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Four-hadron isobar model

Sadhan K. Adhikari
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife, Pernambuco, Brazil
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The constraints of unitarity and analyticity on four-body final states are studied. The danger of implementing unitarity, without analyticity, on four-body final states is stressed. We develop a set of relativistic integral equations that incorporate unitarity and analyticity. The subenergy dependence predicted by these equations should find an important application in the phenomenological analysis of four-hadron final states. We work in terms of quasi-two-body states in the isobar model and the equations we obtain are in one vector variable and in principle will be easy to solve after partial-wave analysis.

I. INTRODUCTION

Over the last few years an enormous amount of intermediate-energy data has accumulated in few-hadron systems such as $\pi\pi\pi$, $\pi\pi N$, $\pi\pi K$, $\pi\pi NN$, $\pi\pi\pi N$, etc. Most of the three-hadron final-state phenomenologies[1] depended heavily on the isobar model.[1,2] A word of caution on the reliability of these calculations was first sounded by Aaron and Amado.[3] The usual phenomenology treated the isobar amplitudes as slowly varying and independent functions over the three-body phase space. They pointed out that this failed to satisfy unitarity — especially pair-subenergy unitarity — which is considered important for isobar phenomenology. As a consequence this also fails to satisfy full three-body unitarity. Recently there have been some calculations which tried to incorporate exact or approximate unitarity[4] in the three-body isobar model. The unitarity corrections were found to be very large in many cases and the corrected results were even worse than the uncorrected results. Aitchison and Golding[5] pointed out that some of the large, rapidly varying corrections that had been obtained were, in fact, spurious effects. These spurious singularities were present because of the failure to include analyticity while implementing subenergy unitarity. Recently Aaron and Amado[6] showed that the origin of this spurious singularity in the case of a resonant final-state interaction can be traced back to the singularity on the "wrong" sheet — known in the literature as Peierls' singularity.[7,8] In order to eliminate the spurious singularities from the theory one must add analyticity to unitarity. Aaron and Amado[6] developed the isobar model for three hadrons consistent with unitarity and analyticity. The better phenomenology yielded a set of relativistic three-body equations similar to the Blankenbecler-Sugar equations[9] — the solution of which does not contain spurious singularities. Recently Aitchison[10] also gave a three-body relativistic theory consistent with unitarity and analyticity. This method is the same in spirit as the recent method of Ref. 6, but differs from the latter in technical detail.

Here we turn to a similar analysis of four-body final states consistent with unitarity and analyticity using the isobar model. The idea of the present analysis is borrowed from a similar analysis of nonrelativistic four-body final states by the present author.[11] A formally correct theory[12] of four-body final states using the quasiparticle or isobar model and incorporating unitarity and analyticity exists. But its complexity is great because it involves solving an integral equation in two-vector variables. Here we formulate the four-body isobar model in terms of quasi-two-body states, apply unitarity on these amplitudes, and derive important unitarity constraints on them. We then implement these constraints of unitarity with analyticity and arrive at a set of dynamical equations for the quasi-two-body amplitudes. The dynamical equations we get are integral equations in one-vector variables and are easy to solve in principle after partial-wave decomposition. They are also the simplest four-body relativistic equations consistent with unitarity and analyticity. Similar equations[11] have been derived in the nonrelativistic four-body problem.

We consider the amplitude $T_{2,4}$ for a reaction going from a two-body to a four-body state. We postulate a simple form for $T_{2,4}$. This is suggested by the sequential decay or quasiparticle model of nuclear physics and the isobar model of particle physics. The isobar model we use is a straightforward generalization of the one used in Refs. 6 and 12 and is essentially the same as the one used in Ref. 11. In this model we have for the reaction $a(p)+b(p')\rightarrow c(p_1)+d(p_2)+e(p_3)+f(p_4)$ in the four-body center-of-mass frame with center-of-mass energy W:

$$\langle p,p'|T_{2,4}(W)|p_1,p_2,p_3,p_4\rangle = \frac{1}{8}\sum_{i,j,k,l=1}^{4}\langle p,p'|F_{ij,kl}(W)|p_i+p_j\rangle \frac{v_{ij}(\vec{M}_{ij}^{\,2})}{D_{ij}(\sigma_{ij})}\frac{v_{kl}(\vec{M}_{kl}^{\,2})}{D_{kl}(\sigma_{kl})}$$

$$+\frac{1}{2}\sum_{i,j,l,k=1}^{4}\langle p,p'|G_{i,jkl}(W)|p_i\rangle \frac{w_{jk,l}(\vec{M}_{jk,l}^{\,2})}{D'_{jkl}(\sigma_i)}\frac{v_{jk}(\vec{M}_{jk}^{\,2})}{D_{jk}(\sigma_{jk})}\,. \tag{1}$$

Here we have suppressed all internal quantum numbers and have assumed that each pair or three-particle cluster is dominated by a single isobar. The quasi-two-body amplitude $\langle p,p'|F_{ij,kl}(W) \times |p_i+p_j\rangle$ denotes the production from the initial state of the (i,j) and (k,l) isobars. The factors of v/D multiplying F describe the subsequent propagation of the isobars and their decay into particles of energy-momentum p_i, p_j, p_k, and p_l. The quasi-two-body amplitude $\langle p,p'|G_{i,jkl}(W)|p_i\rangle$ denotes the production of particle i and the (j,k,l) isobar. The factors of w/D' and v/D multiplying G denote the subsequent propagation of the (j,k,l) isobar and its decay into particle l and isobar (j,k) which subsequently propagates and decays into particles j and k. Equation (1) is diagramatically represented in Fig. 1. The factors of $\frac{1}{8}$ and $\frac{1}{2}$ in front of the two terms account for the fact that $F_{ij,kl}=F_{kl,ij}=F_{ji,kl}=F_{ij,lk}$, etc., $v_{ij}=v_{ji}$, $D_{ij}=D_{ji}$, $w_{ij,k}=w_{ji,k}$, etc., for distinguishable particles. Consequently there are three terms of the first type and twelve terms of the second type in Eq. (1). Here $\vec{M}$'s are the relative three-momentum between the decay products in the center-of-mass frame and are discussed in Ref. 6 and will again be discussed to the present context in Sec. II. Here the total center-of-mass energy W is defined by $W^2=(p+p')^2$, while $\sigma_i=(P-p_i)^2$, $\sigma_{ij}=(P-p_k-p_l)^2$ with $P=p+p'$. σ_i and σ_{ij} are squares of (j,k,l) and (i,j) isobar masses. In any application of the isobar model it is assumed that two- or three-body interactions are usually dominated by a few (resonant) partial waves. Then the amplitudes F and G are expanded in terms of partial-wave amplitudes $F^J_{ij,kl}(W,\sigma_{ij},\sigma_{kl},l',l)$ and $G^J_{i,jkl}(W,\sigma_i,l',l)$, where J is the total angular momentum and l and l' are partial waves for the initial and final channels. Now the usual isobar assumption borrowed from three-body isobar phenomenology is that $F_{ij,kl}$ and $G_{i,jkl}$ are slowly varying functions of σ_{ij}, σ_{kl}, and σ_i, respectively, and hence can be approximated by constants. We shall show that in general the constraints of unitarity prohibit us from making such assumptions. As in the three-body problem,[6] if the isobar resonances are very narrow the assumption of constant isobar amplitudes is a reasonable one. But in case of overlapping wide resonance bands the isobar amplitudes $F_{ij,kl}$ and $G_{i,jkl}$ will have strong singular dependence on σ_{ij}, σ_{kl}, and σ_i, respectively. The purpose of this paper is to develop a better phenomenology to include these ideas.

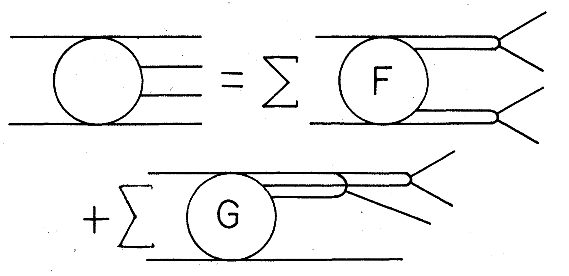

FIG. 1. Schematic representation of isobar production and decay corresponding to Eq. (1).

Let us point out in brief, following Refs. 5, 6, what goes wrong in taking the isobar amplitudes to be constants. We shall show that the constraints of unitarity force the amplitudes F and G to have physical branch cuts in the subenergies. In the case of G, it is a branch cut in σ_i. Then we take (suppressing the indices j, k, and l)

$$G_i(\sigma_i)=\text{Disp}G_i(\sigma_i)+i\,\text{Abs}G_i(\sigma_i)\,, \tag{2}$$

where $\text{Abs}G_i(\sigma_i)$ is the discontinuity of G_i required by this particular branch cut. The constraints of unitarity can be written schematically as

$$\text{Abs}G_i(\sigma_i)=\sum_{j\neq i}K(\sigma_i,\sigma_j)G_j(\sigma_j)\,. \tag{3}$$

In Ref. 6 it has been pointed out that in case of resonant final-state interactions, $\text{Abs}G_i$ acquires via Eq. (3) rapid variations that come from the singularities on the unphysical sheet of σ_i. In fact $\text{Disp}G_i$ contains a rapidly varying part which cancels the rapid variations of $\text{Abs}G_i$. However, if a simple input guess, e.g., a constant, is used for $\text{Disp}G_i$ and if no attention is paid to analyticity this spurious rapid variation of $\text{Abs}G_i$ will propagate into the physical amplitude. The same conclusion is true for the amplitude F.

Here we apply unitarity to these quasi-two-body states F and G, focusing particularly on a special type of two-body "subenergy" unitarity — "independent-pair two-body unitarity" (see Sec. III for detail), where there is no interaction between two independent pairs in four-body final states — and on three-body subenergy unitarity. Constraints of unitarity on F are derived by a consideration of independent-pair two-body unitarity, and those on G are derived by considering three-body subenergy unitarity. As in other similar few-body problems,[6,11-13] we find that unitarity forces the amplitudes to vary over their phase spaces, be singular on their edges, and be coherent and interrelated. The unitarity relation itself can be used to determine the numerical importance of these effects in any problem. From the preceeding discussion it is clear that if they are important they must be implemented by considering analyticity as well. Since we are taking into account a part of the full four-body unitarity, the implementation of unitarity is ambiguous. Out of these various ambiguous ways we choose the one that preserves total energy analyticity as well, in view of the various problems one faces upon neglecting it.[13] In this way we get a set of relativistic dynamical equations for the isobar amplitudes F and G. In Sec. II we define two- and three-body t matrices to be used in subsequent sections. We also develop the unitarity relations for independent-pair four-body unitarity and three-body subenergy unitarity in four-body final states. In Sec. III we derive the unitarity constraints for the four-body amplitudes F and G. In Sec. IV we discuss their implementation, stressing the importance of the "arbitrary" choices that are made there. In Sec. V we summarize our results, discuss possible applications, and make some concluding remarks.

II. UNITARITY

A. Two-body unitarity

Before applying unitarity to the four-body system, we review some of the conventions and definitions for two-body unitarity (for a complete review see Ref. 6).

The S and T matrices are related by

$$S=1+(2\pi)^4 i\delta^4(P)T\,, \tag{4}$$

so that in the case of the elastic scattering of two spinless particles of masses m_1 and m_2 the unitarity relation becomes

$$2\,\mathrm{Im}\langle q_{12}|T(W)|q_{12}'\rangle=\frac{1}{(2\pi)^2}\int d^4p_1''d^4p_2''\,\delta^4(P_{12}-P_{12}'')\delta^+(p_1''^2-m_1^2)\delta^+(p_2''^2-m_2^2)\langle q_{12}|T^\dagger(W)|q_{12}''\rangle\langle q_{12}''|T(W)|q_{12}'\rangle\,, \tag{5}$$

where $P_{12}=p_1+p_2$, $P_{12}^2=W^2$, $2q_{12}=p_1-p_2$, etc. Here we have suppressed all internal quantum numbers. In Eq. (5) p_1, p_2, p_1', p_2' are on their mass shells, but the amplitudes are off the energy shells in the sense that q_{12}, q_{12}', and W are considered independent.

In general in the case of n-body intermediate states we have

$$\begin{aligned}\langle\alpha|[T(W)-T^\dagger(W)]|\beta\rangle &= i\int\langle\alpha|T(W)|n\rangle\rho(n)\langle n|T^\dagger(W)|\beta\rangle\\ &= i\int\langle\alpha|T^\dagger(W)|n\rangle\rho(n)\langle n|T(W)|\beta\rangle\,,\end{aligned} \tag{6}$$

where $\rho(n)$ the n-body phase space is given by

$$\rho(n)=(2\pi)^4\delta^4\Big(P-\sum_{i=1}^n q_i\Big)\times\prod_{i=1}^n\left[\frac{d^4q_i}{(2\pi)^3}(2m_i)^F\delta^+(q_i^2-m_i^2)\right], \tag{7}$$

where $F=0$ for bosons and $F=1$ for fermions.

Decomposition into partial waves of the two-body t matrix gives

$$\langle q_{12}|T(W)|q_{12}'\rangle=\sum_{l,m}Y_{lm}^*(\hat M_{12})\tau_l(|\vec{M}_{12}|,|\vec{M}_{12}'|,W)\times Y_{lm}(\hat M_{12}')\,. \tag{8}$$

Here $\vec{M}_{12}$ is a special vector and is a function of $\vec{p}_1$ and $\vec{p}_2$, and is in fact the relative momentum of the two particles in the center-of-mass frame. Then the partial-wave unitarity relation gives

$$\mathrm{Im}\tau_l(W)=\frac{|\vec{q}_{12}|}{32\pi^2W}\,|\tau_l(W)|^2\,. \tag{9}$$

We assume that the two-body interaction proceeds through coupling to a definite isobar in a particular partial wave only. To make the algebra simple we will assume from now on that all the particles concerned are spinless and two- and three-body isobars occur always in relative S waves only. We make a separable representation of the two-body interaction that leads to the formation of the isobar. The interaction can be written as

$$\langle p|V(W)|q\rangle=v(p^2)\frac{1}{W^2-m_\Delta^2}v(q^2)\,, \tag{10}$$

where m_Δ is the bare mass of the isobar and the two factors of v are the form factors for the production and decay of isobars. We solve the two-body Blankenbecler-Sugar[9] equation with this potential to obtain the t matrix, dominated by the presence of the isobar:

$$\langle p|t(W)|q\rangle = v(p^2)\frac{1}{D(s)}v(q^2). \quad (11)$$

Here $s=W^2$ and $D(s)=s-m_\Delta'^2$, where m_Δ' is the dressed mass of the isobar and can be found from an actual solution of the Blankenbecler-Sugar equation. Consequently, the unitarity relation (9) takes the form

$$\mathrm{Im}\frac{1}{D(s)}=\frac{|\vec{q}_{12}|}{32\pi^2\sqrt{s}}\frac{v^2(\vec{M}_{12}{}^2)}{|D(s)|^2}. \quad (12)$$

It is important to recall that $D(s)$ has a zero at the isobar mass, carries the scattering phase, and has the unitarity cut, whereas N $(=v^2)$ has the left-hand cut.

In the four-body amplitude there is a term which involves no interaction between two independent pairs and hence is disconnected. The unitarity relation for this disconnected piece of amplitude is just a manipulation of two-body unitarity and can be written for four particles of equal mass as

$$\mathrm{Im}\frac{1}{D_{12}(\sigma_{12})D_{34}(\sigma_{34})}=\frac{|\vec{q}_{12}||\vec{q}_{34}|}{(32\pi^2)^2(\sigma_{12}\,\sigma_{34})^{1/2}} \times\frac{v_{12}{}^2(\vec{M}_{12}{}^2)v_{34}{}^2(\vec{M}_{34}{}^2)}{|D_{12}(\sigma_{12})D_{34}(\sigma_{34})|^2}, \quad (13)$$

where

$$s=W^2=(p_1+p_2+p_3+p_4)^2$$
$$=\sigma_{12}+\sigma_{34}+2(p_1+p_2)\cdot(p_3+p_4) \quad (14)$$

and refers to the four-body system; σ_{12} and σ_{34} refer to the two independent pairs (1, 2) and (3, 4).

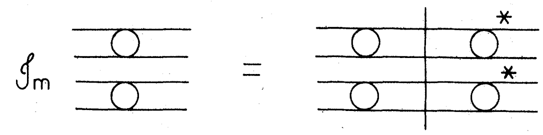

FIG. 2. Schematic representation for "independent-pair two-body unitarity". This is a disconnected part of four-body unitarity — where each of the disconnected pieces involve a two-body t matrix.

Unitarity relation (13) is represented schematically in Fig. 2.

In this article we shall be considering four distinguishable spinless bosons of equal mass m. We also neglect isospin and other internal quantum numbers. These restrictions keep the algebra simple, but otherwise have no effect on the result. If needed, these restrictions can be removed in principle without too much trouble.

B. Three-body unitarity

To proceed to the four-body problem with no further approximation would lead to the numerical difficulties inherent in multivariable integral equations; to keep the algebra managable we assume that the three-body interaction proceeds through coupling to a three-body isobar in a particular partial wave. This is a prelude to the introduction of a separable three-body interaction, which keeps the four-body problem managable. For simplicity we assume that all the two- and three-body isobars appear in the partial wave $l=0$. The t-matrix $T_{3,3}$ for the $3\to 3$ process can be written as[11]

$$\langle\vec{p}_1,\vec{p}_2,\vec{p}_3|T_{3,3}(W)|\vec{q}_1,\vec{q}_2,\vec{q}_3\rangle=\frac{1}{2}\sum_{a,b,c=1}^{3}\frac{1}{2}\sum_{d,e,f=1}^{3}\frac{v_{ab}(\vec{M}_{ab}{}^2)}{D_{ab}(\sigma_{ab})}w_{ab,c}(\vec{M}_{ab,c}{}^2)\frac{1}{D'(\sigma)}w_{de,f}(\vec{M}_{de,f}{}^2)\frac{v_{de}(\vec{M}_{de}{}^2)}{D_{de}(\sigma_{de})}. \quad (15)$$

In this model, as shown in Fig. 3, in the initial state, two particles a and b in a relative momentum state $\vec{M}_{ab}$ form the isobar (a, b) which then interacts with particle c, in relative momentum state $\vec{M}_{ab,c}$ with respect to isobar (a, b), and forms a three-body correlated state or isobar. The three-body correlated state then propagates and decays first to a free particle f and isobar (d, e) which subsequently decays to particles d and e. Here $w_{ab,c}$ is the vertex function for the interaction between c and the correlated state (a, b) in the S wave; and $\vec{M}_{ab,c}$ is the relative momentum between particle c and isobar (a, b). $D'(\sigma)$ is the three-body denominator function at energy W $(=\sigma^{1/2})$. v and D are the corresponding quantities for the two-body system and have already been introduced in the last subsection. An expression for $D'(\sigma)$ can be easily found by summing the series of self-energy bubbles. It has been discussed in detail elsewhere[14] for the nonrelativistic case. The generalization to the relativistic case is straightforward and we shall not consider it here. Another

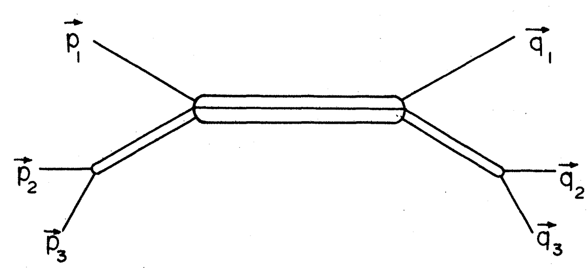

FIG. 3. Schematic representation of the three-body t matrix. The internal isobar lines are fully dressed.

way to calculate it is to approximate the Born term of the three-body Blankenbecler-Sugar equation[9] by a simple separable term of the form given by Eq. (10) with the v's replaced by w's and then to solve the three-body relativistic equation with this approximate Born term. Then the solution is of the form of Eq. (11) with v and D replaced by w and D', respectively, and hence we get an expression for D'. The factors of $\frac{1}{2}$ in front of the summations in Eq. (9) are there to take care of symmetries such as $v_{ab}=v_{ba}$, $w_{ab,c}=w_{ba,c}$, etc., for distinguishable particles.

The connected three-body unitarity relation for the amplitude (15) has the form[11,15]

$$\frac{1}{2}\sum_{a,b,c=1}^{3}\frac{1}{2}\sum_{d,e,f=1}^{3} 2\,\mathrm{Abs}\left[\frac{v_{ab}(\vec{M}_{ab}{}^2)}{D_{ab}(\sigma_{ab})}\, w_{ab,c}(\vec{M}_{ab,c}{}^2)\,\frac{1}{D'_{jkl}(\sigma_i)}\, w_{de,f}(\vec{M}_{de,f}{}^2)\,\frac{v_{de}(\vec{M}_{de}{}^2)}{D_{de}(\sigma_{de})}\right]$$

$$= \frac{1}{2}\sum_{a,b,c=1}^{3}\frac{1}{2}\sum_{d,e,f=1}^{3}\frac{1}{2}\sum_{s,t,u=1}^{3}\int \frac{v_{ab}(\vec{M}_{ab}{}^2)}{D_{ab}(\sigma_{ab})}\, w_{ab,c}(\vec{M}_{ab,c}{}^2)\,\frac{1}{D'_{jkl}(\sigma_i)}\, w_{st,u}(\vec{M}'^2_{st,u})\,\frac{v_{st}(\vec{M}'_{st}{}^2)}{D_{st}(\sigma'_{st})}\,(2\pi)^4\delta^4(p_i-p'_i)$$

$$\times(2\pi)^4\delta^4(p_j+p_k+p_l-p'_j-p'_k-p'_l)\left(\prod_{i=1}^{4}\frac{d^4p'_i}{(2\pi)^3}\right)\delta^+(p'^2_j-m^2)\,\delta^+(p'^2_k-m^2)$$

$$\times\delta^+(p'^2_l-m^2)\, w_{jk,l}(\vec{M}'_{jk,l}{}^2)\,\frac{v_{jk}(\vec{M}'_{jk}{}^2)}{D_{jk}(\sigma'_{jk})}\,\frac{1}{D'^{*}_{jkl}(\sigma_i)}\, w_{de,f}(\vec{M}_{de,f}{}^2)\,\frac{v_{de}(\vec{M}_{de}{}^2)}{D_{de}(\sigma_{de})}\;. \tag{16}$$

This unitarity relation is shown diagramatically[15] in Fig. 4. Equation (16) represents the discontinuity across the three-body cut. For the two-body D-function we have used the identity $D=D^*$ across the three-body cut, because D is continuous across the three-body cut we are interested in. Here $\vec{M}_{st,u}$ refers to the relative momentum between particle u and the correlated state (s,t). Cancelling common factors from both sides of Eq. (16) we get

$$2\,\mathrm{Abs}\,\frac{1}{D'_{jkl}(\sigma_i)}=\frac{1}{2}\,\frac{1}{(2\pi)^4}\sum_{s,t,u=1}^{3}\int\frac{1}{D'_{jkl}(\sigma_i)}\, w_{st,u}(\vec{M}'_{st,u}{}^2)\,\frac{v_{st}(\vec{M}'_{st}{}^2)}{D_{st}(\sigma'_{st})}\,\delta^4(p_i-p'_i)\,\delta^4(p_j+p_k+p_l-p'_j-p'_k-p'_l)$$

$$\times\delta^+(p'^2_j-m^2)\,\delta^+(p'^2_k-m^2)\,\delta^+(p'^2_l-m^2)\left(\prod_{i=1}^{4}d^4p'_i\right)w_{jk,l}(\vec{M}'_{jk,l}{}^2)\,\frac{v_{jk}(\vec{M}'_{jk}{}^2)}{D_{jk}(\sigma'_{jk})}\,\frac{1}{D'_{jkl}(\sigma_i)}\;. \tag{17}$$

Using these conventions and definitions for two- and three-body unitarity we turn to the problem of four-body unitarity.

III. FOUR-BODY UNITARITY

We shall study the problem of four-body unitarity in the isobar model introduced in Sec. I. The unitarity relation for $T_{2,4}$, which has been discussed in detail elsewhere,[12] contains many terms. ($T_{m,n}$ in general represents the t matrix for going from an m-body initial state to an n-body final state.) Assuming that only two-, three-, and four-body intermediate states are energetically allowed, the unitarity relation for $T_{2,4}$ can be written as[12,15] follows:

$$2\,\mathrm{Im}T_{2,4}=\int T_{2,2'}\,\rho(2')T^{\dagger}_{2',4}+\int T_{2,3'}\,\rho(3')T^{\dagger}_{3',4}+\int T_{2,4'}\,\rho(4')T^{\dagger}_{4',4}\,. \tag{18}$$

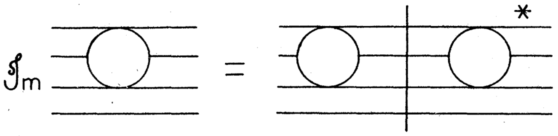

FIG. 4. Schematic representation of connected three-body unitarity in four-body space.

A contribution to the special type of four-body discontinuity we are interested in will come from the disconnected parts of the unitarity relation. Hence we decompose the amplitude in Eq. (18) into disconnected and totally connected parts. Equation (18) and this decomposition are shown in Fig. 5. Each term in unitarity implies a singularity at the threshold of that term and in the variable with that threshold. Strictly speaking, each term in the unitarity relation contributes to the discontinuity across the singularity beginning at that particular threshold.[6,15] We are interested in exploiting (18) to obtain the dependence of $T_{2,4}$ on the

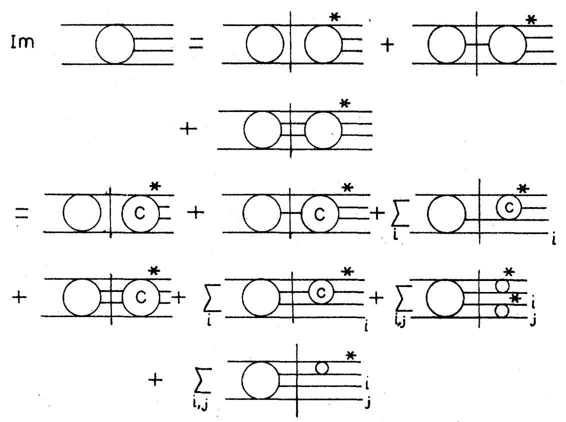

FIG. 5. Schematic representation of the unitarity relation in Eq. (18). The second line shows the amplitudes decomposed into fully connected (represented by a C) and disconnected parts.

three-body subenergy and the two pair subenergies. The only terms in unitarity having these thresholds will be related to these physical subenergy singularities. We are interested in finding the constraints of unitarity on the amplitudes F and G defined by Eq. (1). Because the two terms in Eq. (1) have distinct singularity structures Eq. (18) will be satisfied by the two terms in Eq. (1).

A. Unitarity constraints on F

First let us consider constraints of unitarity on F. This is a quasi-two-body amplitude where two independent pairs exist in the form of two isobars in the final state. The information about the independent pair interaction in the final state or, in other words, about the two pair-subenergy dependences of the amplitude F at a fixed total energy will be contained in the last but one term in Fig. 5. In this term a given pair threshold depends on the energy of the other pair.[11] As this involves only two-particle interaction, the last term in Fig. 5 also appears to contribute to this singularity. But we can neglect this term because it has a different threshold — the simple pair subenergy threshold — which was considered in detail in a previous work,[12] where we concentrated on this term to analyze the problem of four-body final states. Hence to find unitarity constraints on F we shall be limited to the consideration of the peculiar term — where two independent pairs interact among themselves in the final state — of four-body unitarity.[11] So the last but one term in Fig. 5 does, in fact, contain the entire physical subenergy discontinuity we are interested in. This term alone will give the constraints of unitarity on F. Keeping this term alone, we no longer have $\mathrm{Im}T_{2,4}$, but only the discontinuity across this subenergy cut, where a given pair threshold depends on the energy of the other pair. This will be called the absorptive part of $T_{2,4}$ ($\mathrm{Abs}T_{2,4}$) to stress the fact that, following Refs. 6, 12, we are dividing the amplitude into absorptive and dispersive parts, each of which can be complex (the absorptive part contains the physical discontinuity).

If we keep only this term in four-body unitarity (18), schematically we have the following for the discontinuity across this cut:

$$2\,\mathrm{Abs}T_{2,4}=\int T_{2,4'}\,\rho(4')T^{\dagger}_{2',2}T^{\dagger}_{2',2}\,, \tag{19}$$

where the two $T^{\dagger}_{2',2}$'s refer to the two two-body t matrices in the last but one term in Fig. 5. We now substitute the first term of Eq. (1) in Eq. (19) to obtain

$$\begin{aligned}
&\tfrac{1}{8}\sum_{\substack{i,j,k,l\\=1}}^{4} 2\,\mathrm{Abs}\left[\langle\vec{\mathrm{q}}\,|F_{ij,kl}(W)|\,\vec{\mathrm{p}}_i+\vec{\mathrm{p}}_j\rangle\frac{v_{ij}(\vec{\mathrm{M}}_{ij}{}^2)}{D_{ij}(\sigma_{ij})}\frac{v_{kl}(\vec{\mathrm{M}}_{kl}{}^2)}{D_{kl}(\sigma_{kl})}\right]\\
&\quad=\tfrac{1}{8}\sum_{\substack{i,j,k,l\\=1}}^{4}\sum_{\substack{a,b,c,d\\=1}}^{4}\int\langle \mathrm{q}\,|F_{ab,cd}(W)|\vec{\mathrm{p}}'_a+\vec{\mathrm{p}}'_b\rangle\frac{v_{ab}(\vec{\mathrm{M}}'_{ab}{}^2)}{D_{ab}(\sigma'_{ab})}\frac{v_{cd}(\vec{\mathrm{M}}'_{cd}{}^2)}{D_{cd}(\sigma'_{cd})}(2\pi)^4\delta^4(p_i+p_j-p'_i-p'_j)(2\pi)^4\delta^4(p_k+p_l-p'_k-p'_l)\\
&\qquad\times\delta^+(p'^2_j-m^2)\delta^+(p'^2_k-m^2)\delta^+(p'^2_l-m^2)\left[\prod_{i=1}^{4}\frac{d^4p'_i}{(2\pi)^3}\right]\frac{v_{kl}(\vec{\mathrm{M}}'_{kl}{}^2)v_{kl}(\vec{\mathrm{M}}_{kl}{}^2)}{D^*_{kl}(\sigma_{kl})}\frac{v_{ij}(\vec{\mathrm{M}}'^2_{ij})v_{ij}(\vec{\mathrm{M}}_{ij}{}^2)}{D_{ij}(\sigma_{ij})}\\
&\qquad\times\{1+[\delta^+(p'^2_i-m^2)-1](\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi}+\delta_{ak}\delta_{bl}+\delta_{al}\delta_{bk})\}\;.
\end{aligned}\tag{20}$$

The last factor in curly bracket puts all the four particles on their mass shell when $ab=ij$ or $ab=kl$. In other words, it accounts for the fact that in the unitary relation in Fig. 2 relative energies of the two independent pairs are separately conserved. The extra factor of $\frac{1}{8}$ on the right-hand side comes from the symmetry under interchange of the pairs ij and kl as well as ab and cd. Using

$$\mathrm{Abs}\left(\frac{F}{D_{ij}D_{kl}}\right)=F\,\mathrm{Abs}\left(\frac{1}{D_{ij}D_{kl}}\right)+\frac{1}{D^*_{ij}D^*_{kl}}\,\mathrm{Abs}\,F \tag{21}$$

as in the nonrelativistic case[11] and noting that $\mathrm{Abs}[1/(D_{ij}D_{kl})]=\mathrm{Im}[1/(D_{ij}D_{kl})]$ and using Eq. (13) for $\mathrm{Im}[1/(D_{ij}D_{kl})]$, we find that $\mathrm{Im}[1/(D_{ij}D_{kl})]$ terms on the left-hand side of Eq. (20) just cancel the terms on the right-hand side with $ab=ij$, $cd=kl$, $ab=kl$, and $cd=ij$ and permutations. Equating appropriate coefficients, taking account of symmetries, and cancelling common factors, then gives

$$2\,\mathrm{Abs}\langle \vec{q}\,|F_{ij,kl}(W)|\vec{p}_i+\vec{p}_j\rangle$$

$$=\frac{1}{8}\frac{1}{(2\pi)^4}\int\left(\prod_{i=1}^{4}d^4p_i'\right)\delta^4(p_i+p_j-p_i'-p_j')\delta^4(p_k+p_l-p_k'-p_l')\delta^+(p_j'^2-m^2)\delta^+(p_k'^2-m^2)\delta^+(p_l'^2-m^2)$$

$$\times v_{ij}(\vec{M}_{ij}'^2)v_{kl}(\vec{M}_{kl}'^2)\sum_{\substack{a,b,c,d=1\\ ab,cd\neq ij,kl}}^{4}\langle \vec{q}\,|F_{ab,cd}(W)|\vec{p}_a'+\vec{p}_b'\rangle\frac{v_{ab}(\vec{M}_{ab}'^2)}{D_{ab}(\sigma_{ab}')}\frac{v_{cd}(\vec{M}_{cd}'^2)}{D_{cd}(\sigma_{cd}')}\,. \tag{22}$$

This result, that the $F\,\mathrm{Im}[1/(D_{ij}D_{kl})]$ term must cancel with a corresponding term on the right-hand side by independent-pair two-body unitarity (13), is generally valid in all such calculations, identical particle or not, relativistic or not, and serves as a useful check on these calculations. This result is a very general result and holds for three-body and other similar problems.[6] Now it is clear that the last term in curly brackets in Eq. (20) must be there for this cancellation to be done, and the four-body intermediate state in Eq. (20) breaks into two independent two-body intermediate states as it should. When the term in small curly brackets in Eq. (20) does not contribute we can easily perform one of the energy-momentum integrations and effectively obtain a three-body intermediate state. It is easy to perform two of the energy-momentum integrations with two of the four-momentum conserving δ functions and two more of the energy integrations with two of the δ^+ functions. Then Eq. (22) becomes

$$2\,\mathrm{Abs}\langle \vec{q}\,|F_{ij,kl}(W)|\vec{p}_i+\vec{p}_j\rangle=\frac{1}{8}\frac{1}{(2\pi)^4}\int\frac{d^3p_j'}{2w_j'}\frac{d^3p_k'}{2w_k'}\delta^+(p_l'^2-m^2)v_{kl}(\vec{M}_{kl}'^2)v_{ij}(\vec{M}_{ij}'^2)$$

$$\times\sum_{\substack{a,b,c,d=1\\ ab,cd\neq ij,kl}}^{4}\langle \vec{q}\,|F_{ab,cd}(W)|\vec{p}_a'+\vec{p}_b'\rangle\frac{v_{ab}(\vec{M}_{ab}'^2)}{D_{ab}(\sigma_{ab}')}\frac{v_{cd}(\vec{M}_{cd}'^2)}{D_{cd}(\sigma_{cd}')} \tag{23}$$

where $w_n'^2=\vec{p}_n'^2+m^2$, and it is understood that the right-hand side of Eq. (23) has to be evaluated at the zeros of the arguments of the δ functions that have been integrated over. Next we make a transformation of integration variables in Eq. (23) to $\vec{k}$ and $\vec{t}$ defined by $\vec{p}_i'=\frac{1}{2}(\vec{p}+\vec{k}+\vec{t})$, $\vec{p}_j'=\frac{1}{2}(\vec{p}-\vec{k}-\vec{t})$, $\vec{p}_k'=\frac{1}{2}(-\vec{p}+\vec{k}-\vec{t})$, and $\vec{p}_l'=\frac{1}{2}(-\vec{p}-\vec{k}+\vec{t})$. Then we have $\vec{p}_i+\vec{p}_j=-(\vec{p}_k+\vec{p}_l)=\vec{p}$, and in these variables the conditions of the zeros of the arguments of the δ functions can be easily implemented on the right-hand side of Eq. (23), which takes the elegant form in the special case of identical particles each of mass m as follows:

$$2\,\mathrm{Abs}\langle \vec{q}\,|F(W)|\vec{p}\rangle=\frac{1}{(2\pi)^4}\int\frac{d^3k\,d^3t}{4w_j'w_k'}\delta^+(p_l'^2-m^2)v(\vec{M}_{ij}'^2)v(\vec{M}_{lk}'^2)$$

$$\times\left[\langle \vec{q}\,|F(W)|\vec{k}\rangle\frac{v(\vec{M}_{ik}'^2)}{D(\sigma_{ik}')}\frac{v(\vec{M}_{jl}'^2)}{D(\sigma_{jl}')}+\langle \vec{q}\,|F(W)|\vec{t}\rangle\frac{v(\vec{M}_{il}'^2)}{D(\sigma_{il}')}\frac{v(\vec{M}_{jk}'^2)}{D(\sigma_{jk}')}\right]. \tag{24}$$

Here we have

$$w_k'^2=m^2+\vec{p}_k'^2=m^2+\tfrac{1}{4}(-\vec{p}+\vec{k}-\vec{t})^2$$
$$w_j'^2=m^2+\vec{p}_j'^2=m^2+\tfrac{1}{4}(\vec{p}-\vec{k}-\vec{t})^2 \tag{25}$$

$$\sigma_{ik}'=2m^2+2(m^2+\vec{p}_i'^2)^{1/2}(m^2+\vec{p}_k'^2)^{1/2}-2\vec{p}_i'\cdot\vec{p}_k'\ \text{etc.}, \tag{26}$$

where the $\vec{p}_j'$'s have been already defined in terms of $\vec{p}$, $\vec{k}$, and $\vec{t}$. In order that Eq. (24) has a compact and simple form it is written in terms of old and new variables, but if needed all the quantities can be written in terms of new variables with help of Eqs. (25) and (26). It is easy to see that the two terms in the square bracket in Eq. (24) go into each other under the change $k\leftrightarrow t$. Hence the two terms under the integral in Eq. (24) are equal and Eq. (24) becomes

$$2\,\mathrm{Abs}\langle \vec{q}\,|F(W)|\vec{p}\rangle=\frac{2}{(2\pi)^4}\int d^3k\langle \vec{q}\,|F(W)|\vec{k}\rangle\int\frac{d^3t}{4w_j'w_k'}\delta^+(p_l'^2-m^2)v(\vec{M}_{ij}'^2)v(\vec{M}_{lk}'^2)\frac{v(\vec{M}_{ik}'^2)}{D(\sigma_{ik}')}\frac{v(\vec{M}_{jl}'^2)}{D(\sigma_{jl}')}\,. \tag{27}$$

B. Unitarity constraints on G

Next we would like to find the constraints of unitarity on G. This forms a quasi-two-body state, where one of the four particles is free and the three others appear in a correlated isobar state. Here we are interested in exploiting Eq. (18) to obtain the dependence of G on three-body subenergy at a fixed total energy. We are particularly interested in its physical-region singularities. Only terms in the unitarity relation having three-body subenergy threshold will be related to this subenergy singularity. This subenergy dependence of the amplitude G at a fixed total energy will be contained in the last but two terms in the unitarity relation in Fig. 5, because this term has the appropriate threshold. This term contributes to the three-body-subenergy unitarity cut which will give the correct unitarity constraint on G.[11] As before in the last subsection, keeping this term alone in Eq. (18) we no longer have $\mathrm{Im}T_{2,4}$, but only the discontinuity across this three-body subenergy cut. We again call this $\mathrm{Abs}T_{2,4}$ as in Sec. III A.

If we keep only this term in four-body unitarity (18), schematically we have the following for discontinuity across this cut:

$$2\,\mathrm{Abs}T_{2,4}=\int T_{2,4'}\,\rho(4')T^{\dagger}_{3',3}\delta\,, \tag{28}$$

where $T_{3',3}$ is the connected three-body amplitude and δ refers to the particle which does not interact. Equation (28) has contributions corresponding only to the last but two terms in Fig. 5. We now substitute the second term of Eq. (1) into Eq. (28) to obtain

$$\sum_{\substack{i,j,k,l\\=1}}^{4} 2\,\mathrm{Abs}\left\{\langle\vec{q}\,|\,G_{i,jkl}(W)\,|\vec{p}_i\rangle\frac{w_{jk,l}(\vec{M}_{jk,l}{}^2)}{D'_{jkl}(\sigma_i)}\frac{v_{jk}(\vec{M}_{jk}{}^2)}{D_{jk}(\sigma_{jk})}\right\}$$

$$=\sum_{\substack{i,j,k,l\\=1}}^{4}\tfrac{1}{2}\sum_{\substack{a,b,c,d\\=1}}^{4}\int\langle\vec{q}\,|\,G_{a,bcd}(W)\,|\vec{p}\,'_a\rangle\frac{w_{bc,d}(\vec{M}'_{bc,d}{}^2)}{D'_{bcd}(\sigma'_a)}\frac{v_{bc}(\vec{M}'_{bc}{}^2)}{D_{bc}(\sigma'_{bc})}(2\pi)^4\delta^4(p_i-p'_i)(2\pi)^4\delta^4(p_j+p_k+p_l-p'_j-p'_k-p'_l)$$

$$\times\,\delta^+(p'^2_j-m^2)\delta^+(p'^2_k-m^2)\delta^+(p'^2_l-m^2)\left(\prod_{i=1}^{4}\frac{d^4p'_i}{(2\pi)^3}\right)$$

$$\times\, w_{jk,l}(\vec{M}'^2_{jk,l})\frac{v_{jk}(\mathrm{M}'_{jk}{}^2)}{D_{jk}(\sigma'_{jk})}\frac{1}{D'^{*}_{jkl}(\sigma_i)}w_{jk,l}(\vec{M}_{jk,l}{}^2)\frac{v_{jk}(\vec{M}_{jk}{}^2)}{D_{jk}(\sigma_{jk})}\quad. \tag{29}$$

Here as in Eq. (16), we have used $D=D^*$ across the three-body cut, because the two-body D function is continuous across this cut. Now using $\mathrm{Abs}(G/D')=G\,\mathrm{Abs}(1/D')+(1/D'^*)\,\mathrm{Abs}G$ as before and using Eq. (17) we find that the $\mathrm{Abs}(1/D')$ term on the left-hand side of Eq. (29) cancels the term on the right-hand side with $a=i$. Equating appropriate coefficients and cancelling common factors then gives

$$2\,\mathrm{Abs}\langle\vec{q}\,|\,G_{i,jkl}(W)\,|\vec{p}_i\rangle=\frac{1}{2}\,\frac{1}{(2\pi)^4}\int\left(\prod_{i=1}^{4}d^4p'_i\right)\delta^4(p_i-p'_i)\delta^4(p_j+p_k+p_l-p'_j-p'_k-p'_l)\delta^+(p'^2_j-m^2)$$

$$\times\,\delta^+(p'^2_k-m^2)\delta^+(p'^2_l-m^2)\frac{v_{jk}(\vec{M}'_{jk}{}^2)}{D_{jk}(\sigma'_{jk})}w_{jk,l}(\vec{M}'_{jk,l}{}^2)$$

$$\times\sum_{\substack{a,b,c,d=1\\a\neq i}}^{4}\langle\vec{q}\,|\,G_{a,bcd}(W)\,|\,\vec{p}\,'_a\rangle\frac{w_{bc,d}(\vec{M}'_{bc,d}{}^2)}{D_{bcd}(\sigma'_a)}\frac{v_{bc}(\vec{M}'_{bc}{}^2)}{D_{bc}(\sigma'_{bc})}\,. \tag{30}$$

In the special case of identical bosons it is easy to see that the contributions of terms with $a=j$ are exactly the same as those with $a=k$. Also, in the case of $a=l$, the contributions of terms with $d=k$ are the same as those with $d=j$. In the case of identical particles only the distinct terms in Eq. (30) will contribute, because the unitary relation for identical particles will contain only the distinct terms. Hence the nine terms in the summation in Eq. (30) will reduce to five terms only because four of the terms are repeated twice. Then for identical spinless bosons after the change of variable $\vec{p}\,'_i=\vec{p}$, $\vec{p}\,'_j=\vec{t}$, and $\vec{p}\,'_k=\vec{k}$, Eq. (30) becomes (see Appendix for details)

$$2\,\mathrm{Abs}\langle\vec{q}\,|G(W)|\vec{p}\rangle = \frac{2}{(2\pi)^4}\int\frac{d^3k}{2w_k}\frac{\langle\vec{q}\,|G(W)|\vec{k}\rangle}{D'(\sigma_k')} \times\int\frac{d^3t}{2\omega_t}\delta^+(p_l'^2-m^2)\left\{\left[\frac{w(\vec{M}'^{\,2}_{jl,i})v(\vec{M}'^{\,2}_{jl})}{D(\sigma'_{jl})}+\frac{w(\vec{M}'^{\,2}_{ij,l})v(\vec{M}'^{\,2}_{ij})}{D(\sigma'_{ij})}+\frac{w(\vec{M}'^{\,2}_{il,j})v(\vec{M}'^{\,2}_{il})}{D(\sigma'_{il})}\right]\frac{w(\vec{M}'^{\,2}_{jk,l})v(\vec{M}'^{\,2}_{jk})}{D(\sigma'_{jk})} + \frac{w(\vec{M}'^{\,2}_{jl,k})v(\vec{M}'^{\,2}_{jl})}{D(\sigma'_{jl})}\left[\frac{w(\vec{M}'^{\,2}_{jl,i})v(\vec{M}'^{\,2}_{jl})}{2D(\sigma'_{jl})}+\frac{w(\vec{M}'^{\,2}_{il,j})v(\vec{M}'^{\,2}_{il})}{D(\sigma'_{il})}\right]\right\}. \tag{31}$$

Here σ'_{ab}'s are defined by Eq. (26) with $\vec{p}$ defined just before Eq. (31) and $w_k^{\,2}=\vec{k}^2+m^2$, $w_t^{\,2}=\vec{t}^{\,2}+m^2$, and $\sigma_k'=(P-p_k')^2=s+m^2-2\sqrt{s}\,(m^2+\vec{k}^2)^{1/2}$. Before we go to actual implementation of these constraints of unitarity, we give a brief discussion concerning their physical significance.

C. Discussions

As in other similar few-body problems[12] in nuclear[11,13] and particle physics,[6] constraints of unitarity (27) and (31) show that constraints of quantum mechanics introduce coherence and variation on amplitudes usually taken as independent and constant in isobar phenomenology. In particular this forces F and G to have complicated branch points in the pair and three-body subenergies at the edge of the phase spaces. These correspond to "independent pair" and three-body thresholds. In the nonrelativistic case[11] F has two square-root branch points in the subenergies of the two independent pairs. The term G has a singularity that corresponds to a simple three-body threshold and in the nonrelativistic case[11] this corresponds to the $\epsilon^2\ln(-\epsilon)$ singularity, where ϵ is the three-body subenergy. The positions of these branch points make them particularly important in cases such as threshold enhancements in nuclear physics and resonant final-state interactions in nuclear and particle physics. A test of the importance of the unitarity constraints in a particular problem is easily made with unitarity constraints (27) and (31) themselves. The F's and G's are assumed to be constants and the right-hand sides of Eqs. (27) and (31) are calculated. If they generate small Abs F or Abs G (measured against the assumed scale of F or G, respectively) the assumption of constant F or G is a good approximation. If Abs F or Abs G are large by the scale of F or G, the constraints of unitarity are important and must be implemented. It is true that if there is a resonant final-state interaction this simple way of calculating Abs F or Abs G will violate analyticity and will introduce spurious rapid variations in Abs F or Abs G. This has been pointed out numerically[5] and analytically[6] in the case of the three-body problem. A similar demonstration is possible in the case of the four-body problem and will be an interesting future project. But the simple test to determine the importance of the constraints of unitarity holds good even in cases where Abs F or Abs G have large rapid spurious variations.

IV. IMPLEMENTATION

In the last subsection we saw unitarity forces F and G to be singular with the discontinuities across the singularities given by Eqs. (27) and (31), respectively. When we turn to the implementation of these constraints, we find two cases of practical interest. The first is final-state two-body and three-body threshold enhancements such as we encounter in nuclear physics. We hope from our experience in the three-body problem[6] that a simple input guess for Disp G in Eq. (2) can serve as an excellent phenomenology which deals with the subenergy dependence at a fixed total energy. In the case of the three-body problem it has been demonstrated that subenergy unitarity can represent the shape of the amplitude and can serve as a basis for phenomenology.[16] The most interesting case in particle physics is the one with two- and three-body resonant final-state interactions. As in the three-body case[6] a simple input guess for Disp G (or Disp F) in Eq. (2) can be disastarous. Here unitarity must be implemented with analyticity.

In order to be able to implement the constraints of unitarity (to make the algebra simple) we shall be limited to the consideration of four identical spinless bosons each of mass m. The correct method of implementation is to write a dispersion relation for the amplitudes F and G in terms of their discontinuities given by Eqs. (27) and (31), respectively. From experience in three-body[6] and four-body[11] problems we find that the best way to implement unitarity constraints (27) and (31), in order that they give us information about the subenergy discontinuities, without introducing other spurious singularities is to disperse in total energy. Because the essential feature of the discontinuities (27) and (31) is a simple δ function in $s=W^2$ (as we shall see later) it is then trivial to disperse in s and maintain total energy analyticity. The functions F and G have other singularities that arise from thresholds corresponding to other terms in the unitary relation in Fig. 5. This method of implementation will not interfere with other singularities of the functions F and G.

We write the dispersion relation for $F(s)$ or $G(s)$ in schematic partial-wave form. Let us call the function $A(s)$ and assume that $A(s)$ goes to zero sufficiently rapidly as $s\to\infty$. Then we can write the dispersion relation for partial wave $A(s)$ as[11]

$$A(s)=R(s)+\frac{1}{\pi}\int\frac{ds'}{s'-s}\mathrm{Abs}A(s'), \tag{32}$$

where $R(s)$ is a term that does not have the discontinuity. Now before dispersing the singularities in Eqs. (27) and (31) we note that the δ function occurring there can be written as

$$\begin{aligned}\delta^+(p_l'^2-m^2)&=\delta^+((P-p_i'-p_j'-p_k')^2-m^2)\\&=\delta^+((W-w_i'-w_j'-w_k')^2-w_l'^2)\\&=\frac{1}{2w_l'}\delta(W-w_i'-w_j'-w_k'-w_l')\\&=\frac{w_i'+w_j'+w_k'+w_l'}{w_l'}\\&\quad\times\delta(s-(w_i'+w_j'+w_k'+w_l')^2),\end{aligned} \tag{33}$$

where $w_i'^2=m^2+\vec{p}_i'^2$, etc. $\vec{p}_l'=-\vec{p}_i'-\vec{p}_j'-\vec{p}_k'$ and $\vec{p}\,''$s used in Eqs. (27) and (31) are defined in terms of $\vec{p}$, $\vec{k}$, and $\vec{t}$ in Subsecs. III A and III B, respectively. Hence the essential feature of the discontinuities given by the constraints of unitarity is a simple δ function in s. Schematically the discontinuity is represented by

$$\mathrm{Abs}A(s')=\mathcal{a}(s')\delta(s'-s_0), \tag{34}$$

apart from multiplying kinematic factors and phase-space integrals. Substituting (34) into (32) we get

$$A(s)=R(s)+\frac{1}{\pi}\frac{\mathcal{a}(s_0)+a(s,s_0)}{s_0-s}, \tag{35}$$

where $a(s_0,s_0)=0$ and hence does not contribute to the discontinuity of $A(s)$ because of the δ function in Eq. (34). Function a is arbitrary except for this condition and could be included in $R(s)$, but it is more convenient to keep it explicitly. This is because, as we shall see later, by choosing function a cleverly it is easy to maintain total energy analyticity with subenergy unitarity and analyticity. For example, if we take $a(s,s_0)=\mathcal{a}(s)-\mathcal{a}(s_0)$, Eq. (35) becomes

$$A(s)=R(s)+\frac{1}{\pi}\frac{\mathcal{a}(s)}{s_0-s}. \tag{36}$$

In general if in Eq. (34) we have

$$\mathcal{a}(s')=\mathcal{a}_1(s')\mathcal{a}_2(s')\mathcal{a}_3(s')\ldots\mathcal{a}_N(s') \tag{37}$$

then after implementation with proper choices of $a(s,s_0)$ we can have s as an argument in some of the factors and s_0 as an argument in the rest of the factors of $\mathcal{a}$ in Eq. (37). The way to achieve this has been demonstrated in Ref. 12. The dispersion integral essentially puts the argument of the δ function in the denominator and in the multiplying factors we may or may not implement the constraint of the δ function (i.e., write in terms of s_0 or s).

From this discussion it is clear that if we disperse the discontinuity of Eq. (27) in s by a dispersion relation, we get

$$\begin{aligned}\langle\vec{q}\,|F(W)|\,\vec{p}\rangle=\langle\vec{q}\,|R_1(W)|\,\vec{p}\rangle+\frac{2}{(2\pi)^5}\int d^3k\langle\vec{q}\,|F(W)|\,\vec{k}\rangle\int\frac{d^3t}{4w_j'w_k'}\frac{w_i'+w_j'+w_k'+w_l'}{w_l'[(w_i'+w_j'+w_k'+w_l')^2-W^2]}\\\times v(\vec{M}_{ij}'^2)v(\vec{M}_{lk}'^2)\frac{v(\vec{M}_{ik}'^2)}{D(\sigma_{ik}')}\frac{v(\vec{M}_{jl}')}{D(\sigma_{jl}')},\end{aligned} \tag{38}$$

where

$$\begin{aligned}\sigma_{ik}'&=(P-p_j'-p_l')^2=s+\sigma_{jl}'-2P\cdot(p_j'+p_l')\\&=s+2m^2+2(m^2+\vec{p}_j'^2)^{1/2}(m^2+\vec{p}_l'^2)^{1/2}-2\vec{p}_j'\cdot\vec{p}_l'-2\sqrt{s}\,[(m^2+\vec{p}_j'^2)^{1/2}+(m^2+\vec{p}_l'^2)^{1/2}],\ \text{etc.},\end{aligned} \tag{39}$$

$$w_n'^2=(m^2+p_n'^2),\quad n=i,j,k,l, \tag{40}$$

and where $\vec{p}\,''$s are defined in Sec. III A in terms of $\vec{p}$, $\vec{k}$, and $\vec{t}$.

Similarly dispersing the discontinuity of Eq. (31) in s, we get

$$\langle \vec{q} | G(W) | \vec{p} \rangle = \langle \vec{q} | R_2(W) | \vec{p} \rangle + \frac{2}{(2\pi)^5} \int \frac{d^3k}{2w_k} \frac{\langle \vec{q} | G(W) | \vec{k} \rangle}{D'(\sigma_k')} \int \frac{d^3t}{2w_t} \frac{w_k + w_t + w_p + w_l'}{w_l'[(w_k + w_t + w_p + w_l')^2 - W^2]}$$
$$\times \left\{ \left[\frac{w(\vec{M}_{jl,i}'^2) v(\vec{M}_{jl}'^2)}{D(\sigma_{jl}')} + \frac{w(\vec{M}_{ij,l}'^2) v(\vec{M}_{ij}'^2)}{D(\sigma_{ij}')} + \frac{w(\vec{M}_{il,j}'^2) v(\vec{M}_{il}'^2)}{D(\sigma_{il}')} \right] \frac{w(\vec{M}_{jk,l}'^2) v(\vec{M}_{jk}'^2)}{D(\sigma_{jk}')} + \frac{w(\vec{M}_{jl,k}'^2) v(\vec{M}_{jl}'^2)}{D(\sigma_{jl}')} \left[\frac{w(\vec{M}_{jl,i}'^2) v(\vec{M}_{jl}'^2)}{2D(\sigma_{jl}')} + \frac{w(\vec{M}_{il,j}'^2) v(\vec{M}_{il}'^2)}{D(\sigma_{il}')} \right] \right\} \quad (41)$$

where σ_{ik}', etc., are defined as in Eq. (39) with $\vec{p}_i' = \vec{p}$, $\vec{p}_j' = \vec{t}$, $\vec{p}_k' = \vec{k}$, and $\vec{p}_l' = -\vec{p} - \vec{t} - \vec{k}$ and where $w_k^2 = m^2 + \vec{k}^2$, $w_t^2 = m^2 + \vec{t}^2$, $w_p^2 = m^2 + \vec{p}^2$, and $w_l'^2 = m^2 + (\vec{k} + \vec{t} + \vec{p})^2$.

This particular method of implementation in Eqs. (38) and (41) is motivated by the fact that in the final integral equations we wish to get F and G as functions of W, while v's and w's should not depend on W. In other words, in the language of Eq. (37) we have taken v's and w's as functions of relative momenta and F, and G as functions of W and that is what is physically expected of them. This has the added advantage that the left-hand cuts of v's and w's do not get involved in the momentum integrations of Eqs. (38) and (41). Another motivation for the present choice is that we get a set of dynamical equations for F and G and not just an integral representation for them. It is interesting to note that all these choices are, of course, equivalent at the zero of the δ function so that the discontinuities (27) and (31) are independent of these choices. Any way of implementation of unitarity by dispersion relation satisfies that particular "subenergy analyticity", but the present choice also satisfies total energy analyticity.

Equations (38) and (41) are the "minimal" implementation of unitarity and analyticity, provide useful solvable phenomenology for F and G without spurious W singularities. This simple minimal implementation of unitarity and analyticity gives the minimal four-body dynamical scheme with the specific assumptions we made about the interaction. In Eq. (1), through the functions v's and w's we have introduced the assumption of separable interaction in disguise. We chose to make this assumption because otherwise we will encounter multivariable integral equations in case of simple one variable integral Eqs. (38) and (41) without spurious s singularities. Equations (38) and (41) are really integral equations in one vector variable and hence after partial-wave decomposition are simple equations to solve. Using the appropriate choices of driving terms R_1 and R_2, Eqs. (38) and (41) are the simplest form of four-body relativistic equations. With a simple choice of driving terms and the functions v and w, Eqs. (38) and (41) will serve a useful basis for isobar phenomenology. Eqs. (38) and (41) probably will not give the correct s dependence but will give the correct subenergy dependence, which is important for isobar phenomenology.

Equations (38) and (41) are schematically represented in Fig. 6. It is easy to see that the dynamics shown in Fig. 6 do not contain the full dynamics and hence do not satisfy the full content

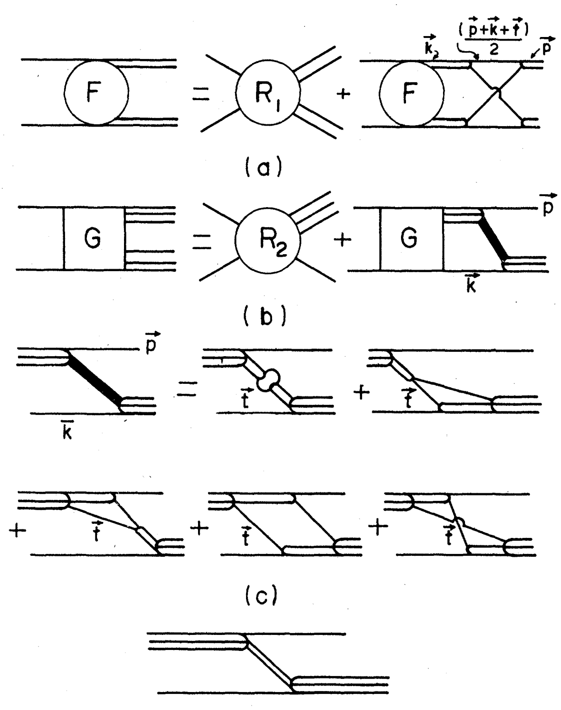

FIG. 6. Schematic representation of (a) Eq. (38), (b) Eq. (41), and (c) the expansion of a term in Eq. (41) with (d) the modified form of the first term on the right in (c).

of unitarity shown in Fig. 5. But this simple model in Fig. 6 — which we derive from the general constraints of quantum mechanics — will give us an idea of the important dynamical aspects of the problem if subenergy unitarity is important, whereas the full dynamical scheme is very complicated and hence is difficult to visualize. All these are a generalization of our analysis of four-body final states in the nonrelativistic case[11] — where we derived useful models for the system of four nucleons.

In other similar implementations of unitarity and analyticity[6,12,13], in few-body problems we find that subenergy unitarity with the associated analyticity implies the full content of unitarity. From the previous analysis, it is clear that if we formulate N-body problems in terms of quasi-$(N-1)$-body amplitudes, simple implementation of unitarity and analyticity will give the full dynamical scheme but may not necessarily give simple equations to solve. The present analysis shows that other formulations of the problem are possible — which do not give the full dynamics but give useful simple equations for making approximations. In the three-body problem only one formulation is possible and it gives the full dynamical scheme — the Faddeev equation with separable interactions, the Amado model,[13] or the Blankenbecler-Sugar equations.[9]

We conclude this section with a brief discussion about the contents of Eqs. (38) and (41). If we look at Fig. 6(c) [or at Eq. (41)] we see that in the first term on the right there is a bubble in the already dressed two-body propagator. In other words, this means a repetition of the two-body t matrices. This is redundant and to do any calculation we should replace this term by a simple two-body isobar exchange without a bubble. This is shown in Fig. 6(d). This term appeared in the present way of implementation of unitarity because by looking at the last but one and last but two terms in the unitary relation in Fig. 5 we find that there must be an intermediate state where four particles are free. This forced us to have this particular term in Fig. 6.

V. SUMMARY AND DISCUSSION

We considered the effects of unitarity — independent-pair two-body unitarity and three-body-subenergy unitarity — on four body final states, derived explicitly the constraints imposed on the isobar amplitudes by these forms of unitarity and obtained Fredholm integral equations for these amplitudes which incorporate unitarity and analyticity. From our experience in solving three-body problems[16] we can say that the approximate solutions (38) and (41) of the four-body problem will give a reasonable picture of subenergy dependence even though they do not describe very well the total energy behavior. In the framework of isobar, sequential decay, or quasiparticle models, the four-body states are thought of as being made of quasi-two-body states, where the quasiparticle states are either two two-body correlated states or one three-body correlated state and a free particle. This model is justified when the two- and three-body t matrices are dominated by the presence of resonance poles in nuclear and particle physics or bound-state poles in nuclear physics. Though the amplitudes for forming these quasi-two-body states — F and G — are usually taken to be constants and independent in phenomenological analysis, we have seen that constraints of quantum mechanics force them to have physical branch points and be coherent and interrelated. The different F amplitudes taken together give us information about independent-pair interaction in four-body final states. The same is true for G in the case of three-body interactions.

From a practical point of view this work permits a systematic study of when unitarity effects are important in a particular case as well as an implementation of these effects. In the three-body case we find situations where unitary corrections are crucial,[16] and where they are reasonable.[4] A similar spectrum of four-body examples exists and the present work provides a foundation for future works in that direction. This type of study will lead to useful approximation techniques for four-body final state. The present formalism is a very simplified statement of the actual state of affairs because it neglects all the internal quantum numbers (spin, isospin, etc.) of the particles. But if needed it is not difficult to generalize the results to include them.

The present analysis should find a very interesting application in reactions such as $\pi N \rightarrow \pi\pi\pi N$. Because of the presence of various resonances that overlap strongly in certain regions of phase space or Dalitz plot we expect that effects of unitarity will be important. These must be implemented with analyticity, otherwise the isobar amplitudes will pick up spurious singularities. In other words a simple assumption for Disp G in Eqs. (2) and (3) — as in done in isobar phenomenology — will grossly violate the required analyticity structure of the amplitude G. In this reaction the present formulation can be used to predict the isobar amplitudes such as $\pi N \rightarrow \epsilon\Delta$, $\pi N \rightarrow \rho\Delta$, $\pi N \rightarrow (\pi\pi\pi)N$, $\pi N \rightarrow \pi(\pi\pi N)$, etc., where $(\pi\pi\pi)$ and $(\pi\pi N)$ represent the various resonances in the 3π and $2\pi N$ systems. A classification of the different states in the 3π system is given in Ref. 10. This method can be used to analyze other four-body final states such as $\pi\pi NN$, $\pi\pi\pi K$, and possibly many other interesting cases. Of these, the $\pi\pi NN$ problem is particularly interesting where the present formalism can be used to predict the isobar amplitudes such as $\pi D\to\Delta\Delta$, $\pi D\to\epsilon D$, $\pi D\to\rho D$, etc. In special problems various approximation schemes may emerge in the future, but at present we have the integral equations if unitarity effects are important.

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APPENDIX

In Eq. (30) it is easy to perform two of the four-momentum integrations with two of the four-dimensional δ functions as well as two of the energy integrals with two of the remaining δ^+ functions. After performing these integrations Eq. (30) can be written as

$$
\begin{aligned}
2\,\mathrm{Abs}\langle\vec{q}\,|G_{i,jkl}(W)|\vec{p}_i\rangle=&\frac{1}{(2\pi)^4}\int\frac{d^3p'_j d^3p'_k}{4w'_jw'_k}\,\delta^+(p_l'^2-m^2)\,\frac{v_{jk}(M_{jk}'^2)}{D_{jk}(\sigma'_{jk})}\,w_{jk,l}(\vec{M}_{jk,l}'^2)\\
&\times\Bigg\{\Bigg[\frac{\langle\vec{q}\,|G_{j,kli}(W)|\vec{p}'_j\rangle}{D'_{kli}(\sigma'_j)}\Bigg(\frac{w_{kl,i}(\vec{M}_{kl,i}'^2)v_{kl}(\vec{M}_{kl}'^2)}{D_{kl}(\sigma'_{kl})}+\frac{w_{il,k}(\vec{M}_{il,k}'^2)v_{il}(\vec{M}_{il}'^2)}{D_{il}(\sigma'_{il})}\\
&\qquad+\frac{w_{ik,l}(\vec{M}_{ik,l}'^2)v_{ik}(\vec{M}_{ik}'^2)}{D_{ik}(\sigma'_{ik})}\Bigg)\Bigg]+[k\leftrightarrow j]\\
&+\Bigg[\frac{\langle q\,|G_{l,ijk}(W)|p'_l\rangle}{D'_{ijk}(\sigma'_l)}\Bigg(\frac{w_{jk,i}(\vec{M}_{jk,i}'^2)v_{jk}(\vec{M}_{jk}'^2)}{D_{jk}(\sigma'_{jk})}+\frac{w_{ij,k}(\vec{M}_{ij,k}'^2)v_{ij}(\vec{M}_{ij}'^2)}{D_{ij}(\sigma'_{ij})}\\
&\qquad+\frac{w_{ik,j}(\vec{M}_{ik,j}'^2)v_{ik}(\vec{M}_{ik}'^2)}{D_{ik}(\sigma'_{ik})}\Bigg)\Bigg]\Bigg\},
\end{aligned}
\tag{A1}
$$

where $[k\leftrightarrow j]$ denotes the terms in the first large square bracket with k and j interchanged. So the two sets of terms, whose coefficients are $G_{j,kli}$ and $G_{k,jli}$, when taken together are symmetric with respect to the $k\leftrightarrow j$ interchange. Similarly the last two terms, whose coefficient is $G_{l,ijk}$, are symmetric with respect to the $k\leftrightarrow j$ interchange. But p'_j and p'_k are integration variables. For identical particles the distinct terms in Eq. (A1) will contribute. That is what has been considered in Eq. (31) after a change of integration variables in last two terms. In case of nonidentical particles all nine terms in Eq. (A1) are supposed to be considered.

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