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Singularities in four-body final-state amplitudes

Sadhan K. Adhikari
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife, PE, Brazil
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Like three-body amplitudes, four-body amplitudes have subenergy threshold singularities over and above total-energy singularities. In the four-body problem we encounter a new type of subenergy singularity besides the usual two- and three-body subenergy threshold singularities. This singularity will be referred to as "independent-pair threshold singularity" and involves pair-subenergy threshold singularities in each of the two independent pair subenergies in four-body final states. We also study the particularly interesting case of resonant two- and three-body interactions in the four-body isobar model and the rapid (singular) dependence of the isobar amplitudes they generate in the four-body phase space. All these singularities are analyzed in the multiple-scattering formalism and it is shown that they arise from the "next-to-last" rescattering and hence may be represented correctly by an approximate amplitude which has that rescattering.

I. INTRODUCTION

Analysis of few-body final states is a very interesting problem because such analysis yields information about two- and three-body interactions. Only recently theories consistent with constraints of quantum mechanics have been developed for such analysis.[1-4] Constraints of quantum mechanics—unitarity and analyticity—force singularity structure and interdependence of amplitudes usually assumed to be constant and independent otherwise. These are the subenergy threshold singularities of the four-body amplitude. In this paper we show how these threshold singularities arise from a consistent analysis of the multiple-scattering series. We also consider some important singularities of the four-body amplitude in the case of resonant final-state interactions. These singularities lead to rapid variations in the isobar amplitudes over the four-body phase space, especially near the physical region.

First we consider the threshold singularities of four-body amplitudes. We show how the threshold singularities arise from a consistent analysis of the multiple-scattering series. Apart from the usual two-body and three-body subenergy singularities there is an important singularity which arises due to interactions between the particles in two noninteracting pairs. In the nonrelativistic case this is a product of square-root singularities each in the two independent pair subenergies. These singularities are also at the boundary of the physical region, like any other threshold singularities and will produce considerable variation of the amplitudes over that region. From our experience in three-body final states we expect that in the four-body problem this particular "independent-pair singularity" will be equally important as other subenergy singularities.

Next we consider the case of resonant final-state interactions. We consider, in particular, a simple model of four spinless bosons in the final state interacting via s-wave two- and three-body resonant interactions. This is the case where the sequential decay or isobar formalism is intended. The purpose of the present work is not to find an exact or even an approximate solution of the problem in this case but rather to investigate some of the rapid variations of the isobar amplitudes over the four-body phase space that the resonant interactions may lead to. In other words, we would like to test the validity of the usual isobar assumption that the isobar amplitudes are constant and independent over the phase space. We find that the resonant interactions may, in fact, lead to rapid variations of the amplitude. We show that this will come from the last rescattering. From the experience in the three-body problem in similar situations,[5] we know that this singularity will be well represented by any approximation that has the last rescattering, for example, the first term in the multiple-scattering series. Hence, we study the first iterate which will contribute to these rapid (singular) variations of the amplitude. The actual isobar amplitude has a very complicated singular structure in this case. But these important rapid variations will be present in the amplitude apart from the threshold singularities. Hence, the usual isobar assumption, that the isobar amplitudes are constant over the phase space, is a shaky one and has been shown to violate unitarity.[2,3] Formal theory of four-body final states implementing unitarity and analyticity has been developed recently.[2,3]

In Sec. II we discuss the singularity analysis of the four-body amplitude necessary to obtain the various threshold singularities in the multiple-scattering formalism. In Sec. III we consider the four-body multiple-scattering formalism in the isobar model and calculate analytically and numerically the rapid variation of some of the terms. Finally in Sec. IV we give a brief discussion and concluding remarks.

II. THRESHOLD SINGULARITIES

In this section we discuss the subenergy threshold singularities of the four-body amplitude, especially the independent pair threshold singularity, which arises due to interactions among the particles in two noninteracting pairs. This will generalize the result obtained from a recent analysis of constraints of unitarity on four-body final states.[2,3] In this analysis we shall fix the total energy and look at the subenergy singularities. In particular we shall look for the singularities that are in or on the boundary of the physical region, since that will produce the rapid dependence of the amplitude.

We consider the weak decay of a particle into four equal mass spinless bosons that interact strongly in final state. We shall work in units $\hbar = m = 1$, where m is the mass of any of the four particles. We shall be using nonrelativistic kinematics in our analysis. But the conclusions of the present and the following section are not limited to this severely restricted model.

We consider a few nth-order multiple-scattering diagrams as shown in Fig. 1. We label the momentum conveniently to serve our purpose. The nature and position of the singularity do not in any way depend on the labeling of momentum variables. First we give a discussion of the simple pair-subenergy singularity. In the derivation of this singularity we shall follow essentially the same steps as in the three-body case.[6] Here we give a brief account of this derivation and refer the interested readers to Ref. 6. Pair-subenergy dependence of the diagrams shown in Fig. 1 will come from considering the four-body propagator next to the last rescattering as shown by a dashed vertical line in Fig. 1 (a). The amplitude associated with this graph is

$$\int h(\vec{p}, \vec{q}_0, \vec{k}')\frac{\langle \vec{k}' | t(k^2) | \vec{k}\rangle}{k^2 - k'^2}\frac{d^3k'}{(2\pi)^3}, \qquad (1)$$

where we have used energy conservation $k^2 = E - p^2 - q_0^2/2$ to write the energy denominator of (1) in the present form. This form demonstrates the subenergy (k^2) dependence of the amplitude we are interested in. Here $\langle \vec{k}' | t(k^2) | \vec{k}\rangle$ is the half-on-shell two-body t matrix and $h(\vec{p}, \vec{q}_0, \vec{k}')$ is everything to the left of the propagator. Since the interesting external-momentum variable k does not penetrate in, our discussion is indepent of what

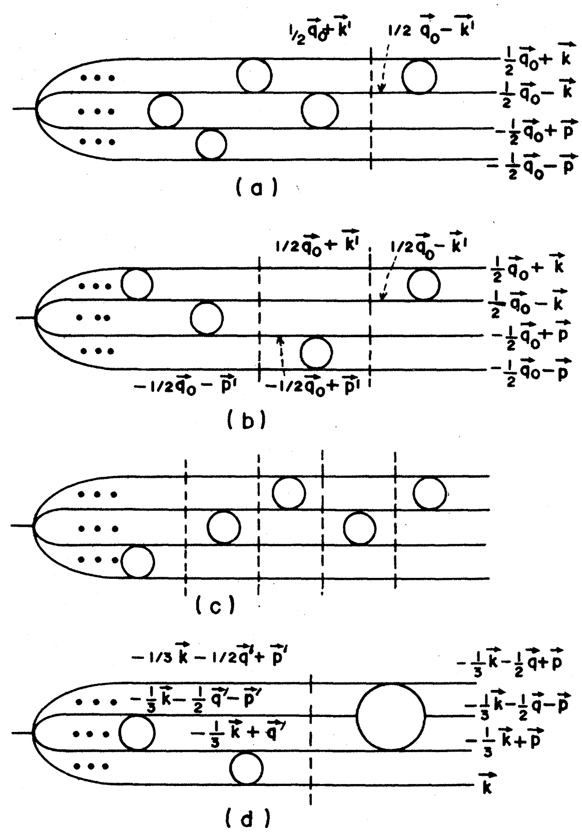

FIG. 1. Schematic representation of some multiple-scattering diagrams. The dashed vertical line(s) represent the propagator that will contribute to physical region singularity.

happens to the left of this point. Hence $h(\vec{p}, \vec{q}_0, \vec{k}')$ will not contribute to any physical-region subenergy singularity. This has been explicitly demonstrated in the case of three-body amplitudes in Ref. 6 and the same reasoning applies here. Also since the $\vec{k}'$ dependence of $\langle \vec{k}' | t(k^2) | \vec{k}\rangle$ involves only unphysical-region singularities, the $\vec{k}'$ integral in Eq. (1) will give physical-region singularity by virtue of the propagator only. This will contribute to the pair-subenergy singularity a square-root branch point $(k^2)^{1/2}$.

If we specialize to a case where the two-body t matrix is dominated by a single partial wave (we take s wave to simplify the kinematics), we can write

$$\langle \vec{k}' | t(k^2) | \vec{k}\rangle = A(k, k')\tau(k^2), \qquad (2)$$

where $\tau(k^2)$ is the on-shell t matrix and $A(k, k')$ is the half-shell function and has only potential or unphysical singularities in k and k'. Then Eq. (1) becomes

$$\tau(k^2)\int \frac{h(p, q_0, k')A(k, k')}{k^2 - k'^2}\frac{d^3k'}{(2\pi)^3}. \qquad (3)$$

This is one of the possible isobar decompositions[2] of the four-body amplitude. The usual isobar assumption that the coefficient of $\tau(k^2)$ is a slowly varying function without any subenergy singularity violates this simple analysis and also constraints of unitarity.[2]

Next we consider a slightly more complicated multiple-scattering graph as shown in Fig. 1(b), where the final interaction involves two t matrices in two independent pairs. This graph is new to four-body problem and gives an interesting physical-region singularity to the four-body amplitude. Only recently we have demonstrated its presence through unitarity.[3] Here we give a simple derivation of this singularity by an analysis of the multiple-scattering graph. The singularity will arise from a consideration of two four-body propagators next to the last rescatterings. The propagators are shown in Fig. 1(b) by dashed lines. The amplitude associated with this graph is

$$\int\int h(\vec{q}_0, \vec{k}', \vec{p}') \frac{\langle \vec{k}'|t(k^2)|\vec{k}\rangle\langle\vec{p}'|t(p^2)|\vec{p}\rangle}{(k^2-k'^2)(k^2+p^2-k'^2-p'^2)} \frac{d^3k'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}, \tag{4}$$

where as before $k^2+p^2=E-q_0^2/2$ by energy conservation. Here the t's are the half-on-shell two-body t matrices of the two independent pairs, and $h(\vec{q}_0, \vec{k}', \vec{p}')$ is everything to the left of the innermost of the two propagators and, by arguments just outlined above and discussed in detail in Ref. 6, will not contribute to physical-region subenergy singularity. As before remembering that $\vec{k}'$ and $\vec{p}'$ dependences of the t matrices in Eq. (4) involves only unphysical-region singularities, the $\vec{k}'$ and $\vec{p}'$ integrals in (4) will give physical-region singularities by virtue of the two propagators. Hence the integral in (4) will give rise to a branch cut $(\sigma_1\sigma_2)^{1/2}$ where σ_1 and σ_2 are the center-of-mass energies of the two t matrices in (4) and are given by $\sigma_1=k^2=E-\sigma_2-q_0^2/2=E-p^2-q_0^2/2$.

When as in Eq. (2) the two t matrices are dominated by a single partial wave (s wave) (4) becomes

$$\left[\int\int h(\vec{q}_0, \vec{k}', \vec{p}') \frac{A(k,k')A(p,p')}{(E-p^2-k'^2-q_0^2/2)(E-k'^2-p'^2-q_0^2/2)} \frac{d^3k'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\right]\tau(p^2)\tau(k^2). \tag{5}$$

As in (3), it is another possible isobar decomposition[3] of the four-body amplitude. The usual isobar assumption that the quantity in the square brackets of (5) does not have any physical-region singularity contradicts this simple analysis as well as unitarity.[3]

Finally in the four-body case there is an interesting diagram as shown in Fig. 1(c), which will give rise to three-body subenergy singularity. Analysis of this diagram is complicated because there is no interaction with one of the particles for few last rescatterings and the three-body subenergy variable k penetrates into the diagram. Hence the three-body subenergy dependence of the integral will come from a consideration of all the propagators shown in Fig. 1(c) by dashed vertical lines. In fact there is an infinite series of diagrams that will contribute to the three-body subenergy singularity. It is clear that all these diagrams can be schematically summed into a diagram shown in Fig. 1(d), where the big circle represents a connected three-body t matrix. Now the three-body subenergy label k does not penetrate in any further and the three-body subenergy singularity will come from the propagator shown by a vertical dashed line in Fig. 1(d). The amplitude associated with this diagram is

$$\int\int \frac{h(\vec{k}, \vec{q}', \vec{p}')\langle\vec{q}', \vec{p}'|T(3q^2/4+p^2)|\vec{q}, \vec{p}\rangle}{E-\frac{2}{3}k^2-\frac{3}{4}q'^2-p'^2} \times \frac{d^3q'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}, \tag{6}$$

where T is the half-on-shell three-body t matrix and we have used energy conservation $E=2k^2/3+\frac{3}{4}q^2+p^2$ to write the energy argument of T. Again h is everything to the left of the propagator shown by the dashed vertical line in Fig. 1(d). The same arguments as in the case of pair subenergy singularity show that h has no physical-region three-body subenergy ($=3q^2/4+p^2$) singularity. Similarly the half-on-shell three-body t matrix T has no physical region singularity in q' and p'. Hence the physical-region singularity of (6) will come from the propagator. Hence the important feature of the three-body subenergy singularity will come from

$$\int^{\Lambda} \frac{q'^2dq'p'^2dp'}{(E-\frac{2}{3}k^2)-\frac{3}{4}q'^2-p'^2}. \tag{7}$$

apart from multiplying factors. The cutoff Λ is introduced to guarantee convergence at the upper limit. In (6), h and T insure convergence at the upper limit. With a transformation of variables $(\frac{3}{4})^{1/2}q'=x^{1/2}\cos\theta$, $p'=x^{1/2}\sin\theta$, and $E-\frac{2}{3}k^2=\epsilon$

the singular part of this integral is given by

$$\int_0^{\Lambda} \frac{x^2 dx}{\epsilon - x}\,, \tag{8}$$

where ϵ is understood to have a small positive imaginary part. The three-body subenergy singularity of this integral will come when the pole of the denominator of the integrand pinches the lower limit and the singular part of the integral has the form (Ref. 7) $\epsilon^2 \log(-\epsilon)$, where ϵ is the three-body subenergy.

We have seen that the amplitude for the decay of one particle to four has three important threshold subenergy singularities over and above those of the two and three-body t matrices. These singularities come from the propagator(s) just before the last t matrix. There are no further physical region two- and three-body subenergy singularities from inside the multiple-scattering diagrams essentially because the external two- and three-body subenergy variables do not penetrate in beyond the propagator(s) considered. In the next section we turn to the problem of the decay of a boson into four spinless bosons in the isobar model and show that final-state resonant interactions may lead to very rapid variations of the isobar amplitudes.

III. RESONANT INTERACTIONS

In this section we consider the decay of a spinless boson into four spinless bosons of mass $m = 1$ ($\hbar = 1$) interacting in the final state through s-wave two- and three-body resonant interactions. We analyze the problem in the isobar model in the multiple-scattering formalism. The isobar model we consider is discussed in detail elsewhere[3] and is represented diagrammatically in Fig. 2. The particles in the final state can be created in two ways. Either they are created as a free particle and a three-body isobar which subsequently propagates and decays successively into three particles as shown in Fig. 2. They can also be created as two two-body isobars which propagate and decay to four particles.

Let us suppose further that the resonant final-state interaction produces an s-wave pair resonance at energy E_{02} with width Γ_{02} and an s-wave three-body resonance at energy E_{03} with width Γ_{03}. Near the respective resonances we parametrize the two- and the three-body t matrices as Breit-Wigner resonances, written as

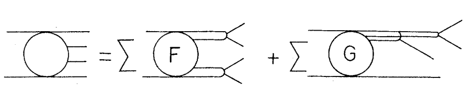

FIG. 2. Diagrammatic representation of the isobar model.

$$t_2(E) = \frac{8\pi^2 \Gamma_{02}}{E - E_{02} + \frac{1}{2} i \Gamma_{02}}\,, \tag{9}$$

and

$$t_3(E) = \frac{8\pi^2 \Gamma_{03}}{E - E_{03} + \frac{1}{2} i \Gamma_{03}} \tag{10}$$

in the narrow-width approximation. Here we are not interested in solving the four-body problem even approximately to find the isobar amplitudes. Rather, we are interested only in some of the rapid variations of the isobar amplitudes which the resonant interaction may lead to. For this purpose we do not need to consider the detailed two-body dynamics that lead to the two- and the three-body t matrices (9) and (10) but can use these t matrices directly in the multiple-scattering formalism without consideration of off-shell-effects which, we know, will not contribute to physical region singularities. Effectively we shall be making separable approximations for t_2 and t_3 with unit vertex form factors. We are motivated to do such a calculation by the success of representing the rapid dependence of the three-body isobar amplitude by the first iterate of the Faddeev equation in the case of resonant pair interactions.[5]

If we study the multiple-scattering expansion of the amplitudes shown in Fig. 2, in the multiple-scattering series among other terms there will be terms as shown in Fig. 3. Figures 3(a) and 3(b) will contribute to amplitude G and Figs. 3(c) and 3(d) will contribute to amplitude F. (F and G are defined in Fig. 2.) From the analysis in the Sec. II and also that in Ref. 6 it is clear that the rapid dependences of these amplitudes will come from the integration of the last loop momentum ($\vec{k}'$) over the propagators. We shall see that this integration gives rise to important rapid variations of the amplitudes. It has been well demonstrated in the three-body problem that this type of very approximate solution represents the rapid dependence of the amplitude very well,[5] and hence we shall be limited to the study of these approximate solutions. In the language of the last section it is justified because the external momentum label does not penetrate in any further.

The contribution of the diagram in Fig. 3(a) is

$$\int \frac{d^3k'}{(2\pi)^3} \frac{h(\vec{k}')}{E - \frac{2}{3}k'^2 - E_3} \frac{1}{E - \frac{3}{4}k^2 - \frac{3}{4}k'^2 - \vec{k}\cdot\vec{k}'/2 - E_2}\,, \tag{11}$$

where $E_2 = E_{02} - \frac{1}{2} i \Gamma_{02}$ and $E_3 = E_{03} - \frac{1}{2} i \Gamma_{03}$ and $h(\vec{k}')$ is everything to the left of the dashed vertical line. Here we have neglected the numerators of the t matrices (9) and (10) which will contribute

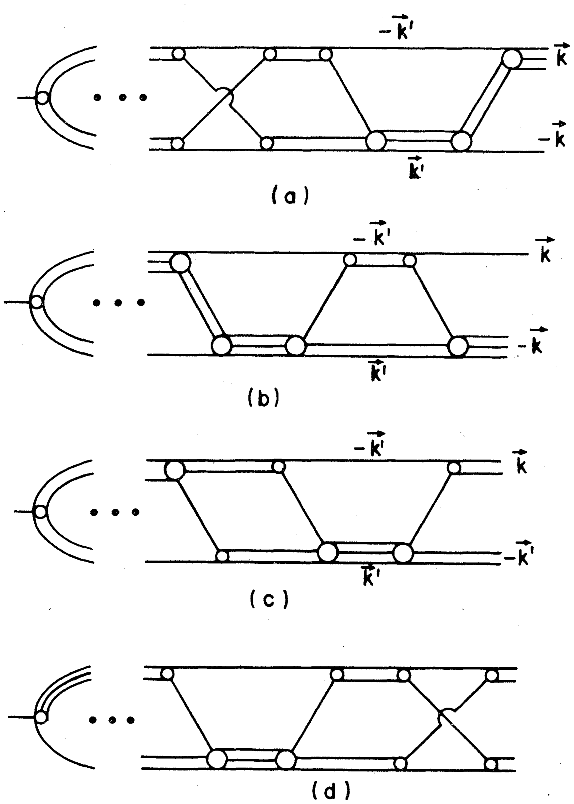

FIG. 3. Schematic representation of some multiple-scattering diagrams in the isobar model.

to uninteresting multiplying constants. After partial-wave projection to s wave (11) becomes

$$\frac{3}{k}\int_0^\infty \frac{k'dk'}{(2\pi)^2}\frac{h(k')}{k'^2-\frac{3}{2}(E-E_3)} \times \ln\frac{k'^2+\frac{2}{3}kk'+k^2-\frac{4}{3}(E-E_2)}{k'^2-\frac{2}{3}kk'+k^2-\frac{4}{3}(E-E_2)}\,. \quad (12)$$

Here $h(k')$ does not contribute to physical region rapid dependence of the amplitude. Hence for our purpose we can take $h(k')=1$. The integral in (12) is readily done by the technique of contour integration[5] and the result is

$$\frac{3i}{4\pi k}\ln\frac{[\frac{3}{2}(E-E_3)]^{1/2}+\frac{2}{3}[3(E-E_2)-2k^2]^{1/2}+k/3}{[\frac{3}{2}(E-E_3)]^{1/2}+\frac{2}{3}[3(E-E_2)-2k^2]^{1/2}-k/3}\,. \quad (13)$$

This will contribute to an important variation in the k dependence of the partial-wave amplitude $G(k)$.

This amplitude will also get an important rapid variation from the diagram in Fig. 3(b). The contribution of this diagram is

$$\int\frac{d^3k'}{(2\pi)^3}\frac{h(\vec{k}')}{E-k^2-\frac{3}{4}k'^2-\vec{k}\cdot\vec{k}'-E_2}\frac{1}{E-k'^2/2-2E_2}\,, \quad (14)$$

where $h(\vec{k}')$ has the same meaning as above. As before putting $h(\vec{k}')=1$, doing the partial-wave projection to the s wave and also the integration gives

$$\frac{i}{2\pi k}\ln\frac{[2(E-2E_2)]^{1/2}+\frac{2}{3}[3(E-E_2)-2k^2]^{1/2}+\frac{2}{3}k}{[2(E-2E_2)]^{1/2}+\frac{2}{3}[3(E-E_2)-2k^2]^{1/2}-\frac{2}{3}k}\,. \quad (15)$$

Next we consider the amplitude of type shown in Fig. 3(c). This will contribute to the k dependence of the isobar amplitude F. The contribution of this multiple-scattering diagram is

$$\int\frac{d^3k'}{(2\pi)^3}\frac{h(\vec{k}')}{E-\frac{2}{3}k'^2-E_3}\frac{1}{E-k'^2-\frac{3}{4}k^2-\vec{k}\cdot\vec{k}'-E_2}\,, \quad (16)$$

where $h(\vec{k}')$ has the same meaning as above. Taking $h(\vec{k}')=1$, doing the partial-wave projection to the s wave as well as the integration gives

$$\frac{3i}{8\pi k}\ln\frac{[\frac{3}{2}(E-E_3)]^{1/2}+\frac{1}{2}[4(E-E_2)-2k^2]^{1/2}+k/2}{[\frac{3}{2}(E-E_3)]^{1/2}+\frac{1}{2}[4(E-E_2)-2k^2]^{1/2}-k/2}\,. \quad (17)$$

The expressions (13), (15), and (17) will produce rapid dependence of amplitudes F and G. We are by no means claiming that we have singled out all the important variations of these amplitudes. Certainly there are many more of them. One particular example is the contribution from the multiple-scattering diagram shown in Fig. 1(d). But the corresponding integrations in this and other cases are complicated and this prohibits us from obtaining a closed analytic expression. Nevertheless we have found out some of the variations over the final-state phase space.

It is true that the arguments of the logarithms in (13), (15), and (17) are never near zero or infinity in the physical region even if the widths of the resonances are put equal to zero, and hence there is no nearby logarithmic singularity. Across the $[3(E-E_2)-2k^2]^{1/2}$ or $[4(E-E_2)-2k^2]^{1/2}$ cut the arguments of the logarithms are changed to arguments with minus signs in front of these square roots. These arguments can introduce logarithmic singularities for physically allowed k, but they are far away from the physical region. These are the analog of Peierls' singularities[8] to the four-body problem. These singularities are important because they may give rise to resonancelike behavior near the physical region even though the individual terms in the multiple-scattering series do not show this behavior. This would be a dynamic

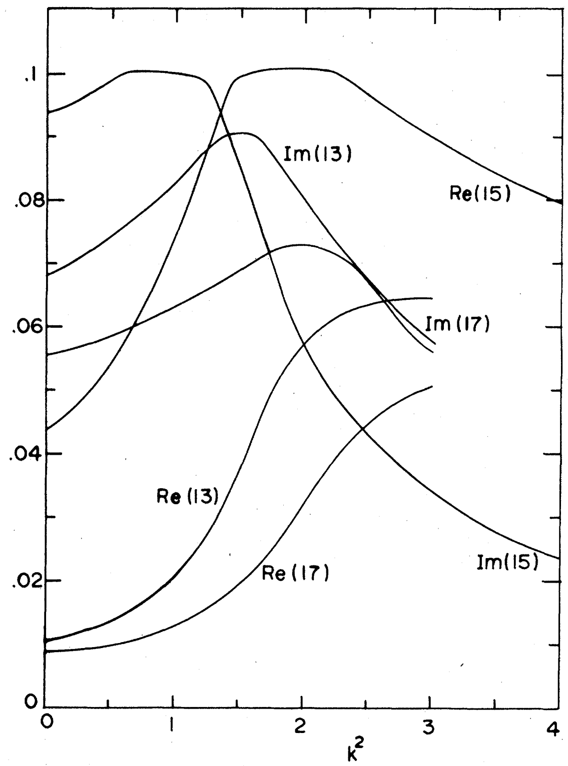

FIG. 4. Graphical representation of real and imaginary parts of expressions (13), (15), and (17) for physical values of k^2. The signs of all the real parts have been changed. The values of the parameters employed are $E=2.0$, $E_2=1.0-0.25i$, $E_3=1.2-0.3i$.

effect and not a kinematic one. This is equivalent to summing the multiple-scattering series. Though highly unlikely, the sum of the series may diverge at these singular energies when the individual terms are finite. Such dynamical effects have been studied and looked for in the three-body case—not through a general framework but in particular models—and so far the result is negative.[9] More work remains to be done in this direction in the three- and the four-body case. However, it is easy to calculate the position of these singularities on the unphysical sheet in a simple way. We take the arguments of the logarithms in (13), (15), and (17), go to the unphysical sheet by changing the sign of the second square root, take the on-shell value of k with all the corelated states resonating and set the arguments of the logarithms to zero to find an expression for E in terms of E_2 and E_3. Relativistically mass and energy are treated on the same footing and this will make the present four-body resonance mechanism equivalent in principle to the three-body Peierls mechanism where the exchanged particle could also be a resonance.

Even though the arguments of the logarithms do not vanish near the physical region they will give rise to rapid variations of the amplitude. The two- and the three-body resonance bands will sweep across the Dalitz plot as E changes. In (13) when $E=E_{03}$ the three-body resonance will just appear at the edge of the Dalitz plot. It is clear from (11) that for interesting variations we must also have $E>E_{02}$ for the two-body resonance to be formed. So (13) is expected to have rapid variations for $E>E_{02}$ and $E>E_{03}$. The same conclusion is true in (17). But it is clear from (14) that rapid variations in (15) are expected to appear near $E>2E_{02}$. To have an idea of these variations we plot these functions on a graph, shown in Fig. 4, which demonstrate the rapid variations beyond doubt. The actual isobar amplitudes will have more complicated variations coming from the threshold singularities and other similar multiple-scattering diagrams one of which is shown in Fig. 3(d). [It is to be noted that simple expressions such as (13), (15), and (17) do not correctly represent the threshold singularities discussed in the last section.] Hence, the usual isobar assumption that treats these amplitudes as constants is not justified.

IV. DISCUSSION AND CONCLUSION

Now let us turn to the question of the generality of our conclusions in view of the rather restrictive model we considered. Clearly the equality of masses keeps the algebra simple and has no essential importance. Furthermore, we have never used the details of the structure of the decay vertex, and hence our results are valid in any production or break-up amplitude into four particles. The detailed form of the interaction is also not important. We analyzed the decay amplitude in the multiple-scattering formalism, but our findings are not dependent on the convergence of the multiple-scattering series. Use of nonrelativistic kinematics simplifies the algebra and our conclusions are true in any relativistic theory as long as the relativistic theory has a proper nonrelativistic limit.

Recently we have demonstrated the presence of the threshold singularities in a more general way through the consideration of unitarity and have shown how to develop a theory of four-body final states that acknowledge the constraints of quantum mechanics—unitarity and analyticity. The usual isobar model does not acknowledge these singularities; hence the better isobar model should be used to analyze the four-body final-states problem in general. The effects of unitarity will in general be important in two cases. The first is the case of threshold enhancement as one may encounter in the four-nucleon problem. The second and more important case is the case of resonant pair interaction as in 4π or $N3\pi$ problems. Much work remains to be done to understand the problem of four-body final states and extract from it informations about two- and three-body interactions.

ACKNOWLEDGMENTS

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