughout.

Theorems 1 to 5 indicate that there are the following important "lines of demarcation" in the

$\xi(E)$ asymptotic energy dependence:

$$\text{(i) } \xi(E)=C\,,$$
$$\text{(ii) } \xi(E)=-\pi/\ln E\,, \quad (4.1)$$
$$\text{(iii) } \xi(E)=-\pi/(2\ln E)\,,$$

C being a constant (negative, positive, or zero).

It is interesting to see which energy dependence of $\xi(E)$ is preferred by the existing high-energy data.[10] Both hadron-hadron total-cross-section measurements and K_S^0 regeneration experiments point toward a negative-power-like decrease of $\Delta\sigma(E)$,

$$\Delta\sigma(E)=\text{const}\times E^{-\lambda}\,, \quad (4.2)$$

and

$$\xi(E) \text{ tending to a positive constant}\,, \quad (4.3)$$

where λ is a constant close to $\frac{1}{2}$. If this is assumed to be valid up to infinite energy we conclude that the measured $\xi(E)$ lies above the lines (ii) and (iii) of (4.1), the position with respect to the line (i) depending on the value of C.

Theorems 1 to 5 derived in the preceding sections lead to the following consequences:

(1) If $\lim_{E\to\infty}\xi(E)\ln E$ (finite or infinite) exists and lies outside the interval $\langle -\pi/2, -\pi\rangle$ (see Fig. 2), then

$$\lim_{E\to\infty} \inf \Delta\sigma(E)=0\,.$$

This is a consequence of theorems 2 and 5. Let us remark that this asymptotic behavior is supported by experimental data.

(2) If $\lim_{E\to\infty}\xi(E)\ln E$ is between $-\pi/2$ and $-\pi$, then either $\Delta\sigma(E)$ does not tend to zero or both integrals $\Sigma_{1,b}(E)$ and $R_{1,b-1}$ must converge with $E\to\infty$ for every $b<-2$.

This follows from corollary 1. Conversely, if $\lim_{E\to\infty}\Delta\sigma(E)\neq 0$ then $\xi(E)$ is asymptotically confined between $-\pi/(2\ln E)$ and $-\pi/\ln E$. Thus, the narrow strip in Fig. 2 represents the only possible (nonoscillating) behavior of $\xi(E)$ which is compatible with $\lim_{E\to\infty}\Delta\sigma(E)\neq 0$. Experimental evidence points toward a very different kind of asymptotic behavior of $\xi(E)$. In accordance with this, $\Delta\sigma(E)$ should approach zero asymptotically.

(3) If $\Delta\sigma(E)$ does not change its sign above some energy and if, in addition, one of the following conditions is satisfied:

$$\text{sign}\Delta\sigma(E)=\text{sign}\,\text{Re}F(E) \quad \text{for } E>E_0\,, \quad (4.4)$$

$$\xi(E)\leqslant -\pi(1+\epsilon)/\ln E \quad \text{for } E>E_0\,, \quad (4.5)$$

$$\lim_{E\to\infty}\xi(E) \text{ exists and } |\text{Re}F(E)/E| \text{ is bounded by a constant for } E>E_0\,, \quad (4.6)$$

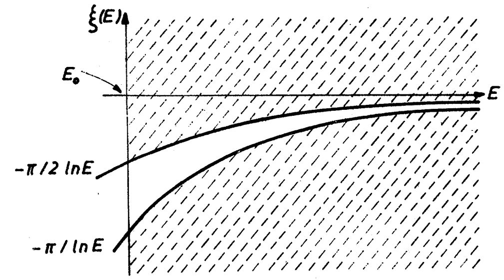

FIG. 2. The assumption that $\lim_{E\to\infty}\Delta\sigma(E)\neq 0$ implies that $\xi(E)$ should, at least on an infinite sequence of energies, fall between the two curves shown in the figure. Experimental data give $\xi(E)$ nearly constant with the value around 1.

then the integral

$$\int_{E_0}^{\infty}\Delta\sigma(E)\frac{dE}{E} \quad (4.7)$$

converges and, moreover, the sum rule (3.2) must be satisfied [see Theorem 2 for the cases (4.4) and (4.5) and Theorem 1 for the case (4.6)]. Consequently, $\text{Lim}_{E\to\infty}\Delta\sigma(E)=0$ in the sense of Meiman.[5] Let us mention that high-energy data give equal signs for $\Delta\sigma(E)$ and $\text{Re}F(E)$ in accordance with relation (4.4) and also indicate that relation (4.6) is satisfied. It should be pointed out that condition (4.6) was not explicitly shown here to be sufficient for the convergence of (4.7). The reason is that a more general statement has been proved already by Grunberg and Truong [see Ref. 2, relation (74)].

(4) For the convergence of the integrals

$$\int_{E_0}^{\infty}\Delta\sigma(E)E^{\alpha}dE \quad\text{and}\quad \int_{E_0}^{\infty}\text{Re}F(E)E^{\alpha-1}dE \quad (4.8)$$

with $\alpha\in(-1,0)$ it is sufficient to require

$$\xi(E)\geqslant(1-\epsilon)\cot\frac{\pi|\alpha|}{2} \quad (4.9)$$

(see Theorem 4). Experimental data on K_S^0 regeneration[11] show that $\xi(E)$ is nearly constant between $E=10$ and $E=50$ GeV, the measured value being 1.1 ± 0.2. If we suppose that these data determine asymptotic behavior, we can infer that the integrals (4.8) converge for all $\alpha<\alpha_0$, α_0 being equal to -0.47 ± 0.06.

(5) If

$$\xi(E)\leqslant -\pi(1+\epsilon)/\ln E\,, \quad (4.10)$$

then the integrals

$$\int_{E_0}^{\infty}\Delta\sigma(E)\frac{dE}{(\ln E)^{1+\epsilon}} \quad\text{and}\quad \int_{E_0}^{\infty}\text{Re}F(E)\frac{dE}{E} \quad (4.11)$$

converge and, moreover, the sum rule (3.9) holds (see theorem 3).

(6) Finally, for the convergence of the integrals

$$\int_{E_0}^{\infty} \Delta\sigma(E)E^{\alpha}dE \quad \text{and} \quad \int_{E_0}^{\infty} \text{Re}(E)E^{\alpha-1}dE \qquad (4.12)$$

with $\alpha \in (0, 1)$, it is sufficient to require

$$-\pi(1+\epsilon)/\ln E \geq \xi(E) \geq -(1+\epsilon)\cot(\tfrac{1}{2}\pi\alpha) \qquad (4.13)$$

(see theorem 4).

Let us discuss the last two correlations in connection with the existing experimental data. Assume that the power law (4.2) is valid up to infinite energies. Then, inserting (4.2) into the relevant integrals in (4.11) and (4.12), we immediately see that they are divergent. Consequently, the inequalities (4.10) and (4.13) should not be supported by experimental data on $\xi(E)$. This is really the case, as is seen if the fit (4.3) is extrapolated toward infinite energies.

V. CONCLUDING REMARKS

It is well known that the possibility of an asymptotically nonvanishing or even diverging total-cross-section difference $\Delta\sigma(E)$ has not been disproved when using the principles of local field theory. In order to prove that $\Delta\sigma(E)$ should asymptotically vanish, additional assumptions have to be imposed either on the real part or on the phase of the antisymmetric forward scattering amplitude (see, e.g., Refs. 1, 2, 5–9, and 12–16). Our formalism has been based on the analyticity of the logarithm of the scattering amplitude. The additional assumptions have the form of asymptotic restrictions on $\xi(E)$ and, by this, also on the phase.

Our approach can give a number of new, physically interesting asymptotic theorems and generalize results which were obtained by some authors earlier. The use of the phase language is advantageous, for instance, in the kaon-nucleon scattering, in which case the phase of the antisymmetric amplitude is directly measured in the K_S^0 regeneration experiments and is, consequently, experimentally better known than the real part, which is obtained by subtracting two real parts measured with great errors.

The phase formalism has allowed us to derive a number of new sufficient conditions of a fast asymptotic vanishing of $\Delta\sigma(E)$. Our results also shed light on the main problem of the Pomeranchuk theorem, by showing what would happen if $\Delta\sigma(E)$ had a nonvanishing high-energy limit. We show in Sec. IV (see also Fig. 2) that $\xi(E)$ is severely restricted in such a case: It either oscillates at infinity or is asymptotically confined in a narrow strip.

A number of interesting consequences can be derived from the theorems obtained. For illustration, let us present here several new high-energy bounds.

Considering Theorem 2 of Sec. III, we see that there are two asymptotic upper bounds on the energy average of $\Delta\sigma(E)$

$$\left| \int_{E_0}^{\infty} \Delta\sigma(E)\frac{dE}{E} \right| < \infty , \qquad (5.1)$$

$$\int_{E_0}^{\infty} [\arctan_{-1}\xi(E) + \pi/\ln E]\frac{dE}{E} > -\infty . \qquad (5.2)$$

Theorem 2 states that either (5.1) or (5.2) must hold. The bound (5.1) implies that $\Delta\sigma(E)$ has to approach zero for $E \to \infty$ at least on some sequence and, moreover, that Meiman's limit $\text{Lim}_{E\to\infty}\Delta\sigma(E)$ exists and equals zero. Relation (5.2), on the other hand, implies that a sequence $\{E_k\}$ of energies with the property $\lim_{k\to\infty}E_k = \infty$ must exist such that

$$-(1+\epsilon)\pi/\ln E < \xi(E_k) \leq 0 .$$

This produces the following bound on $\Delta\sigma$:

$$|\Delta\sigma(E_k)| \leq (1+\epsilon)\pi|\text{Re}F(E_k)|/(E_k \ln E_k); \qquad (5.3)$$

the signs of $\Delta\sigma(E_k)$ and $\text{Re}F(E_k)$ are necessarily opposite. If $\Delta\sigma(E)$ does change sign above every energy, i.e., the assumption of Theorem 2 is not satisfied, relation (5.3) follows trivially from the continuity of $\Delta\sigma(E)$.

We conclude that the total-cross-section difference $\Delta\sigma(E_k)$ is asymptotically bounded either by (5.1) or by (5.3). Being complementary, these two bounds can be looked upon as one unconditional [that is, based on assumptions (A1) to (A4) only] asymptotic bound on $\Delta\sigma(E_k)$. Comparing this result with the bound of Roy and Singh,[12]

$$\lim_{E\to\infty} |\Delta\sigma(E)/\ln E| = \text{const} \qquad (5.4)$$

[which holds if $\Delta\sigma(E)E$ is monotonic for sufficiently large E], we see that the bound (5.1) and (5.3) represents an improvement of (5.4) in all cases in which $\text{Re}F(E)$ does not saturate the Froissart-Martin bound.

An analogous pair of complementary bounds can be derived by combining theorem 2 and theorem 5. It follows that either

$$\lim_{E\to\infty} \inf|\Delta\sigma(E)| = 0 \qquad (5.5)$$

or

$$\frac{\pi(1-\epsilon)|\text{Re}F(E_k)|}{2E_k \ln E_k} < |\Delta\sigma(E_k)| < \frac{\pi(1+\epsilon)|\text{Re}F(E_k)|}{E_k \ln E_k} , \qquad (5.6)$$

the signs of $\Delta\sigma(E)$ and $\text{Re}F(E)$ again being necessarily opposite. Note that the first inequality in (5.6) follows only if the sign of $\Delta\sigma(E)$ remains fixed above some energy.

A wide spectrum of further physical applications of the formalism developed is obtained by considering nonforward scattering. A trivial generalization would mean to introduce the t dependence into each expression such as $F(E)$, $\xi(E)$, etc. contained in the present paper. It is more interesting, however, to apply the method to the function

$$g(E,t)=iE\ln\frac{F_+(E,t)}{F_-(E,t)},\quad E\in D_r,\ -t_0\le t\le 0 \tag{5.7}$$

which will now play the role of $F(E)$ in the formalism, t being considered as a fixed parameter. To avoid additional singularities, we have to make an extra assumption that $F_+(E,t)$ and $F_-(E,t)$ have no zeros in the closure of D_r for some r. Then the results of the present paper can be transferred to $g(E,t)$ by formally substituting $\Delta\sigma(E)$ and $\mathrm{Re}F(E)/E$ with

$$\ln\left|\frac{F_+(E,t)}{F_-(E,t)}\right|$$

and $\arg(F_-(E,t)/F_+(E,t))$, respectively.

Consider Remark 2, (i), for illustration. If for sufficiently high energies $|F_+(E,t)|>|F_-(E,t)|$ and $\arg F_-(E,t)>\arg F_+(E,t)$, then

$$\int_{E_0}^{\infty}\ln\left|\frac{F_+(E,t)}{F_-(E,t)}\right|\frac{dE}{E} \tag{5.8}$$

converges and, accordingly,

$$\lim_{E\to\infty}|F_+(E,t)/F_-(E,t)|=1$$

in Meiman's sense.

Moreover, we obtain an unconditional asymptotic bound on the logarithm $\ln|F_+/F_-|$ in complete analogy with relations (5.1) and (5.3). Indeed, the logarithm must obey one of the following relations: either

$$\left|\int^{\infty}\ln|F_+(E,t)/F_-(E,t)|\frac{dE}{E}\right|<\infty \tag{5.9}$$

or

$$|\ln|F_+(E_k,t)/F_-(E_k,t)||\le\frac{\pi(1+\epsilon)}{2\ln E_k}|\arg(F_-(E_k,t)/F_+(E_k,t))|\,. \tag{5.10}$$

In analogy with (5.5) and (5.6) it also follows that either

$$\liminf_{E\to\infty}|\ln|F_+(E,t)/F_-(E,t)||=0 \tag{5.11}$$

or

$$\frac{\pi(1-\epsilon)}{2\ln E_k}|\arg(F_+(E_k,t)/F_-(E_k,t))|\le|\ln|F_+(E_k,t)/F_-(E_k,t)||\times\frac{\pi(1+\epsilon)}{2\ln E_k}|\arg(F_+(E_k,t)/F_-(E_k,t))|\,. \tag{5.12}$$

It is interesting to compare these relations with those obtained in Refs. 13, 14, and 5.

The results of the present paper prove the method to be powerful for obtaining asymptotic correlations for forward and elastic scattering. It is to be hoped that the use of averaged quantities similar to those introduced in Ref. 4 and applied in Refs. 2 and 14 could lead to some improvement of the inequalities such as (5.3), (5.6), (5.10), and (5.12).

ACKNOWLEDGMENT

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APPENDIX

In Theorem A of this Appendix we find sufficient conditions for the existence of Meiman's limit[5] $\mathrm{Lim}_{E\to\infty+}h(E)$ of a real function $h(E)$.

Let us introduce first the concept of Meiman's limit. Let M denote a subset of the real axis and E_0 a given point of the real axis. The set M will be called full with respect to E_0 if

$$\lim_{y\to0+}\frac{1}{\pi}\int_M\frac{y}{[(E-E_0)^2+y^2]}\,dE=1\,. \tag{A1}$$

Similarly, the set is called right full or left full with respect to E_0 if the set M is a subset of $\langle E_0,\infty)$ (or, respectively, a subset of $(-\infty,E_0\rangle$) and

$$\lim_{y\to0+}\frac{1}{\pi}\int_M\frac{y}{[(E-E_0)^2+y^2]}\,dE=\tfrac{1}{2}\,. \tag{A2}$$

The set M will be called a null set with respect to E_0 if

$$\lim_{y\to0+}\frac{1}{\pi}\int_M\frac{y}{[(E-E_0)^2+y^2]}\,dE=0\,. \tag{A3}$$

If, in addition to (A3), M is a subset of $\langle E_0,\infty)$ [or a subset of $(-\infty,E_0\rangle$], then M will be called right-null set (or left-null set respectively).

Let the function $h(E)$ be defined in some neighborhood of E_0. Denote by $h_M(E)$ the restriction of h to M, i.e., $h_M(E)$ is defined only on M and $h_M(E)=h(E)$ for $E\in M$.

Definition. The function $h(E)$ has Meiman's limit $\mathrm{Lim}_{E\to E_0}h(E)$ [$\mathrm{Lim}_{E\to E_0+}h(E)$ or $\mathrm{Lim}_{E\to E_0-}h(E)$] if there exists a set M which is full with respect to E_0 (right-full or left-full with respect to E_0) such that $\lim_{E\to E_0}h_M(E)$ [$\lim_{E\to E_0+}h_M(E)$ and $\lim_{E\to E_0-}h(E)$, respectively] exists. The value of Meiman's limit is given by

$$\mathrm{Lim}_{E\to E_0}h(E)=\lim_{E\to E_0}h_M(E)\,,$$

$$[\mathrm{Lim}_{E\to E_0\pm}h(E)=\lim_{E\to E_0\pm}h_M(E)]\,.$$

If the point E_0 is 0, then the full sets, null sets, right-full, right-null sets, etc. with respect to 0 will shortly be called full sets, null sets, right-full, right-null sets, etc., respectively.

Theorem A. Let a function $h(E)$ be defined on an interval $(E_0, E_0+\delta)$, $\delta>0$. If

$$\int_{E_0}^{E_0+\delta} |h(E)/(E-E_0)|\,dE<\infty\,, \tag{A4}$$

then $\mathrm{Lim}_{E\to E_0+} h(E)=0$.

Proof. Without loss of generality we can assume $E_0=0$ and $\delta=1$. The proof is split into several lemmas.

Lemma 1. Let sets M_1 and M_2 be subsets of $\langle 0,1\rangle$ such that $M_1\cup M_2=\langle 0,1\rangle$ and $M_1\cap M_2=\emptyset$. Then the set M_1 is right-full if and only if M_2 is a right-null set.

Proof. We have

$$\frac{1}{\pi}\int_{M_1}\frac{y}{(E^2+y^2)}\,dE+\frac{1}{\pi}\int_{M_2}\frac{y}{(E^2+y^2)}\,dE =\frac{1}{\pi}\int_0^1\frac{y}{(E^2+y^2)}\,dE=\frac{1}{\pi}\tan^{-1}\frac{1}{y}\,.$$

Since the right-hand side converges to $\frac{1}{2}$ for $y\to 0+$ the lemma is proved.

Let n be a positive integer. Define

$$A_n=\{E:\ |h(E)|<1/n\}\,,$$
$$B_n=\{E:\ |h(E)|\geq 1/n\}\,.$$

Lemma 2. Let the assumptions of Theorem A be fulfilled with $E_0=0$, $\delta=1$, then B_n are right-null sets for every $n>0$.

Proof. Because of the definition of B_n we have

$$\frac{1}{n}\int_{B_n}\frac{dE}{E}\leq\int_{B_n}|h(E)/E|\,dE \leq\int_{A_n}|h(E)/E|\,dE+\int_{B_n}|h(E)/E|\,dE =\int_0^1|h(E)/E|\,dE<\infty\,.$$

It follows that $\int_{B_n}E^{-1}dE$ is convergent for every $n>0$. Since $Ey/(E^2+y^2)\leq\frac{1}{2}$ and $\lim_{y\to 0}y/(E^2+y^2)=0$ $(E>0)$ we apply the Lebesgue theorem [the majorant of $y/(E^2+y^2)$ is $1/(2E)$ for $E\in B_n$]. The Lebesgue theorem implies

$$\lim_{y\to 0+}\int_{B_n}\frac{y}{(E^2+y^2)}\,dE=0\,.$$

Lemma 2 is proved.

Choose numbers $\delta_n>0$ such that

$$\sum_{i=1}^{n}\int_{B_i}\frac{y}{(E^2+y^2)}\,dE<\frac{1}{n}\quad\text{for } 0<y<\delta_n\,,\qquad \delta_n<\frac{1}{n}\quad\text{and}\quad\cdots<\delta_{n+1}<\delta_n<\cdots\,. \tag{A5}$$

Such numbers exist because of lemma 2. Denote

$$B=\bigcup_{i=1}^{\infty}B_i\cap\langle\delta_{i+1}^2,\delta_i^2\rangle\,. \tag{A6}$$

Certainly B is a subset of $\langle 0,1\rangle$.

Lemma 3. B is a right-null set.

Proof. Let y be a positive number, $y<\delta_1$. Relation (A5) implies that $\{\delta_n\}$ is a decreasing sequence, $\lim_{n\to\infty}\delta_n=0$. It follows that there exists an index n such that $\delta_{n+1}\leq y<\delta_n$. We have

$$\int_B\frac{y}{(E^2+y^2)}\,dE=\sum_{i=1}^{n}\int_{B_i\cap\langle\delta_{i+1}^2,\delta_i^2\rangle}\frac{y}{(E^2+y^2)}\,dE+\sum_{i=n+1}^{\infty}\int_{B_i\cap\langle\delta_{i+1}^2,\delta_i^2\rangle}\frac{y}{(E^2+y^2)}\,dE$$
$$\leq\sum_{i=1}^{n}\int_{B_i}\frac{y}{(E^2+y^2)}\,dE+\int_0^{\delta_{n+1}^2}\frac{y}{(E^2+y^2)}\,dE<\frac{1}{n}+\tan^{-1}\frac{\delta_{n+1}^2}{y}\leq\frac{1}{n}+\tan^{-1}\delta_{n+1}\,.$$

The last two inequalities follow from (A5) and from $\delta_{n+1}\leq y<\delta_n<1/n$. The expression obtained is smaller than $1/n+1/(n+1)$. This proves Lemma 3.

Denote $A=\langle 0,\delta_1^2\rangle-B$. By Lemma 1 and Lemma 3 the set A is right-full. By (A5) the set A can be rewritten

$$A=\bigcup_{i=1}^{\infty}A_i\cap\langle\delta_{i+1}^2,\delta_i^2\rangle\,. \tag{A7}$$

We shall prove that $\lim_{E\to 0+}h_A(E)=0$. Let a number $\epsilon>0$ be given. Choose n such that $1/n\leq\epsilon$ and $\delta=\delta_n^2$. If $0<E<\delta=\delta_n^2$ and $E\in A$ then $E\in A_m$ by (A7) where $m\geq n$, and by the definition of A_m we obtain

$$|h(E)|=|h_A(E)|<\frac{1}{m}\leq\frac{1}{n}\leq\epsilon$$

for every $E\in A\cap(0,\delta)$. This means $\lim_{E\to 0+}h_A(E)=0$ and since A is a right-full set Theorem A is proved.

The notion of Meiman's limit can be modified for the cases $\mathrm{Lim}_{|E|\to\infty}h(E)$, $\mathrm{Lim}_{E\to+\infty}h(E)$, and $\mathrm{Lim}_{E\to-\infty}h(E)$. Let a set M be given. The set M is called full at infinity (right-full or left-full at infinity) if

$$\lim_{y\to 0+}\frac{1}{\pi}\int_M\frac{y}{(1+E^2y^2)}\,dE=1$$

($\lim_{y\to 0+}(1/\pi)\int_M[y/(1+E^2y^2)]dE=\frac{1}{2}$ and $M\subset\langle 0,\infty)$ and $M\subset(-\infty,0\rangle$, respectively). Meiman's limit $\mathrm{Lim}_{|E|\to\infty}h(E)$ ($\mathrm{Lim}_{E\to+\infty}h(E)$ and $\mathrm{Lim}_{E\to-\infty}h(E)$, respectively) exists if there exists a set M full at infinity (right-full or left-full at infinity) such that $\lim_{|E|\to\infty} h_M(E)$ ($\lim_{E\to\infty+} h_M(E)$ or $\lim_{E\to\infty-} h_M(E)$) exists. If the limit exists, the value of Meiman's limit is defined by this limit.

Let a set $M \subset (0,\infty)$ be given. Denote $N = \{-1/E: E \in M\}$. The set M is right-full at infinity if and only if N is left-full. On the other hand let $h(E)$ be a function defined for $E \geq R > 0$. Denote $g(E) = h(-1/E)$; then g is a function defined on the interval $(-1/R, 0)$. Certainly

$$\int_R^\infty |h(E)/E|\, dE = \int_{-1/R}^0 |g(E)/E|\, dE .$$

These two statements together with Theorem A imply the following:

Corollary A. Let a function $h(E)$ be defined on (R,∞). If

$$\int_R^\infty |h(E)/E|\, dE < \infty , \tag{A8}$$

then

$$\operatorname*{Lim}_{E\to\infty+} h(E) = 0 .$$

Coming back to Theorem 2 and the subsequent discussion, we notice that if $\Sigma_{-1,0}(E)$ is bounded for $E\to\infty$, then $\operatorname{Lim}_{|E|\to\infty} \Delta\sigma(E) = 0$. [Note that the crossing-symmetry property (A2) allows $\operatorname{Lim}_{E\to\infty+}$ to be replaced by $\operatorname{Lim}_{E\to\infty}$.]

Certainly the integral conditions (A4) and (A8) are not necessary conditions. But the following Theorem B shows that they are "not far from being" necessary.

Theorem B. Let a function $l(E)$ be defined on (R,∞) and let $\lim_{E\to\infty} |l(E)| = \infty$, then there exists a function $h(E)$ defined on (R,∞), fulfilling (A8) and such that the Meiman limit $\operatorname{Lim}_{E\to\infty+} h(E) l(E)$ does not exist.

Proof. Let a_1 be such a number that $|l(E)| \geq 1$ for $E \geq a_1 > 0$ and $a_1 \geq R$. If a_{n-1} is given let a_n be a number fulfilling

$$|l(E)| \geq n^2 \quad \text{for } E \geq a_n, \quad a_n \geq 3a_{n-1} . \tag{A9}$$

Denote $b_n = 2a_n$. Certainly $b_n < a_{n+1}$ for every n, i.e., the intervals $\langle a_n, b_n\rangle$ are disjoint for different n.

Define the sets

$$K = \bigcup_{n=1}^\infty \langle a_n, b_n\rangle ,$$

$$L = \bigcup_{n=1}^\infty (b_n, a_{n+1}) \cup (R, a_1)$$

and the function $h(E)$, $h(E) = 1/l(E)$ for $E \in K$ and $h(E) = 0$ for $E \in L$. The function $h(E)$ is defined on (R,∞) since $K \cup L = (R,\infty)$ and $K \cap L = \emptyset$.

First, we shall prove that condition (A8) is satisfied. We have

$$\begin{aligned}\int_R^\infty |h(E)/E|\, dE &= \sum_{n=1}^\infty \int_{a_n}^{b_n} \frac{1}{|E l(E)|} dE \\ &\leq \sum_{n=1}^\infty (1/n^2) \int_{a_n}^{b_n} dE/E \\ &= (\ln 2) \sum_{n=1}^\infty 1/n^2 = \tfrac{1}{6}\pi^2 \ln 2 .\end{aligned}$$

The inequality holds because of (A9).

Second, we shall prove that K is not a right-null set at infinity, i.e., that

$$\limsup_{y\to 0+} \int_K \frac{y}{(1+E^2y^2)}\, dE > 0.$$

We have

$$\begin{aligned}\limsup_{y\to 0+} \int_K \frac{y}{(1+E^2y^2)}\, dE &= \limsup_{y\to 0+} \sum_{n=1}^\infty (\tan^{-1} b_n y - \tan^{-1} a_n y) = \limsup_{y\to 0+} \sum_{n=1}^\infty \tan^{-1}\left(y\, \frac{b_n - a_n}{1 + a_n b_n y^2}\right) \\ &= \limsup_{y\to 0+} \sum_{n=1}^\infty \tan^{-1}\left(\frac{a_n y}{1 + 2a_n^2 y^2}\right) \\ &\geq \tan^{-1}(\tfrac{1}{3}) > 0 .\end{aligned}$$

The last but one inequality holds because $\tan^{-1}[a_n y/(1+2a_n^2y^2)] = \tan^{-1}(\frac{1}{3})$ for $y = 1/a_n$.

Similarly we can prove that L is not a right-null set at infinity. We have

$$\begin{aligned}\limsup_{y\to 0+} \int_L \frac{y}{(1+E^2y^2)}\, dE &\geq \limsup_{y\to 0+} \sum_{n=1}^\infty \int_{b_n}^{a_{n+1}} \frac{y}{(1+E^2y^2)}\, dE = \limsup_{y\to 0+} \sum_{n=1}^\infty \tan^{-1}\left(y\, \frac{a_{n+1} - b_n}{1 + a_{n+1} b_n y^2}\right) \\ &\geq \limsup_{y\to 0+} \sum_{n=1}^\infty \tan^{-1}\left(\frac{a_{n+1} y}{3 + 2a_{n+1}^2 y^2}\right) \\ &\geq \tan^{-1}(\tfrac{1}{5}) > 0 .\end{aligned}$$

The second inequality follows from $b_n \leq 2a_{n+1}/3$ [see (A9)], whereas the third one follows, again, from the fact that

$$\tan^{-1}[ya_{n+1}/(3+2a_{n+1}^2y^2)] = \tan^{-1}(\tfrac{1}{5})$$

for $y = 1/a_{n+1}$.

Suppose that $\operatorname{Lim}_{E\to\infty+} h(E) l(E)$ exists. Then there exists a set M which is right-full at infinity and such that either

(i) $M \cap K = \emptyset$ and $\lim_{E\to\infty} h_M(E) l_M(E) = 0$, or

(ii) $M \cap L = \emptyset$ and $\lim_{E\to\infty+} h_M(E) l_M(E) = 1$.

Assume (i). We have $K \subset (R,\infty) - M$ where $(R,\infty) - M$ is not a right-null set at infinity and, by lemma 1, the set M cannot be right-full at infinity. Similarly, $L \subset (R,\infty) - M$ in case (ii) and, by the same argument, we obtain again that M cannot be right-full at infinity. This contradiction proves the theorem.

Although h is not continuous, it is evident that a smooth function satisfying our conditions can be constructed from it.

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