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Little groups, discrete symmetry, and an Sp(4) × U(1) theory of the weak and electromagnetic interactions

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We introduce a graphical method for determining the little groups of vacuum expectation values of Higgs mesons in $G\times U(1)$ gauge theories. This method is particularly useful when rank $G=2$. A general method for natural suppression of intramultiplet mixings of equally charged fermions using discrete symmetries is given. For concreteness we develop these methods using the gauge group Sp(4)×U(1) and present a quasivectorlike theory of the weak and electromagnetic interactions based on this group. This theory insures e-μ and Cabibbo universality, the absence of right-handed currents in neutron, hyperon, and muon decay, suppression of flavor-changing neutral currents and suppression of untenably large contributions to various weak processes such as the K_LK_S mass difference and lepton-number-nonconserving decays. Discrete symmetries are used to discuss the fermion mass spectrum of this theory and, finally, to predict Cabibbo type mixing angles in terms of ratios of fermion masses.

I. INTRODUCTION

Unified gauge models of the weak and electromagnetic interactions based on the group SU(2) × U(1) have been rather successful in explaining various properties of weak decays and charged- and neutral-current neutrino interactions. However, the observation of high-energy trimuons by the Harvard-Pennsylvania-Wisconsin-Fermilab (HPWF) group[1] raises serious questions about the validity of such theories. The HPWF group concludes that the trimuons arise from the production and cascade decay of heavy leptons M^- and B^0. The decay chain is $\nu_\mu+N\to M^-+X$, $M^-\to\mu^-+B^0+\bar\nu_\mu$, and finally $B^0\to\mu^-+\mu^++\nu_\mu$. The measured rate of trimuon production is $R(\nu_\mu\to\mu^-\mu^-\mu^+)/R(\nu_\mu\to\mu^-)\simeq5\times10^{-4}$. An analysis of the invariant-mass distributions in the above sequential decay indicates that $m_{M^-}\simeq7.0^{+3.0}_{-1.0}$ GeV and $m_{B^0}\simeq3.5^{+1.5}_{-0.4}$ GeV. With M^- very massive and the branching ratio $B(M^-\to\mu^-\mu^-\mu^+)$ small it is necessary to have the $\nu_\mu\to M^-$ transition occur at nearly full strength to account for the trimuon production rate. This runs afoul with $e-\mu$ and Cabibbo (quark-lepton) universality in most SU(2) × U(1) models. Several authors[2] have constructed SU(2)×U(1) models in which universality is restored by allowing the electron and light quarks to undergo the appropriate (large) mixing with a heavy lepton and heavy quarks, respectively. These models do not seem to violate any existing experimental constraints. They are, however, rather unnatural and contrived and one is led to ask whether there may not be a simpler and more believable alternative.

In this spirit we note that it is possible to have a full-strength $\nu_\mu\to M^-$ transition if we introduce *new gauge bosons* which couple ν_μ to M^-. This, of course, can only be accomplished by enlarging the gauge group. There have been a number of attempts[3] to implement this data using SU(3) × U(1), SO(4)×U(1), and SU(4)×U(1) as gauge groups with varying degrees of success. Most of these models can account for present phenomenology, including, of course, trimuon production. Their greatest drawback lies in their complexity. Unless otherwise constrained, these theories allow prolific mixings among fermions of equal charge and chirality. These mixings do violence to the notions of e-μ and Cabibbo universality, and lead to right-handed currents in neutron, hyperon, and μ decay. They also allow flavor-changing neutral currents (including the unobserved neutral d-s quark current) and untenably large values for the K_LK_S mass difference, the $K_L\to\mu\bar\mu$ rate, and lepton-number-nonconserving processes such as $\mu\to e\gamma$ and $\mu\to ee\bar e$. We can, of course, set unwanted mixing angles to zero by fiat, thus circumventing the above problems. Unfortunately, this procedure is as artificial as the SU(2)×U(1) models that we are trying to improve upon. What is needed is a group-theoretical way to naturally suppress unwanted mixings. We will examine in Secs. IV and V methods for suppressing such mixings based on the liberal use of discrete symmetry groups. These methods are generally applicable to any $G\times U(1)$ gauge theory but, for concreteness, we will implement them for the group Sp(4)×U(1). A second complexity that arises in gauge theories with enlarged gauge groups is the difficulty of determining the "little groups" of the vacuum expectation values (VEV's) of Higgs mesons. Though a trivial undertaking in most SU(2)×U(1) theories, this is a serious problem when dealing with enlarged gauge groups and Higgs mesons in representations other than the fundamental one.[4] In Sec. III we introduce a graphical method for determining little groups based on the Cartan weight diagrams. This method is, in theory, applicable to any $G\times U(1)$ gauge theory, but, in practice, is most useful when rank $G=2$. Here again, we illustrate these techniques for the gauge group $Sp(4)\times U(1)$. It is our contention that once one knows how to find little groups and how to naturally suppress unwanted mixings, gauge theories based on enlarged gauge groups become tractable and are rich in their theoretical implications.

The $Sp(4)\times U(1)$ gauge model naturally ensures (with the proviso stated in Sec. V) e-μ and Cabibbo universality, the absence of right-handed currents in neutron, hyperon, and muon decay, suppression of flavor-changing neutral currents, and suppression of untenably large contributions to the $K_L K_S$ mass difference, the $K_L\to\mu\bar{\mu}$ rate, the weak contributions to the e and μ anomalous magnetic moments, and lepton-number-nonconserving processes. The phenomenological implications of the model will be fully discussed in a subsequent paper. We note that our model contains an absolutely stable, neutral heavy lepton with interesting cosmological implications.[5] The existence of a stable, heavy fermion would appear to be a general feature of any gauge model based on an extended gauge group provided that certain intramultiplet fermion mixings are absolutely suppressed. In our model such mixings are naturally suppressed by the RU discrete symmetry. (See Sec. IV.) The stable, heavy fermion can be charged or neutral, lepton or quark depending on the relative magnitudes of the fermion masses. In this paper the acceptance of the leptonic cascade explanation of trimuon events demands that this fermion be a neutral lepton. In Sec. VII we modify our model to show how mixing angles (in particular the Cabibbo angle) can be predicted in terms of fermion mass ratios by the use of discrete symmetries.[6] The various properties of the group Sp(4) necessary in our analysis are derived and catalogued in Appendixes A, B, and C.

II. FERMION ASSIGNMENTS AND HIGGS MESONS

In this section we discuss the fermion assignments and Higgs mesons for a quasivectorlike theory of the weak and electromagnetic interactions based on the gauge group $Sp(4)\times U(1)$. The charge operator is chosen to be

$$Q=T_3^{\alpha}+T_3^{\gamma}+\tfrac{1}{2}Y\,, \tag{2.1}$$

where Y is the generator for U(1). Attempts to put μ-related leptons into the 5_c or 10_c representations either cannot account for trimuon production (via leptonic cascade), allow for unobserved "wrong sign" $(\mu^+\mu^+\mu^-)$ trimuons at the same rate as $\mu^-\mu^-\mu^+$ trimuons, or introduce exotic doubly charged leptons. We therefore assign left and right chiral fermions in the μ family to the fundamental 4_c representation of $Sp(4)\times U(1)$. Considerations of e-μ and Cabibbo universality then require us to assign all other left and right chiral fermion families to the 4_c representation. The absence of right-handed currents in neutrino and hyperon β decay necessitates the introduction of both left and right chiral quark singlets. Similarly, the purely left-handed chirality of the e and μ neutrinos implies left-handed lepton singlets. Fermion assignments are shown in Fig. 1 and anticipate the results of Sec. IV. The primes in Fig. 1 indicate possible mixings between equally charged, same chirality fermions. These mixings can be broken into two categories: (1) intermultiplet mixings (as between u and c) and (2) intramultiplet mixings (as between u and t_1).

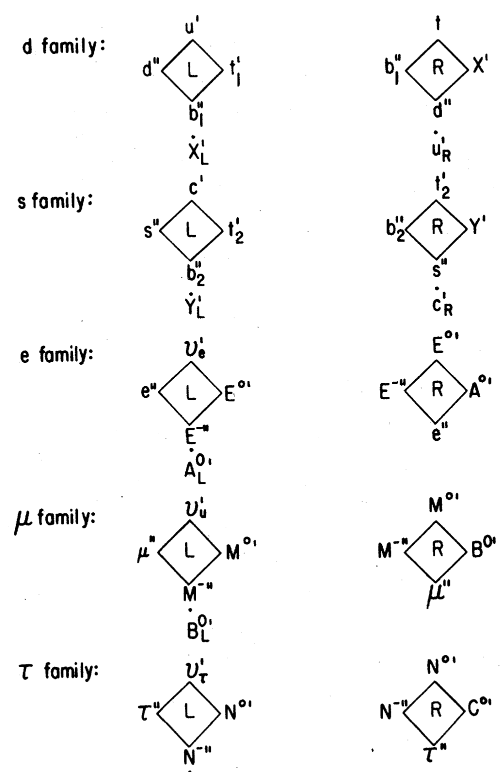

FIG. 1. Fermion assignments for a quasivectorlike gauge model based on $Sp(4)\times U(1)$. The primes indicate allowed mixings of equally charged, same chirality, fermions.

The first type of mixings occurs (as they must for Cabibbo universality) in the usual SU(2) × U(1) gauge models. The latter arise only in theories based on enlarged gauge groups. Intramultiplet mixings vastly increase the number of mixing angles and CP-violating phases in a model. More importantly, if such mixings are not forbidden they lead to flavor-changing transitions in the neutral current of *at least one* (and usually all) Z^0-like vector boson. It is the first task of any theory that aspires to natural suppression[7] of flavor-changing neutral currents to prevent intramultiplet mixings group theoretically. This can be done in several ways. In Sec. IV we expand on one such method and apply it to the Sp(4) × U(1) gauge group. Our model will incorporate this group-theoretical suppression. Intermultiplet mixings still can (and do) occur. We want to forbid such mixings between b_1 and b_2 and E^- and M^- in order to suppress large contributions to the $K_L K_S$ mass difference, the $K_L \to \mu\bar{\mu}$ rate, and lepton-number-nonconserving processes. This can be done naturally using discrete symmetry and will be discussed in Sec. V.

We will consider Higgs mesons in the 4_c $Y=1$, $5_c(5_r)Y=0$, and $10_c(10_r)Y=0$ representations only. We want to emphasize the essential differences between real and complex representations. Real Higgs mesons are desirable in that they have half the number of fields as their complex counterparts. However, their VEV's are usually too restrictive to lead to a viable zeroth-order fermion mass spectrum or to allow simplifying discrete symmetries in the theory. This will be discussed in detail in Secs. V and VII. In ending this section we would like to point out that our model, being quasivectorlike, is free of Adler-Bell-Jackiw triangle anomalies.

III. CONTINUOUS LITTLE GROUPS

We now determine the *continuous part* of the little group of *selected* vacuum expectation values of Sp(4) × U(1) Higgs mesons. The reason for considering only certain VEV's will become clear in Sec. IV.

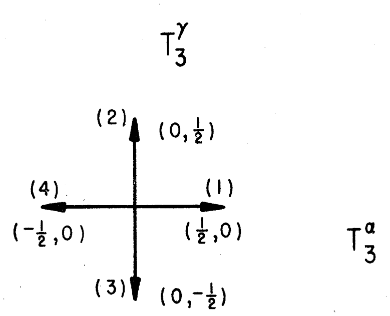

FIG. 2. The weight diagram for the fundamental representation (4_c) of Sp(4).

Continuous little groups will be found using a graphical method that exploits the algebraic structure of Cartan weight diagrams.[8] This method is convenient *for any gauge group* and leads to a pictorial representation of the little groups. For concreteness we develop the method within the framework of our Sp(4) × U(1) gauge model.

First, note that from Eq. (B3) it follows that any element of the *real* Lie algebra of Sp(4) can be written

$$RT_3^\alpha + R'T_3^\gamma + c_1E_\alpha + \bar{c}_1E_{-\alpha} + c_2E_\beta + \bar{c}_2E_{-\beta} + c_3E_\gamma + \bar{c}_3E_{-\gamma} + c_4E_\xi + \bar{c}_4E_{-\xi} + R''\tfrac{1}{2}Y, \quad (3.1)$$

where R, R', R'' are real and c_1, c_2, c_3, c_4, are complex numbers. Let h^A be an element of 4_c, $Y=-1$ (see Fig. 2). Let its VEV be

$$\langle h^A \rangle = v\xi_1^A. \quad (3.2)$$

Apply (3.1) to $\langle h^A \rangle$, use part (3) of Appendix C, and set the result equal to zero. We find that

$$\tfrac{1}{2}(R-R'')\xi_1^A + c_4\xi_2^A + \bar{c}_2\xi_3^A + \bar{c}_1\xi_4^A = 0. \quad (3.3)$$

Since the ξ_1^A's are linearly independent, we must have

$$R = R'', \qquad c_1 = c_2 = c_4 = 0. \quad (3.4)$$

Therefore, an arbitrary element in the little algebra of $\langle h^A \rangle$ is given by

$$R(T_3^\alpha + \tfrac{1}{2}Y) + R'T_3^\gamma + c_3E_\gamma + \bar{c}_3E_{-\gamma}. \quad (3.5)$$

From (B9) we know that $T_3^\gamma, E_{\pm\gamma}$ are generators for the SU(2) subgroup of Sp(4) associated with the γ direction on the root diagram. Using Part 1 of Appendix C it is easy to show that $T_3^\alpha + \frac{1}{2}Y$ commutes with T_3^γ and $E_{\pm\gamma}$. Therefore, the continuous little group of (3.2) is SU(2) × U(1), where SU(2) is generated by $T_3^\gamma, E_{\pm\gamma}$, and U(1) is generated by $T_3^\alpha + \frac{1}{2}Y$. Now assume

$$\langle h^A \rangle = v\xi_2^A. \quad (3.6)$$

In precisely the same manner as above, we find that the continuous little group of (3.6) is SU(2) × U(1), where SU(2) is generated by $T_3^\alpha, E_{\pm\alpha}$ and U(1) is generated by $T_3^\gamma + \frac{1}{2}Y$. Before drawing any conclusions, let us try one more example. Let

$$\langle h^A \rangle = v(\xi_1^A + \xi_2^A). \quad (3.7)$$

Proceeding as above, we find the most general element in the little algebra of (3.7) to be

$$R(T_3^\alpha + T_3^\gamma + \tfrac{1}{2}Y) + r(E_\xi + E_{-\xi} + Y) + c_1(E_\alpha + E_\beta + E_\gamma) + \bar{c}_1(E_{-\alpha} + E_{-\beta} + E_{-\gamma}), \quad (3.8)$$

where $r=c_4$ is real. The last two terms in (3.8) are obviously the raising and lowering operators for an SU(2) subgroup. Commuting these two terms using part 1 of Appendix C, we find that the third generator of this SU(2) subgroup is

$$2(T_3^\alpha+T_3^\gamma)-(E_\xi+E_{-\xi}). \qquad (3.9)$$

Expression (3.8) can be rewritten

$$r_1[2(T_3^\alpha+T_3^\gamma)-(E_\xi+E_{-\xi})]$$
$$r_2[2(T_3^\alpha+T_3^\gamma)+(E_\xi+E_{-\xi})+2Y]$$
$$+c_1(E_\alpha+E_\beta+E_\gamma)+\bar{c}_1(E_{-\alpha}+E_{-\beta}+E_{-\gamma}), \qquad (3.10)$$

where r_1, r_2 are real. Using part 1 of Appendix C, it can be shown that $2(T_3^\alpha+T_3^\gamma)+(E_\xi+E_{-\xi})+2Y$ commutes with the other terms in (3.10). Therefore, the continuous little group of (3.7) is SU(2) × U(1), where SU(2) is generated by $2(T_3^\alpha+T_3^\gamma)-(E_\xi+E_{-\xi}), E_{\pm\alpha}+E_{\pm\beta}+E_{\pm\gamma}$, and U(1) is generated by $2(T_3^\alpha+T_3^\gamma)+(E_\xi+E_{-\xi})+2Y$. From these results we extract the following rules:

(1) Draw the weight diagram for the Higgs meson of interest. Indicate the VEV on the diagram by marking the appropriate coefficient next to each weight vector.

(2) Determine all linear combinations of step operators that annihilate the VEV. This can be done by inspection, using part 3 of Appendix C and (B34) and (B35) to get the correct signs and taking account of the coefficients in front of each weight vector. From these linear combinations choose only those that (a) when multiplied by a phase are Hermitian (b) come in Hermitian-conjugate pairs.

(3) Determine the remaining Hermitian operators that annihilate the VEV by commuting the Hermitian conjugate pairs of step operators found in (2). If these do not exhaust all such Hermitian operators the remaining ones can easily be found using linear combinations of $T_3^\alpha, T_3^\gamma, Y$ and Hermitian combinations of step operators.

(4) The generators so found span the little algebra of the VEV. The various subalgebras of this set may not, unfortunately, mutually commute. It can, therefore, be difficult to decide what little groups they generate. In many cases it is easy to take linear combinations and arrive at new, mutually commuting subalgebras. Even when this cannot be done, general algebraic considerations can usually decide which little group the operators generate.

(5) The action of the above generators on the weight diagrams gives a pictorial representation of the continuous little groups.

As an example, we now determine the little group of a Higgs meson in the 5_c, $Y=0$ representation with VEV

$$\langle T_{5c}^{A\dot{B}}\rangle = V\psi_3^{A\dot{B}}, \qquad (3.11)$$

where $V\neq 0$. It is clear from the weight diagram in Fig. 3 that the only linear combinations of step operators that annihilate (3.11) are $E_{\pm\alpha}, E_{\pm\gamma}$. Evaluating $[E_\alpha, E_{-\alpha}]$ and $[E_\gamma, E_{-\gamma}]$ using part 1 of Appendix C, we find Hermitian generators T_3^α and T_3^γ respectively. Both T_3^α, T_3^γ as well as Y annihilate (3.11). Since $T_3^\alpha, E_{\pm\alpha}$ all commute with $T_3^\gamma, E_{\pm\gamma}$, it is clear that the little group of (3.11) is SU(2) × SU(2) × U(1). One SU(2) subgroup is generated by $T_3^\alpha, E_{\pm\alpha}$ the other by $T_3^\gamma, E_{\pm\gamma}$, and U(1) by Y.

For a more complicated example, consider a Higgs meson in the 10_c, $Y=0$ representation with VEV

$$\langle T_{10c}^{A\dot{B}}\rangle = V\eta_4^{A\dot{B}} + V'\eta_6^{A\dot{B}}, \qquad (3.12)$$

where V, V' are not both zero. From part 3 of Appendix C and Eqs. (B35), we have

$$E_{-\alpha}\eta_4^{A\dot{B}}=\eta_8^{A\dot{B}},$$
$$E_\gamma\eta_4^{A\dot{B}}=-\eta_2^{A\dot{B}},$$
$$E_\xi\eta_4^{A\dot{B}}=\eta_5^{A\dot{B}}-\eta_{10}^{A\dot{B}}, \qquad (3.13)$$
$$E_\alpha\eta_6^{A\dot{B}}=-\eta_2^{A\dot{B}},$$
$$E_{-\gamma}\eta_6^{A\dot{B}}=\eta_8^{A\dot{B}},$$
$$E_{-\xi}\eta_6^{A\dot{B}}=-\eta_5^{A\dot{B}}+\eta_{10}^{A\dot{B}}.$$

From the weight diagram in Fig. 4 it is obvious that the only linear combinations of step operators that annihilate the VEV are multiples of

$$\text{(a)}\quad \frac{1}{V'}E_\alpha-\frac{1}{V}E_\gamma,$$
$$\text{(b)}\quad \frac{1}{V}E_{-\alpha}-\frac{1}{V'}E_{-\gamma}, \qquad (3.14)$$
$$\text{(c)}\quad \frac{1}{V}E_\xi+\frac{1}{V'}E_{-\xi}.$$

By rule 2, (a) and (b) are in the little algebra only if some multiple of (b) is the Hermitian conjugate

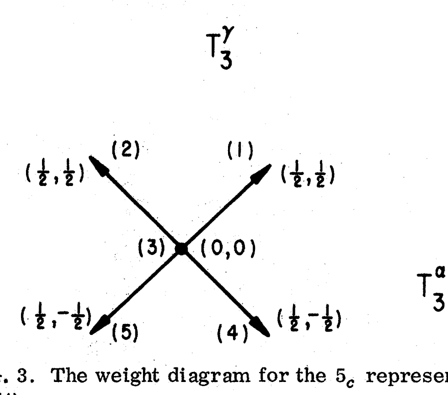

FIG. 3. The weight diagram for the 5_c representation of Sp(4).

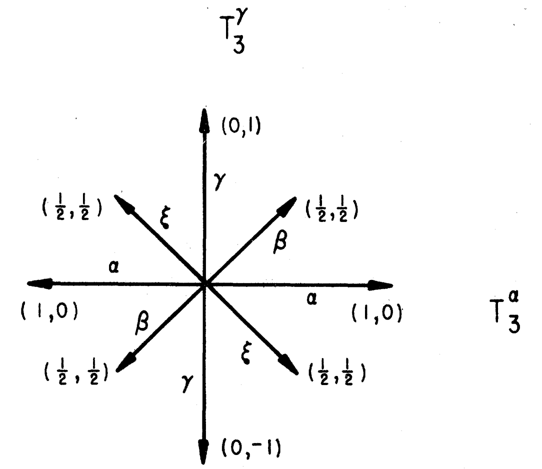

FIG. 4. The weight diagram for the 10_c representation of Sp(4).

of (a). This will be true if and only if

$$|V|=|V'|. \tag{3.15}$$

Also, by rule 2, (c) is in the little algebra only if there is a phase $e^{i\delta}$ such that $e^{i\delta}$ times (c) is Hermitian. This will also be true if and only if (3.15) is true. Therefore, for V, V' where $|V| \neq |V'|$, the little group of (3.12) can only be U(1) × U(1), where one U(1) is generated by $T_3^\alpha + T_3^\gamma$ and the other is generated by Y. Now consider V, V' where $|V|=|V'|=r$. Let $V=re^{-i\phi}$ and $V'=re^{-i\psi}$. Then the above generators of the little algebra can be written (after multiplying with appropriate factors)

$$\begin{aligned}&(\text{a}')\quad e^{i\psi}E_\alpha - e^{i\phi}E_\gamma\,,\\ &(\text{b}')\quad e^{-i\psi}E_{-\alpha} - E^{-i\phi}E_{-\gamma}\,,\\ &(\text{c}')\quad e^{i(\phi-\psi)/2}E_\xi + e^{-i(\phi-\psi)/2}E_{-\xi}\,.\end{aligned} \tag{3.16}$$

Commuting (a′) with (b′), we get

$$2(T_3^\alpha + T_3^\gamma)\,. \tag{3.17}$$

This Hermitian operator annihilates (3.12). Y also annihilates (3.12). Operators (c′), (3.17), and Y exhaust all such Hermitian operators since the rank of SP(4) × U(1) is three and therefore the rank of any little group must be ≤ 3. Using part (1) of Appendix C it can be shown that (c′) commutes with all other generators in the little algebra. Therefore, the little group of (3.12) is SU(2) × U(1) × U(1), where SU(2) is generated by $T_3^\alpha + T_3^\gamma$, $e^{\pm i\psi}E_{\pm\alpha} - e^{\pm i\phi}E_{\pm\gamma}$, one U(1) subgroup is generated by (c′) and the other generated by Y. The action of this little group on, say, the 4_c representation is evident from the weight diagram in Fig. 2. It obviously groups ξ_1^A, ξ_4^A and ξ_2^A, ξ_3^A into independent doublets under SU(2).

The continuous little groups of selected VEV of 4_c, 10_c, 5_c, 10_r, and 5_r representations are tabulated in Table I.

IV. RU DISCRETE SYMMETRY

As discussed in Secs. I and II, there is, in general, undesirable mixing, *within the same multiplet*, of fermions of equal charge. In this section we discuss one method of naturally suppressing[9] such mixings. Again, we will work with the gauge group Sp(4) × U(1) for concreteness. First, note that in the 4_c, $Y=-1$ representation, the charge operator (2.1) is given by

$$Q_B^A = \xi_3^A\xi_{2B} - \xi_4^A\xi_{1B}\,, \tag{4.1}$$

or, in matrix notation

$$[Q_B^A] = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{4.2}$$

The one parameter group that Q_B^A generates is found by exponentiation. That is

$$[\Lambda_B^A(\theta)] = \exp(-i[Q_B^A]\theta) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e^{i\theta} & \\ & & & e^{i\theta} \end{bmatrix}. \tag{4.3}$$

Since det $[\Lambda_B^A]$ is not 1, it follows from (B6) that Λ_B^A is not in Sp(4) alone. However,

$$[\Lambda_B^A(\theta)] = \begin{bmatrix} e^{-i\theta/2} & & & \\ & e^{-i\theta/2} & & \\ & & e^{i\theta/2} & \\ & & & e^{i\theta/2} \end{bmatrix} e^{-i(-\theta/2)} \tag{4.4}$$

clearly displays the relation of Λ_B^A to Sp(4) × U(1). Any realistic model of the weak and electromagnetic interactions must have $[\Lambda_B^A(\theta)]$ [≈ U(1)] as the *continuous* little group of all the VEV's of its Higgs mesons. However, nothing prevents us from having a discrete symmetry group [not contained in U(1)] of the VEV's. If such a discrete symmetry contained Z_2, and if Z_2 had as its matrix representation one of the matrices

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \quad (4.5)$$

then the unwanted mixings of equal charge fermions within the same multiplet would be suppressed. It is not hard to show that we can restrict Z_2 to be a subgroup of Sp(4). Using Eqs. (B1) and (B22), we find that only the last two matrices in (4.5), that is, only

$$A = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \quad (4.6)$$

are in Sp(4). We now examine all possible patterns of symmetry breaking using Higgs mesons in the 4_c, $Y=-1$, $5_c(5_r)$ $Y=0$, $10_c(10_r)$ $Y\neq 0$ representations. We will show that it is impossible, within the confines of an Sp(4) × U(1) gauge group to have such a discrete symmetry. First, consider the 10_c $Y=0$ representation. The matrix representation for this Higgs meson is given in Eq. (B40). For VEV

$$\langle T^{A\dot{B}}_{10c}\rangle = \sqrt{2} \times (V\eta_5^{A\dot{B}} + V'\eta_{10}^{A\dot{B}}),$$

TABLE I. Continuous little groups and their generators for physically relevant VEV's of Sp(4) × U(1) Higgs mesons.

Representation	VEV	Continuous little group	Generators of the little group
4_c, $Y=-1$	ξ_1^A	SU(2) × U(1)	T_3^γ, $E_{\pm\gamma}$; $T_3^\alpha + \frac{1}{2}Y$
	ξ_2^A	SU(2) × U(1)	T_3^α, $E_{\pm\alpha}$; $T_3^\gamma + \frac{1}{2}Y$
	$\xi_1^A + \xi_2^A$	SU(2) × U(1)	$2(T_3^\alpha + T_3^\gamma) - (E_\xi + E_{-\xi})$, $E_{\pm\alpha} + E_{\pm\beta} + E_{\pm\gamma}$; $2(T_3^\alpha + T_3^\gamma) + (E_\xi + E_{-\xi}) + 2Y$
5_c 5_r, $Y=0$	ψ_3^{AB}	SU(2) × SU(2) × U(1)	T_3^α, $E_{\pm\alpha}$; T_3^γ, $E_{\pm\gamma}$;
	$V\psi_2^{AB} + V'\psi_4^{AB}$	(a) $\lvert V\rvert \neq \lvert V'\rvert$: SU(2) × U(1)	$T_3^\alpha + T_3^\gamma$, $E_{\pm\beta}$; Y
		(b) $\lvert V\rvert = \lvert V'\rvert$, $V=re^{-i\phi}$, $V'=re^{-i\eta}$: SU(2) × SU(2) × U(1)	$T_3^\alpha + T_3^\gamma$, $E_{\pm\beta}$;$e^{\pm i\phi}E_{\pm\alpha} - e^{\pm i\eta}E_{\pm\gamma}$, $i(e^{i(\phi-\eta)/2}E_\xi - e^{-i(\phi-\eta)/2}E_{-\xi})$; Y
10_c 10_r, $Y=0$	$r_2\eta_5^{AB} + r_2\eta_{10}^{AB}$, r_1, r_2 real	(a) $r_1 \neq 0, r_2 = 0$: SU(2) × U(1)ˆ × ˆU(1)	$T_3^\gamma, E_{\pm\gamma}; T_3^\alpha; Y$
		(b) $r_1 = 0, r_2 \neq 0$: SU(2) × U(1) × U(1)	$T_3^\alpha, E_{\pm\alpha}; T_3^\gamma; Y$
		(c) $r_1 = r_2 \neq 0$: SU(2) × U(1) × U(1)	$-T_3^\alpha + T_3^\gamma, E_{\pm\xi}; T_3^\alpha + T_3^\gamma; Y$
		(d) $r_1 = -r_2 \neq 0$: SU(2) × U(1) × U(1)	$T_3^\alpha + T_3^\gamma, E_{\pm\beta}; -T_3^\alpha + T_3^\gamma; Y$
		(e) all other r_1, r_2: U(1) × U(1) × U(1)	$T_3^\alpha; T_3^\gamma; Y$
	$V\eta_4^{AB} + V'\eta_6^{AB}$, V, V' complex	(a) $\lvert V\rvert \neq \lvert V'\rvert$: U(1) × U(1)	$T_3^\alpha + T_3^\gamma; Y$
		(b) $\lvert V\rvert = \lvert V'\rvert$, $V=re^{-i\phi}$, $V'=re^{-i\psi}$: SU(2) × U(1) × U(1)	$T_3^\alpha + T_3^\gamma, e^{\pm i\psi}E_{\pm\alpha} - e^{\pm i\phi}$; $e^{i(\phi-\psi)/2}E_\xi e^{-i(\phi-\psi)/2}E_{-\xi}; Y$

we have

$$\langle T^{A\dot{B}}_{10c}\rangle = \begin{bmatrix} -V & & & \\ & -V' & & \\ & & V' & \\ & & & V \end{bmatrix}. \tag{4.7}$$

This VEV admits both A and B of (4.6) as symmetries for any values of V, V'. However, the VEV $\langle T^{A\dot{B}}_{10c}\rangle = \sqrt{2}\,(V\eta_4^{A\dot{B}} + V'\eta_6^{A\dot{B}})$, that is,

$$\langle T^{A\dot{B}}_{10c}\rangle = \begin{bmatrix} 0 & V & & \\ V' & 0 & & \\ & & 0 & V \\ & & V' & 0 \end{bmatrix}, \tag{4.8}$$

does not admit either A or B as symmetries (V, V' are not both zero). Since all other VEV's that have (4.4) as a little group are linear combinations of (4.7) and (4.8), it is clear that to have the above discrete symmetry, $\langle T^{A\dot{B}}_{10c}\rangle$ must be of the form (4.7). From the discussion in Sec. III it follows that for any values of V, V' the little group of (4.7) contains $U(1)\times U(1)\times U(1)$ generated by T_3^{α}; T_3^{γ}; Y. A similar discussion goes through for the 10_r, $Y=0$ representation. In this case, V and V' are real in (4.7) and $V'=\bar{V}$ in (4.8). The little groups of (4.7) are given in Table I. They too contain $U(1)\times U(1)\times U(1)$ generated by T_3^{α}; T_3^{γ}; Y.

For the $5_c(5_r)$ $Y=0$ representation, we determine, in precisely the same way, that the only VEV that admits the above discrete symmetry is $\langle T^{A\dot{B}}_{5c}\rangle = 2\tilde{V}\psi_3^{A\dot{B}}$, that is,

$$\langle T^{A\dot{B}}_{5c}\rangle = \begin{bmatrix} -\tilde{V} & & & \\ & \tilde{V} & & \\ & & \tilde{V} & \\ & & & -\tilde{V} \end{bmatrix} \tag{4.9}$$

(where $\tilde{V}$ is real for $\langle T^{A\dot{B}}_{5r}\rangle$). This VEV admits both A and B. The little group of (4.9) is listed in Table I. Note that the intersection of this little group with any little group of (4.7) must contain $U(1)\times U(1)\times U(1)$ generated by T_3^{α}, T_3^{γ}, and Y. Therefore, in order to break the symmetry down to U(1), we *must* introduce 4_c, $Y=-1$ Higgs mesons. Now it is clear that

$$h^A = \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.10}$$

admits A (but not B) as a symmetry, and that

$$h'^A = \begin{bmatrix} 0 \\ V' \\ 0 \\ 0 \end{bmatrix} \tag{4.11}$$

admits B (but not A) as a symmetry. Any linear combination of (4.10) and (4.11) admits neither A nor B. However, if we use 4_c Higgs mesons with VEV (4.10) only, the little group must contain $U(1)\times U(1)$ generated by T_3^{γ} and $T_3^{\alpha}+\frac{1}{2}Y$. Similarly, if we use 4_c Higgs mesons with VEV (4.11), only the little group contains $U(1)\times U(1)$ generated by T_3^{α}, $T_3^{\gamma}+\frac{1}{2}Y$. Since we must break the symmetry down to U(1), we must use 4_c Higgs mesons, the sum of whose VEV's is a linear combination of (4.10) and (4.11). The continuous little group of these VEV's is indeed U(1) (generated by Q) but they admit *neither* of the discrete symmetries A nor B above. This difficulty cannot be avoided by using $Y=+1$ 4_c Higgs mesons. Therefore, within our $Sp(4)\times U(1)$ gauge theory it is *not possible* to have such a discrete symmetry prevent fermion mixings.

The simplest way to overcome this difficulty is to extend the gauge group to $Sp(4)\times U(1)\times R$, where R is a discrete group isomorphic to Z_2. As we will show shortly, it is now possible to have a discrete symmetry of the type (4.6) in the little group of a realistic theory. We would like to emphasize that a discrete symmetry such as R appended to the continuous gauge group is very useful in limiting the number of couplings in the Lagrangian. These proliferate rapidly in theories involving enlarged gauge groups and a discrete symmetry such as R would have to be applied simply to make such theories tractable. In this sense, the introduction of the discrete symmetry R is a *simplification* of the gauge theory. In this paper we pick the fermions representations in such a way as to forbid $\bar{4}_c 4_c$ type "bare" (non-Higgs) mass couplings. We let all 4_c left chiral fermions transform as -1 under R, and all 4_c right chiral fermions transform as $+1$ under R. Therefore, any 10_c (10_r) or 5_c (5_r) Higgs multiplet that has mass couplings with fermions must transform as -1 under R. In this paper we assume that all Higgs mesons have mass couplings with fermions.

We now return to the possibility of having a discrete symmetry such as (4.6) in the little group of a realistic theory. Again, ignoring the U(1) part of the final little group, we have, in place of (4.6), four possible discrete symmetries which would prevent fermion mixings. These are

$$(A,1),\quad (A,-1),\quad (B,1),\quad (B,-1), \tag{4.12}$$

where A, B are as in (4.6) and $1, -1 \in R$. Returning to 10_c, $Y=0$ Higgs mesons, it is easy to see that (4.7) admits $(A,1)$ and $(B,1)$ as symmetries, but not $(A,-1)$ and $(B,-1)$. On the other hand, (4.8) admits $(A,-1)$ and $(B,-1)$ as symmetries. Any linear combination of (4.7) and (4.8) has no such discrete symmetry. Similarly, the 5_c, $Y=0$ Higgs meson with VEV (4.9), admits $(A,1)$, $(B,1)$ as symmetries. VEV $\langle T_{5c}^{A\dot{B}}\rangle = \sqrt{2}\,(\tilde{V}\psi_2^{A\dot{B}} + \tilde{V}'\psi_4^{A\dot{B}})$, that is

$$\langle T_{5c}^{A\dot{B}}\rangle = \begin{bmatrix} 0 & \tilde{V}' & & \\ -\tilde{V} & 0 & & \\ & & 0 & -\tilde{V} \\ & & \tilde{V} & 0 \end{bmatrix} \tag{4.13}$$

admits $(A,-1)$ $(B,-1)$ as symmetries. Again, any linear combination of (4.9) and (4.13) has no such discrete symmetry. Exactly similar arguments hold for the 10_r and 5_r representations. For 5_r, $\tilde{V}' = -\tilde{V}$ in (4.13). Therefore, we have two possibilities. First, consider theories in which the VEV's of 10_c (10_r) and 5_c (5_r) Higgs mesons are like (4.7) and (4.9), respectively. These both have $(A,1)$ and $(B,1)$ as discrete symmetries. We know from the previous discussion that the continuous little group of these VEV's contains $U(1)\times U(1) \times U(1)$. We must therefore consider 4_c, $Y=-1$ Higgs mesons. The VEV (4.10) admits only $(A,1)$ as a symmetry *independently* of how h^A transforms under R. Similarly, VEV (4.11) has only $(B,1)$ symmetry independently of how h'^A transforms under R. Any linear combination of these two VEV's has no discrete symmetry. However, in order to have final little group U(1) it is necessary to have such a linear combination. This model, therefore, has no discrete symmetry of type (4.12) to prevent fermion mixings. Now consider the second possibility. Let the VEV's of the 10_c (10_r) and 5_c (5_r) Higgs mesons be (4.8) and (4.13), respectively. These have both $(A,-1)$ and $(B,-1)$ as symmetries. From Table I we see that the little groups of such VEV's always contain $U(1)\times U(1)$ generated by $T_3^\alpha + T_3^\gamma$ and Y. Therefore, we must introduce 4_c, $Y=-1$ Higgs mesons. The VEV (4.10) admits $(A,-1)$ $[(B,-1)]$ if R acts on h^A as $+1$ (-1). Similarly, VEV (4.11) admits $(A,-1)$ $[(B,-1)]$ if R acts on h'^A as -1 $(+1)$. Clearly, by choosing (4.10)-type Higgs mesons as $R=+1$ (-1) representations and (4.11)-type Higgs mesons as $R=-1$ $(+1)$ representations, the final little group will contain $(A,-1)$ $[(B,-1)]$. Both choices lead to the same physics. In this paper we will take the second alternative. This immediately tells us that R acts as -1 on all fermion singlets since we want these to contribute to the mass matrix. The continuous little group is U(1) and the discrete little group, denoted RU, is given by

$$\mathrm{RU} = \{(I,1),\ (B,-1)\}. \tag{4.14}$$

RU is isomorphic to Z_2. The matrix expression for $(B,-1)$ on left chiral fermions is

$$\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \quad X-1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}. \tag{4.15}$$

For right chiral fermions, $(B,-1)$ acts as

$$\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \tag{4.16}$$

To summarize, we have shown that if (1) the gauge group is enlarged to $Sp(4)\times U(1)\times R(\approx Z_2)$ acts as -1 $(+1)$ on left (right) chiral 4_c fermions, (2) all Higgs mesons couple to some fermions, there is a *unique* pattern of symmetry breaking which admits a discrete symmetry of type (4.12). In this model all $10_{c(r)}$ and $5_{c(r)}$ Higgs mesons transform as -1 under R and have VEV's (4.8) and (4.13), respectively. There must be at least one 4_c Higgs meson. The 4_c mesons with VEV (4.10) (4.11) transform as -1 $(+1)$ under R. All fermion singlets transform as -1 under R. The little group of this model is $U(1)\times \mathrm{RU}$, where U(1) is generated by Q and RU is given in (4.14). The RU symmetry forbids intramultiplet mixings of equal charge fermions in the zeroth-order mass matrix. This model will be used exclusively throughout the remainder of this paper.

We note that in any $G\times U(1)$ gauge theory, the continuous little group generated by Q [call it $U_0(1)$] is not, as a rule, identical to U(1). Rather, it is a mixture of G and U(1), of which the Weinberg angle is a measure. In exactly the same way, RU is not identical to R but instead is a nontrivial mixture of R with Sp(4). In the above theories, R is spontaneously broken but RU is not. It is to be emphasized that the mere addition of R to the gauge group does *not* guarantee that RU will be in the little group. The Higgs potential must be chosen so that this is the case.

V. STRUCTURE OF THE MODEL

We now construct an explicit model for the weak and electromagnetic interactions based on the gauge group $\mathrm{Sp}(4)\times \mathrm{U}(1)$ and the fermion assignments in Fig. 1. Using the results of Sec. IV we expand the gauge group to $\mathrm{Sp}(4)\times \mathrm{U}(1)\times R$ and assume that the parameters in the Higgs potential are such that RU is in the little group.

First, we determine the minimal number of Higgs mesons necessary to give a realistic zeroth-order fermion mass spectrum. Let us denote left and right chiral fermions in the 4_c representation by ξ^A_{iL} and ξ^A_{jR}, respectively. Let $T^{A\dot{B}}$ be any traceless tensor. Then we can form precisely two Yukawa couplings

$$\bar{\xi}^{\dot{A}}_{iL}\, \xi^B_{jR}\, T^{C\dot{D}} g_{C\dot{A}}\, g_{B\dot{D}} \tag{5.1}$$

and

$$\bar{\xi}^{\dot{A}}_{iL}\, \xi^B_{jR}\, T^{C\dot{D}} J_{\dot{A}\dot{D}}\, J_{BC}\,, \tag{5.2}$$

In general, these two expressions are independent. However, when $T^{A\dot{B}}$ is in 10_c or 5_c this is no longer the case. Assuming that $T^{A\dot{B}}$ is in 10_c (5_c), we know from Appendix B that

$$T^{CD} = T^{C\dot{D}} g_{F\dot{D}}\, J^{DF} \tag{5.3}$$

is a symmetric (antisymmetric) tensor. Inverting Eq. (5.3), substituting for $T^{C\dot{D}}$ in Eq. (5.2), and using Eq. (A9), we find that (5.2) becomes

$$\bar{\xi}^{\dot{A}}_{iL}\, \xi^B_{jR}\, T^{CD} g_{D\dot{A}}\, J_{BC}\,. \tag{5.4}$$

Using the symmetry (antisymmetry) of T^{CD} and Eq. (A8), it follows that (5.2) is equal to 1 (−1) times expression (5.1). Since we restrict tensor Higgs mesons to 10_c and 5_c representations, we need only consider (5.1). For 10_c Higgs mesons with VEV (4.8), the mass coupling takes the form

$$V\bar{\xi}^1_{iL}\,\xi^2_{jR} + V'\bar{\xi}^2_{iL}\,\xi^1_{jR} + V\bar{\xi}^3_{iL}\,\xi^4_{jR} + V'\bar{\xi}^4_{iL}\,\xi^3_{jR}\,, \tag{5.5}$$

(where $V' = \bar{V}$ for 10_r representations). Similarly, for 5_c Higgs mesons with VEV (4.13), the mass coupling is

$$\tilde{V}'\bar{\xi}^1_{iL}\,\xi^2_{jR} - \tilde{V}\bar{\xi}^2_{iL}\,\xi^1_{jR} - \tilde{V}'\bar{\xi}^3_{iL}\,\xi^4_{jR} + \tilde{V}\bar{\xi}^4_{iL}\,\xi^3_{jR} \tag{5.6}$$

($\tilde{V}' = \bar{\tilde{V}}$ for 5_r representations). The 4_c Higgs mesons with $R=-1$ (+1) and VEV (4.10) (4.11) are denoted by h^A (h'^A). Their Yukawa couplings are obvious. Using expressions (5.5) and (5.6) it is not hard to show that the simplest theory with a realistic mass spectrum (all fermion mass with the exception of neutrinos nonzero) requires one 10_c and one 5_c Higgs meson and both h^A and h'^A. In this case, nonzero fermion masses are arbitrary. At this point we note that cross couplings between the d and s quark families and e and μ lepton families induce mixings of b_1 with b_2 and E^- with M^-. These mixings lead to intolerably large contributions to the $K_L K_S$ mass difference, the $K_L \to \mu\bar{\mu}$ rate and lepton-number-violating decays. The mixings can be naturally suppressed by introducing yet another discrete symmetry. Let Z_8 be the discrete group of eight elements generated by $e^{i\pi/4}$. Its action on the various fermion families is shown in Table II. By allowing Z_8 to act trivially on the above four Higgs mesons we naturally suppress the unwanted mixing angles but, unhappily, also suppress the phenomenologically necessary Cabibbo angle and possible $\mu\to e\gamma$, $\mu \to ee\bar{e}$ events. This can be easily remedied by introducing two new 4_c $Y=-1$ Higgs mesons $\tilde{h}^A$ and $\tilde{h}'^A$. These transform as −1 and +1, respectively, under R and are assumed to have VEV's (4.10) and (4.11) so that RU remains in the little group. They transform the same way as the s and μ fermion families under Z_8. This restores mixing between u, X, c, and Y quarks and between ν_e, A^0, ν_μ, and B^0, but naturally suppresses all other mixings. Our standard model will have as its invariance group $\mathrm{Sp}(4)\times \mathrm{U}(1)\times R\times Z_8$ and will allow only the above six Higgs mesons $T^{A\dot{B}}_{10c}$, $T^{A\dot{B}}_{5c}$, h^A, h'^A, $\tilde{h}^A$, and $\tilde{h}'^A$. The fermion assignments of the standard model indicating the allowed mixings are given in Fig. 5. From Table I we see that for $|\tilde{V}'| \neq |\tilde{V}|$ the continuous little group of the 5_c VEV is $\mathrm{SU}(2)\times\mathrm{U}(1)$, where SU(2) is generated by $T^\alpha_3 + T^\gamma_3, E_{\pm\beta}$ and U(1) by Y. It is clear from the fermion assignments in Fig. 5 that this little group is the $\mathrm{SU}(2)\times\mathrm{U}(1)$ group for a quasivector-like Weinberg-Salam model.[10] In the limit $|\tilde{V}|(|\tilde{V}'|) \gg |\tilde{V}'|(|\tilde{V}|)$ and all other VEV's, the results of our model reduce to those of Weinberg

TABLE II. The action of Z_8 on fermion families of the $\mathrm{Sp}(4)\times\mathrm{U}(1)$ gauge model.

Fermion family (including singlets)	Action of Z_8 ± 1	$\pm e^{i\pi/4}$	$\pm i$	$\pm i e^{i\pi/4}$
d, e	1	1	1	1
s, μ	1	i	−1	$-i$
τ	± 1	$\pm e^{i\pi/4}$	$\pm i$	$\pm i e^{i\pi/4}$

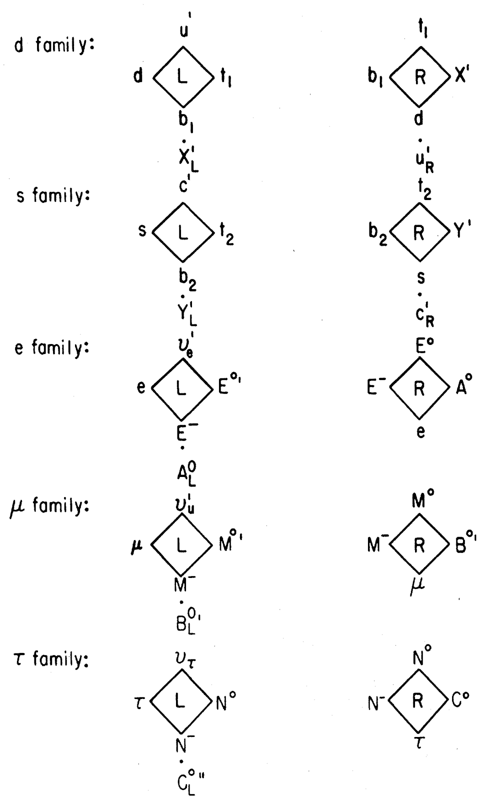

FIG. 5. Fermion assignments for the standard Sp(4) ×U(1) gauge model. The primes indicate the allowed fermion mixings after the application of both RU and Z_8 discrete symmetries.

and Salam. It is important to note that $\tilde{h}^A$ and $\tilde{h}'^A$ actually restore more mixing than is desirable. The arbitrary mixing of ν_e, A^0, ν_μ, and B^0 does not account for e-μ, universality, and leads to disastrously large branching ratios for the processes $\mu \to e\gamma$ and $\mu \to ee\bar{e}$. Similarly, the arbitrary mixing of u, X, c, and Y does not account for Cabibbo universality, leads to large contributions to the $K_L K_S$ mass difference, induces a right-handed u-d current, and, since X' and Y' are singlets, does not naturally suppress charm-changing neutral currents. Though it would be desirable to naturally suppress all these mixings with the exception of the Cabibbo angle, we find the discrete symmetries that must be invoked to do so both aritifical and overly complicated. We therefore take the point of view that the above mixings must be determined experimentally. Consider the leptons first. It is not hard to show that the upper bound on the branching ratio for the $\mu^+ \to e^+\gamma$ decay

$[\Gamma(\mu^+ \to e^+\gamma)/\Gamma(\mu^+ \to e^+\nu_e\bar{\nu}_\mu) < 3.6\times 10^{-9}$ (Ref. 11)]

essentially decouples ν_e, ν_μ from A^0, B^0. Anticipating this result, we set the mixing of ν_e, ν_μ with A^0, B^0 to be strictly zero. Since ν_e, ν_μ are massless they can always be defined to equal ν_e' and ν_μ', respectively. This restores e-μ universality. A^0 and B^0 can mix arbitrarily. Similarly, in the quark sector, absence of charm-changing neutral currents evidenced by the small branching ratio $B(\psi(3772)\to e^+e^-) = (1.3\pm 0.2)\times 10^{-5}$ in the 3.772-GeV ψ resonance[12] essentially decouples u, c from X, Y. Anticipating this result, we set the mixing of u, c with X, Y to be strictly zero. This restores Cabibbo universality, suppresses the right-handed u-d current, and leads to a realistic prediction of the $K_L K_S$ mass difference and the $K_L \to \mu\bar{\mu}$ rate. Therefore, *with the proviso that fermion singlets be prevented by fiat from mixing with 4c fermions* (with good experimental justification), our model naturally ensures all the desirable properties listed in Sec. I. We would like to point out that similar models based on SU(3) × U(1) automatically prevent singlets from mixing with 4_c fermions and thus are free of our slightly unsatisfying proviso. Assuming that E^0 is less massive than any other *heavy* lepton (with the possible exception of A^0 and B^0), it is easy to see that E^0 is absolutely stable.

We now turn to the determination of the mass-eigenstate vector bosons and their mass spectrum. The gauge-covariant derivative in our Sp(4) × U(1) model is given by

$$D^\mu = \partial^\mu - ig\left(T_3^\alpha N_\alpha^\mu + T_3^\gamma N_\gamma^\mu + \frac{E_{\pm\alpha}}{\sqrt{2}} W_{\pm\alpha}^\mu + \frac{E_{\pm\beta}}{2} W_{\pm\beta}^\mu + \frac{E_{\pm\gamma}}{\sqrt{2}} W_{\pm\gamma}^\mu + \frac{E_{\pm\xi}}{2} W_{\pm\xi}^\mu\right) - ig'\frac{Y}{2}B^\mu\,, \tag{5.8}$$

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where N_α^μ, N_γ^μ, and B^μ are real fields and $W_{-i}^\mu = W_{+i}^{\mu\dagger}$ for all i. The VEV's of T_{10c}^{AB}, T_{5c}^{AB}, h^A, and h'^A are given by (4.8), (4.13), (4 10), and (4.11), respectively. The VEV's of $\tilde{h}^A$ and $\tilde{h}'^A$ are

$$\langle\tilde{h}^A\rangle = \begin{bmatrix} \tilde{v} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.9}$$

and

$$\langle \tilde{h}'^A \rangle = \begin{bmatrix} 0 \\ \tilde{v} \\ 0 \\ 0 \end{bmatrix} . \tag{5.10}$$

We can now solve for the mass-eigenstate vector bosons and their masses This is best done graphically using Eq. (5.8) and the properties of the modified Cartan basis. Let

$$\begin{aligned} &A = |V|^2 + |V'|^2, \quad B = |\tilde{V}|^2 + |\tilde{V}'|^2, \\ &Y = |v|^2 + |\tilde{v}|^2, \quad z = |v'|^2 + |\tilde{v}'|^2, \\ &p = \frac{B}{A}, \quad q = \frac{y+z}{2A}, \quad q' = \frac{y-z}{2A}, \\ &\delta = \frac{2|\bar{V}'V + \bar{\tilde{V}}\tilde{V}'|}{A}, \quad \delta' = \frac{2|\bar{V}'V - \bar{\tilde{V}}\tilde{V}'|}{A}, \\ &w = \frac{g'^2}{g^2 + 2g'^2} . \end{aligned} \tag{5.11}$$

Note that in the Weinberg-Salam (WS) limit

$$\delta \cong \delta' \cong \frac{2|\tilde{V}||\tilde{V}'|}{A}, \quad p \gg q, q', \delta, \delta', w . \tag{5.12}$$

We find the physical vector bosons and their masses to be as follows.

(1) $W^{\mu}_{\pm\beta}$ (charge $\mp$):

$$m_\beta{}^2 = \tfrac{1}{2} g^2 A (1+q). \tag{5.13}$$

(2) $W'^{\mu}_{\pm\alpha}, W'^{\mu}_{\pm\gamma}$ (charge $\mp, \mp$):

$$m_{\alpha(\gamma)}{}^2 = \left[\frac{1+p+q_{\mp}(q^2+\delta'^2)^{1/2}}{1+q}\right] m_\beta{}^2 , \tag{5.14}$$

$$\begin{pmatrix} W'^{\mu}_{\alpha} \\ W'^{\mu}_{\gamma} \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon e^{i\phi} \\ \sin\epsilon & \cos\epsilon e^{i\phi} \end{pmatrix} \begin{pmatrix} W^{\mu}_{\alpha} \\ W^{\mu}_{\gamma} \end{pmatrix}, \tag{5.15}$$

where

$$\cos\epsilon = \frac{q'}{\sqrt{2}\,[1+q'^2-(1+q'^2)^{1/2}]^{1/2}} \tag{5.16}$$

and

$$\bar{\tilde{V}}\tilde{V}' - V\bar{V}' = |\bar{\tilde{V}}\tilde{V}' - V\bar{V}'|e^{i\phi} . \tag{5.17}$$

Note that

$$\lim_{q' \to 0} \cos\epsilon = 1 \tag{5.18}$$

and

$$\lim_{q' \to \infty} \cos\epsilon = \frac{1}{\sqrt{2}} .$$

In the Weinberg-Salam limit we find that

$$m_{\alpha(\gamma)}{}^2 \to \left(\frac{p}{1+q}\right) m_\beta{}^2 . \tag{5.19}$$

(3). $W^{\mu}_{\xi 1}, W^{\mu}_{\xi 2}$ (charge zero):

$$m_{\xi_1(\xi_2)} = \left(\frac{1+p+q_{\mp}\delta}{1+q}\right) m_\beta{}^2 , \tag{5.20}$$

$$\begin{bmatrix} W^{\mu}_{\xi 1} \\ W^{\mu}_{\xi 2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{e^{i\phi}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{i}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} W^{\dagger\mu}_{\xi} \\ W^{\mu}_{\xi} \end{bmatrix} , \tag{5.21}$$

where

$$\bar{\tilde{V}}\tilde{V}' + \bar{V}'V + |\bar{\tilde{V}}\tilde{V}' + \bar{V}'V|e^{i\phi} . \tag{5.22}$$

In the WS limit we find that

$$m_{\xi_1(\xi_2)}{}^2 \to \left(\frac{p}{1+q}\right) m_\beta{}^2 . \tag{5.23}$$

(4) $Z^{\mu}_1, Z^{\mu}_2, A^{\mu}$ (charge zero):

$$\begin{aligned} &m_{Z_1(Z_2)}{}^2 = \left\{ \frac{1+p+\left(\frac{1-w}{1-2w}\right)2q_{(\mp)}\left\{\left[1+p-\left(\frac{2w}{1-2w}\right)q\right](1+p)+\left(\frac{1-w}{1-2w}\right)q^2\right\}^{1/2}}{2(1+q)} \right\} m_\beta{}^2 , \\ &m_A = 0 , \end{aligned} \tag{5.24}$$

where we have used the approximation $z \ll y$ in deriving $m_{Z_1(Z_2)}{}^2$. We will use this approximation for the remainder of the paper Also,

$$\begin{bmatrix} Z^{\mu}_1 \\ Z^{\mu}_2 \\ A^{\mu} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \sqrt{w} & \sqrt{w} & (1-2)^{1/2} \end{bmatrix} \begin{bmatrix} N^{\mu}_{\alpha} \\ N^{\mu}_{\gamma} \\ B^{\mu} \end{bmatrix} , \tag{5.25}$$

where

$$x_{1(2)}=\frac{1+p-\left(\frac{2w}{1-2w}\right)q}{\left[\frac{2(1+p+\tau_{(\mp)})(1+p+2q)+(1-w)\tau_{(\mp)}^2}{1-2w}+\frac{4(1+2w)}{(1-2w)^2}q^2\right]^{1/2}},$$

$$y_{1(2)}=\left[\frac{1+p+2\left(\frac{1-w}{1-2w}\right)q-\tau_{(\mp)}}{1+p-\left(\frac{2w}{1-2w}\right)q}\right]x_{1(2)},\tag{5.26}$$

$$z_{(2)}=-\left(\frac{w}{1-2w}\right)^{1/2}(x_{1(2)}+y_{1(2)}),$$

and

$$\tau_{(\mp)}=2m_{Z_1(Z_2)}{}^2/g^2A .$$

Note that in the WS limit

$$m_{Z_1}{}^2\to g^2/2\left(\frac{1-w}{1-2w}\right)q^A=\left(\frac{g^2+g'^2}{4}\right)(y+z),\tag{5.27}$$

which is the familiar SU(2)×U(1) expression. Also

$$m_{Z_2}{}^2\to\left(\frac{p}{1+q}\right)m_\beta{}^2,\tag{5.28}$$

so that the masses of all vector bosons with the exception of the WS bosons $W^\mu_{\pm\beta}$ and Z_1^μ approach each other and become infinitely large in the WS limit. We also find that expression (5.24) becomes

$$\begin{bmatrix}Z_1^\mu\\ Z_2^\mu\\ A^\mu\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ \left(\frac{1-2w}{2}\right)^{1/2} & \left(\frac{1-2w}{2}\right)^{1/2} & -\sqrt{2w}\\ \sqrt{w} & \sqrt{w} & \sqrt{1-2w}\end{bmatrix}\begin{bmatrix}N_\alpha^\mu\\ N_\gamma^\mu\\ B^\mu\end{bmatrix}\tag{5.29}$$

in this limit. From the coupling of the photon to the electromagnetic current, we find that where

$$e=g\sqrt{w},\tag{5.30}$$

where e is the electron charge. From the coupling of $W^\mu{}_\beta$ to its charged current, we determine that

$$\frac{g^2}{16m_\beta{}^2}=\frac{G_F}{\sqrt2},\tag{5.31}$$

Combining these two equations, we have, finally

$$m_\beta=\left(\frac{\sqrt2\,\pi\alpha}{G_Fw}\right)^{1/2}=\frac{26.37}{\sqrt{w}}\ \text{GeV} .\tag{5.32}$$

VI. $\sqrt{R}\,\sqrt{U}$ DISCRETE SYMMETRY

Notice that in our standard model, the u, d, s, and c quarks and e, μ, and τ leptons are the "light" fermions. In fact, they would be massless if V' vanished in (4.8) $\tilde V$ vanished in (4.13), and all u'_R, c'_R mass couplings were disallowed. If these conditions could be guaranteed by a symmetry of the Lagrangian, then the relatively small masses of the above fermions might have a theoretical explanation in terms of soft breaking of this symmetry. In this section we determine the simplest such symmetry. The most general diagonal element of Sp(4) is given by

$$C=\begin{bmatrix}a&&&\\&b&&\\&&b^{-1}&\\&&&a^{-1}\end{bmatrix},\tag{6.1}$$

where a, b are complex numbers of unit modulus. The triplet $(C,e^{i\theta},\pm1)$ is in Sp(4)×U(1)× R (we ignore Z_8 since it acts trivially on the 10_c and 5_c Higgs mesons). The action of this triplet on VEV (4,8) yields

$$\begin{bmatrix}0 & Va\bar b & & \\ V'\bar ab & 0 & & \\ & & 0 & V(\bar ab)^{-1}\\ & & V'(a\bar b)^{-1} & 0\end{bmatrix}\times(\pm1) .\tag{6.2}$$

We demand that $a\bar b=\pm1$. This, however, implies that $\bar ab=\pm1$, so that a $(C,\ e^{i\theta},\ \pm1)$ discrete symmetry in the little group of an $Sp(4)\times U(1)\times R$ theory *cannot* guarantee that $V'=0$. We reach the same conclusion for VEV (4.13). It is clear that we get the desired result if we have $\pm i$ in (6.2). We therefore introduce a new discrete symmetry $\sqrt{R}$, where $\sqrt{R}$ is isomorphic to Z_4. The $\sqrt{R}$ group acts as $\{1,-1,i,-i\}$ ($\{1,1,1,1\}$) on all left (right) chiral fermions. On Higgs mesons, $\sqrt{R}$ acts as $\{1,-1,i,-i\}$ on the 10_c and 5_c representations, and as $\{1,-1,-i,i\}$ ($\{1,1,-1,-1\}$) on 4_c $R=-1$ (+1) representations. This immediately forbids all u_R', c_R' couplings. For simplicity we ignore the U(1) and R parts of the gauge group and concentrate on $Sp(4)\times\sqrt{R}$. The doublet (C, $\{\pm 1, \pm i\}$) is in $Sp(4)\times\sqrt{R}$ and acts on VEV (4.8) as

$$\begin{bmatrix} 0 & Va\bar{b} & & \\ V'\bar{a}b & 0 & & \\ & & 0 & V(\bar{a}b)^{-1} \\ & & V'(a\bar{b})^{-1} & 0 \end{bmatrix} \times(\pm 1, \pm i)\,, \tag{6.3}$$

If (6.3) is to be invariant under (C, $\{\pm 1, \pm i\}$), we must have $a\bar{b}=\pm i$, respectively. Choosing phases for C such that the VEV's of the 4_c Higgs meson are invariant under (C, $\{\pm 1, \pm i\}$), we find that $a=1, -1, i, -i$ and $b=1, 1, -1, -1$ respectively. For $a\bar{b}=\mp i$ we have $V'\bar{a}b=\pm i V'$. It is clear that (6.3) will be invariant under (C, $\pm i$) if and only if $V'=0$. The set (C, $\{\pm 1, \pm i\}$), where

$$C=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

$$\begin{bmatrix} i & & & \\ & -1 & & \\ & & -1 & \\ & & & -i \end{bmatrix}, \begin{bmatrix} -i & & & \\ & -1 & & \\ & & -1 & \\ & & & i \end{bmatrix} \tag{6.4}$$

for ± 1 and $\pm i$, respectively, forms a discrete group isomorphic to Z_4 which we denote $\sqrt{R}\ \sqrt{U}$ for obvious reasons. In precisely the same manner as above, it is not hard to show that 5_c VEV (4.13) is invariant and under $\sqrt{R}\ \sqrt{U}$ if and only if $\vec{V}=0$. Thus we have shown that if we (1) enlarge the gauge group to $Sp(4)\times U(1)\times R\times Z_8\times\sqrt{R}$, where $\sqrt{R}$ acts as $\{1,-1,i,-i\}$ ($\{1,1,1,1\}$) on left (right) chiral fermions, (2) let $\sqrt{R}$ act as $\{1,-1,i,-i\}$ on 10_c and 5_c Higgs mesons and as $\{1,-1,-i,i\}$ ($\{1,1,-1,-1\}$) on 4_c $R=-1$ (+1) Higgs mesons, (3) require that $\sqrt{R}\ \sqrt{U}$ be in the little group, then the masses of the u, d, s, c quarks and e, μ, τ leptons (but no other fermions except neutrinos) vanish. Note that if $\sqrt{R}\ \sqrt{U}$ is in the little group then so is RU. The addition of $\sqrt{R}$ to the gauge group does not guarantee that $\sqrt{R}\ \sqrt{U}$ is in the little group. The Higgs potential must be chosen so that this is the case.

VII. PREDICTING CABIBBO ANGLES

In the standard model, defined in Sec. V, mixing occurs between u and c, X and Y quarks and between A and B, ν_τ, and C leptons. Mixings between u, c and X, Y quarks and between ν_e, ν_μ and A, B leptons are suppressed by fiat with strong experimental justification. All other mixings are suppressed naturally using RU and Z_8 discrete symmetry. No attempt is made to predict the magnitudes of the allowed mixing angles. In this section we will modify the standard model in such a way as to specify all mixing angles *in the quark sector* (including the d-s Cabibbo angle) in terms of quark mass ratios. The absence of right chiral neutrino singlets precludes a similar result in the lepton sector. For simplicity we will ignore leptons in this section. To begin, we lift the restriction of Z_8 symmetry. This obviates the need for the $\tilde{h}^A$ and $\tilde{h}'^A$ Higgs mesons (we now discard them) and, while retaining the above quark mixings, introduces new mixings between d and s, b_1 and b_2 and, finally, t_1 and t_2. *We continue to suppress by fiat u, c mixings with X, Y.* If we denote the 10_c and 5_c Higgs mesons by H_{10} and H_5, respectively, then the most general Yukawa couplings consistent with $Sp(4)\times R$ are

$$\begin{aligned} &G_1\bar{L}_1H_5R_1+G_2\bar{L}_1H_5R_2+G_3\bar{L}_2H_5R_1+G_4\bar{L}_2H_5R_2+\mathcal{G}_1\bar{L}_1H_{10}R_1+\mathcal{G}_2\bar{L}_1H_{10}R_2+\mathcal{G}_3\bar{L}_2H_{10}R_1+\mathcal{G}_4\bar{L}_2H_{10}R_2\\ &\quad+g_1\bar{X}_Lh^\dagger R_1+g_2\bar{X}_Lh^\dagger R_2+g_3\bar{Y}_Lh^\dagger R_1+g_4\bar{Y}_Lh^\dagger R_2+g_5\bar{L}_1h'u_R+g_6\bar{L}_2h'u_R\\ &\quad+g_7\bar{L}_1h'c_R+g_8\bar{L}_2h'c_R+\text{Hermitian conjugate} \end{aligned} \tag{7.1}$$

Terms such as $\bar{X}_Lu_R$ are forbidden by fiat. There are 16 coupling constants and only 12 fields into which to absorb their complex phases. Therefore, 12 coupling constants can be chosen to be real whereas four must remain complex. We choose these four to be G_3, G_3, g_1, and g_4. Note that complex phases in the VEV's of any Higgs meson can no longer be defined away. We now demand that the Lagrangian be invariant under the following discrete exchange operations.

$$L_1 \leftrightarrow R_1, \quad L_2 \leftrightarrow -iR_2,$$
$$X_L \leftrightarrow -iu_R, \quad Y_L \leftrightarrow c_R, \tag{7.2}$$
$$H_5 \leftrightarrow -iH_5^\dagger, \quad H_{10} \leftrightarrow -iH_{10}^\dagger,$$
$$h \leftrightarrow ih'.$$

Applying these operations in (7.1), remembering that all coupling constants with the exception of G_3, $\mathcal{G}_3$, g_1, and g_4 are real, we find that

$$G_1 = G_4 = \mathcal{G}_1 = \mathcal{G}_4 = g_2 = g_3 = g_6 = g_7 = 0,$$
$$G_3 = G_2, \quad \mathcal{G}_3 = \mathcal{G}_2, \quad g_1 = g_5, \quad g_4 = g_8. \tag{7.3}$$

Therefore, the most general Yukawa couplings become

$$G_2\bar{L}_1H_5R_2 + G_2\bar{L}_2H_5R_1 + \mathcal{G}_2\bar{L}_1H_{10}R_2 + \mathcal{G}_2\bar{L}_2H_{10}R_1 + g_1\bar{X}_Lh^\dagger R_1 + g_1\bar{L}_1h'u_R + g_4\bar{Y}_Lh^\dagger R_2 + g_4\bar{L}_2h'c_R + \text{Hermitian conjugate}, \tag{7.4}$$

where all coupling constants are real. Using Fig. 1 and Eqs. (5.5) and (5.6), we compute the d'-s' mass couplings

$$(\bar{d}', \bar{s}')_L \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \begin{pmatrix} d' \\ s' \end{pmatrix}_R + \text{Hermitian conjugate}, \tag{7.5}$$

where $A = G_2\tilde{V} + \mathcal{G}_2V'$. It is easy to show that this mass matrix implies that $m_d = m_s$. Similarly, this theory predicts $m_{t_1} = m_{t_2}$ and $m_{b_1} = m_{b_2}$. Therefore, the theory, as it stands, is untenable. We find it necessary to add two new Higgs meson H_{10}' and H_5' to the model. They transform as -1 under R and we continue to assume that RU is in the little group. H_{10}' and H_5' have Yukawa couplings identical to those in (7.1) (put primes on the new coupling constants). However, there are now only two fields to absorb all eight phases. Therefore, all couplings constants are complex with the exception of two which we choose to be G_1' and $\mathcal{G}_1'$. We demand that when all previous fields transform as (7.2), H_{10}' and H_5' transform as

$$H_5' \leftrightarrow -H_5'^\dagger, \quad H_{10}' \leftrightarrow -H_{10}'^\dagger. \tag{7.6}$$

Demanding that the Lagrangian be invariant under (7.2) and (7.6), we find that

$$G_1' = \mathcal{G}_1' = 0,$$
$$G_3' = i\bar{G}_2', \quad \mathcal{G}_3' = i\bar{\mathcal{G}}_2', \tag{7.7}$$
$$G_4 = \bar{G}_4, \quad \mathcal{G}_4 = \bar{\mathcal{G}}_4.$$

The new Yukawa couplings become

$$G_2'\bar{L}_1H_5'R_2 + i\bar{G}_2'\bar{L}_2H_5'R_1 + G_4'\bar{L}_2H_5'R_2$$
$$\mathcal{G}_2'\bar{L}_1H_{10}'R_2 + i\bar{\mathcal{G}}_2\bar{L}_2H_{10}'r\,1 + \mathcal{G}_4\bar{L}_2H_{10}'R_2 + \text{Hermitian conjugate}, \tag{7.8}$$

where only G_4' and $\mathcal{G}_4'$ are real. Computing the d'-s' mass matrix, we find

$$\begin{pmatrix} 0 & A \\ A' & B \end{pmatrix}, \tag{7.9}$$

where

$$A = G_2\tilde{V} + \mathcal{G}_2V' + G_2'\tilde{V}_0 + \mathcal{G}_2'V_0', \quad A' = G_2\tilde{V} + \mathcal{G}_2V' + i\bar{G}_2'\tilde{V}_0 + i\bar{\mathcal{G}}_2'V_0', \quad B = G_4'\tilde{V}_0 + \mathcal{G}_4'V_0', \tag{7.10}$$

and the subscript 0 implies an element in the VEV of H_5' or H_{10}'. Since A' is not equal to A, there are three parameters in the mass matrix which are to be determined by two mass eigenvalues m_d and m_s. Therefore, the mixing angle will not be completely specified by a ratio of quark masses. To overcome this difficulty, we introduce yet another discrete symmetry which we denote by R'. R' acts as -1 on L_1, R_2, X_L, c_R, H_5', H_{10}', h, and h', and as $+1$ on everything else. Demanding that the Lagrangian be invariant under R', we find that

$$G_2' = \mathcal{G}_2' = 0. \tag{7.11}$$

Then $A = A'$ in (7.10). We will shortly show that such a mass matrix leads to the correct d-s Cabibbo angle. The remaining mass couplings are given by

$$(\bar{t}_1', \bar{t}_2')_L \begin{pmatrix} 0 & G_2\tilde{V}' + \mathcal{G}_2 V \\ G_2\tilde{V} + \mathcal{G}_2 V & G_2\tilde{V}_0' + \mathcal{G}_4 V_0 \end{pmatrix} \begin{pmatrix} t_1' \\ t_2' \end{pmatrix}_R + (\bar{b}_1', \bar{b}_2')_L \begin{pmatrix} 0 & -G_2\tilde{V}' + \mathcal{G}_2 V \\ -G_2\tilde{V}' + \mathcal{G}_2 V & -G_4'\tilde{V}_0' + \mathcal{G}_4' V_0 \end{pmatrix} \begin{pmatrix} b_1' \\ b_2' \end{pmatrix}_R$$

$$+ (\bar{u}', \bar{c}')_L \begin{pmatrix} g_1 v' & 0 \\ 0 & g_4 v' \end{pmatrix} \begin{pmatrix} u' \\ c' \end{pmatrix}_R + (\bar{X}', \bar{Y}')_L \begin{pmatrix} g_1 v & 0 \\ 0 & g_4 v \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix}_R + \text{Hermitian conjugate}. \quad (7.12)$$

Note that all quark masses are independent and arbitrary due to fortuitous minus signs and the fact that Higgs messons are all in comples representations. How consider the d'-s' mass matrix (7.9) with $A' = A$. Let

$$\begin{pmatrix} d' \\ s' \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} d \\ s \end{pmatrix}_{L,R}, \quad (7.13)$$

where $U_{L,R}$ is unitary and d, s are mass eigenstates. Then

$$U_L^\dagger M U_R = M_D, \quad (7.14)$$

where M_D is diagonal. It follows that

$$U_L^\dagger M M^\dagger U_L = M_D{}^2, \quad U_R^\dagger M^\dagger M U_R = M_D{}^2. \quad (7.15)$$

First consider $MM^\dagger$:

$$MM^\dagger = \begin{pmatrix} |A|^2 & A\bar{B} \\ \bar{A}B & |A|^2 + |B|^2 \end{pmatrix}. \quad (7.16)$$

Diagonalizing (7.16) we find that

$$U_L = \begin{pmatrix} \cos\theta_C & -\sin\theta_C \\ \sin\theta_C e^{i\delta} & \cos\theta_C e^{i\delta} \end{pmatrix}, \quad (7.17)$$

where

$$\tan^2\theta_C = \frac{m_d}{m_s} \quad (7.18)$$

and, if $A = |A| e^{i\psi}$, $B = |B| e^{i\eta}$, then

$$\delta = \eta - \psi. \quad (7.19)$$

Diagonalizing $M^\dagger M$ we find that

$$U_R = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C e^{-i\delta} & \cos\theta_C e^{-i\delta} \end{pmatrix}. \quad (7.20)$$

Similar results hold for t_1, t_2 and b_1, b_2:

$$\tan^2\theta_t = \frac{m_{t_1}}{m_{t_2}},$$

$$\tan^2\theta_b = \frac{m_{b_1}}{m_{b_2}}, \quad (7.21)$$

with one arbitrary *CP* phase in each sector. It is clear from the last two terms of (7.12), that u', c', X', Y' are all mass eigenstates. That is, there is no u-c or X-Y mixing. Note also that

$$\frac{m_u}{m_c} = \frac{m_X}{m_Y} \quad (7.22)$$

of approximately $\frac{1}{20}$. Computing θ_C in (7.18), we find $\theta_C \simeq 12.60°$, very close to the experimental value of $\theta_C \simeq 13°$. Since the d-s Cabibbo angle experimentally is measured relative to the u quark, we emphasize the importance of the zero u-c mixing in this theory.

APPENDIX A: NOTATION

Let J_{AB} be an *antisymmetric* tensor (over a four-dimensional, complex vector space V). Denote by J^{AB} the unique, antisymmetric tensor satisfying

$$J^{AB} J_{CB} = \delta_C^A. \quad (A1)$$

Note that $\bar{J}_{AB}$ and $\bar{J}^{AB}$ (complex conjugation denoted $J_{\dot{A}\dot{B}}$ and $J^{\dot{A}\dot{B}}$, respectively) are antisymmetric and satisfy the relation

$$J^{\dot{A}\dot{B}} J_{\dot{C}\dot{B}} = \delta_{\dot{C}}^{\dot{A}}. \quad (A2)$$

We can use J_{AB}, J^{AB} and $J_{\dot{A}\dot{B}}, J^{\dot{A}\dot{B}}$ to raise and lower indices according to

$$\xi_B = J_{AB}\xi^A, \quad \psi^A = J^{AB}\psi_B \quad (A3)$$

and

$$\bar{\xi}_{\dot{B}} = J_{\dot{A}\dot{B}}\bar{\xi}^{\dot{A}}, \quad \bar{\psi}^{\dot{A}} = J^{\dot{A}\dot{B}}\bar{\psi}_{\dot{B}}. \quad (A4)$$

Note that since J_{AB} is antisymmetric,

$$J_{AB}\xi^B = -J_{BA}\xi^B = -\xi_A, \quad (A5)$$

so one must be careful to contract with the correct index when raising and lowering. Now consider a tensor $g_{A\dot{B}}$ that is both Hermitian

$$\bar{g}_{A\dot{B}} = g_{B\dot{A}} \quad (A6)$$

and positive definite

$$g_{A\dot{B}}\xi^A\bar{\xi}^{\dot{B}} \geq 0 \quad (A7)$$

for all ξ^A, and is zero if and only if $\xi^A = 0$. Denote the unique inverse of $g_{A\dot{B}}$ by $g^{A\dot{B}}$. Then

$$g^{A\dot{B}}g_{C\dot{B}} = \delta_C^A, \quad g^{A\dot{B}}g_{A\dot{C}} = \delta_{\dot{C}}^{\dot{B}}. \quad (A8)$$

There are many such metric tensors. Metric tensor g_{AB} is said to be compatible with J_{AB} if it has the property that

$$g^{A\dot{B}} = J^{AC}J^{\dot{B}\dot{D}}g_{C\dot{D}}. \quad (A9)$$

There are, as a rule, many metric tensors compatible with a given J_{AB}. Using Eqs. (A1) and (A8) it is easy to show that $g_{A\dot{B}}$ is compatible with J_{AB} if and only if

$$J_{AB}=J^{\dot{C}\dot{D}}g_{A\dot{C}}g_{B\dot{D}}\,. \tag{A10}$$

APPENDIX B: THE GROUP Sp(4)

Fix J_{AB} and a compatible metric tensor $g_{A\dot{B}}$ on V. The set of all linear mappings Λ^A_B of V to V with the property that

$$\Lambda^A_C\Lambda^B_D J_{AB}=J_{CD} \tag{B1}$$

and

$$\Lambda^A_C\bar{\Lambda}^{\dot{B}}_{\dot{D}}g_{A\dot{B}}=g_{C\dot{D}} \tag{B2}$$

forms a ten-parameter Lie group. This group is called the unitary, symplectic group in four complex dimensions. We denote it by Sp(4). Define the completely antisymmetric tensor

$$\epsilon_{ABCD}=-3J_{[AB}J_{CD]}\,. \tag{B3}$$

This tensor has the property that

$$\epsilon^{ABCD}\epsilon_{ABCD}=4!\,. \tag{B4}$$

Let Λ^A_B be in Sp(4). Then

$$\det(\Lambda^A_B)=\tfrac{1}{4}!\,\epsilon^{ABCD}\epsilon_{EFGH}\Lambda^E_A\Lambda^F_B\Lambda^G_C\Lambda^H_D\,. \tag{B5}$$

Using Eqs. (B1), (B3), and (B4), it is clear that

$$\det(\Lambda^A_B)=+1\,. \tag{B6}$$

Therefore, Sp(4) is a subgroup of SU(4). The complexified Lie algebra of Sp(4) is called C_2. Denote its canonical Cartan basis by $H_1, H_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm\gamma}, E_{\pm\xi}$. The commutation relations for this basis are given in many references.[13] We find it convenient to use a new basis, related to the canonical basis as follows:

$$T_3^\alpha=\sqrt{3}H_1\,,\quad T_3^\gamma=\sqrt{3}H_2\,,\quad E'_{\pm\alpha}=\sqrt{6}E_{\pm\alpha}\,,$$
$$E'_{\pm\beta}=2\sqrt{3}E_{\pm\beta}\,,\quad E'_{\pm\gamma}=\sqrt{6}E_{\pm\gamma}\,,\quad E'_{\pm\xi}=2\sqrt{3}E_{\pm\xi}\,. \tag{B7}$$

The root diagram for the new basis is shown in Fig. 6. The new commutation relations (from now on we drop the primes) are given in part 1 of Appendix C. The normalization of the new generators is chosen so that the action of any step operator (e.g., E_α) on a weight vector of the fundamental representation of C_2 yields ±1 or 0 multiple of another weight vector. The relation of elements of C_2 to elements of the *real* Lie algebra of Sp(4) is (1) T_3^α, T_3^γ are in the real algebra and (2) for step operators $E_{\pm\psi}$

$$T_1^\psi=\frac{E_\psi+E_{-\psi}}{\sqrt{2}}$$
$$T_2^\psi=\frac{E_\psi+E_{-\psi}}{i\sqrt{2}} \tag{B8}$$

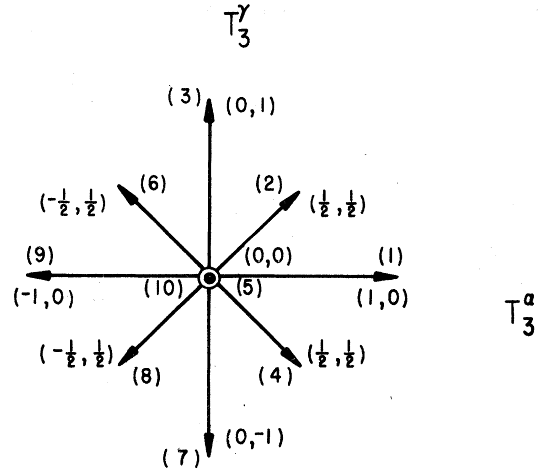

FIG. 6. The root diagram for the group Sp(4).

are in the real Lie algebra. Using part 2 of Appendix C and the commutation relations, we immediately identify four SU(2) subgroups of Sp(4). These are generated by

$$\begin{aligned}&T_3^\alpha\,, E_{\pm\alpha}\,,\\ &T_3^\alpha+T_3^\gamma\,, E_{\pm\beta}\,,\\ &T_3^\gamma\,, E_{\pm\gamma}\,,\\ &-T_3^\alpha+T_3^\gamma\,, E_{\pm\xi}\,,\end{aligned} \tag{B9}$$

respectively, and correspond to the four directions of the root vectors in Fig. 6. The fundamental representation of Sp(4) (and therefore of C_2) is in four complex dimensions and is denoted by 4_c. Its weight diagram is given in Fig. 2. The diagram is self-conjugate, which implies that $\bar{4}_c$ is equivalent to 4_c. Note, however, that 4_c is not equivalent to a real, four-dimensional representation. The weight vectors are written ξ^A_i, ordered as in Fig. 2. They are eigenstates of T_3^α and T_3^γ with eigenvalues λ^α_i and λ^γ_i, respectively. These eigenvalues can be read off the weight diagram. The matrix representations of the complexified generators in the ξ^A_i basis is given in part 3 of Appendix C. Let J_{AB} and $g_{A\dot{B}}$ be the defining tensors of Sp(4). We now find an explicit representation of these tensors in terms of the weight vectors ξ^A_i. To this end we note that for infinitesimal ϵ,

$$\Lambda^A_B=\delta^A_B-iT^{\alpha A}_{3B}\epsilon \tag{B10}$$

is an element of Sp(4). Using Eq. (B1) we find that

$$J_{CA}T^{\alpha C}_{3B}=J_{CB}T^{\alpha C}_{3A}\,. \tag{B11}$$

Now evaluating the expression

$$J_{CA}T^{\alpha C}_{3B}\xi^A_i\xi^B_j \tag{B12}$$

in two different ways using Eq. (B11), we find that

$$(\lambda_i^\alpha + \lambda_j^\alpha)\xi_{iA}\xi_j^A = 0\,. \tag{B13}$$

Similarly, for $T_{3B}^{\gamma A}$ we find that

$$(\lambda_i^\gamma + \lambda_j^\gamma)\xi_{iA}\xi_j^A = 0\,. \tag{B14}$$

Finally, since $E_\beta + E_{-\beta}$ is in the real Lie algebra of Sp(4),

$$\Lambda_B^A = \delta_B^A - i(E_\beta + E_{-\beta})_B^A \epsilon \tag{B15}$$

is in Sp(4) for infinitesimal ϵ. Evaluating

$$\xi_{3A}\xi_4^A = J_{BA}\Lambda^{-1B}{}_C\Lambda^{-1A}{}_D\Lambda_E^C\Lambda_F^D\xi_3^E\xi_4^F\,, \tag{B16}$$

using Eqs. (B1) and (B15) and the matrix representations for $E_{\pm\beta}$ in part 3 of Appendix C, we find that

$$\xi_{1A}\xi_A^4 = -\xi_{2A}\xi_3^A = 0\,. \tag{B17}$$

Since for a given Sp(4) Lie group J_{AB} is defined only up to a nonzero multiple, we choose J_{AB} such that

$$\xi_{1A}\xi_4^A = -\xi_{2A}\xi_3^A = 1\,. \tag{B18}$$

From Eqs. (B13) and (B14) it follows that

$$\xi_{iA}\xi_j^A = 0\,, \tag{B19}$$

when i,j are not 1,4 or 4,1 and 2,3 and 3,2. Now J_{AB} can be written as

$$J_{AB} = \sum_{i=1}^{4}\sum_{j>i}\alpha^{ij}\xi_{i[A}\xi_{jB]}\,. \tag{B20}$$

Contracting J_{AB} with ξ_1^A, ξ_2^A, and ξ_3^A separately, we have finally that

$$J_{AB} = 2(\xi_{1[A}\xi_{4B]} - \xi_{2[A}\xi_{3B]})\,, \tag{B21}$$

which is the desired expression. In matrix notation, with respect to basis ξ_i^A,

$$[J_{AB}] = [J^{AB}] = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{bmatrix}. \tag{B22}$$

In the same manner as above, and using the compatibility of $g_{A\dot B}$ with J_{AB}, it can be shown that

$$g_{A\dot B}\xi_i^A\bar\xi_j^{\dot B} = \delta_{ij} \tag{B23}$$

and

$$g_{A\dot B} = \xi_{1A}\bar\xi_{1\dot B} + \xi_{2A}\bar\xi_{2\dot B} + \xi_{3A}\bar\xi_{3\dot B} + \xi_{4A}\bar\xi_{4\dot B}\,. \tag{B24}$$

In matrix notation, with respect to basis ξ_i^A

$$[g_{AB}] = [g^{AB}] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{B25}$$

Higher dimensional irreducible representations of Sp(4) are obtained by taking tensor products of 4_c and $\bar 4_c$, subtracting out the $g_{A\dot B}$ and J_{AB} trace and, finally, reducing the tensor products using the symmetry properties of the indices. Consider $T'^{A\dot B} \in 4_c \otimes \bar 4_c$. Then

$$T'^{A\dot B} = T^{A\dot B} + \frac{(T'^{C\dot D}g_{C\dot D})}{4}\,g^{A\dot B}\,. \tag{B26}$$

The ($g_{A\dot B}$ traceless) tensors $T^{A\dot B}$ can be further reduced as follows. Consider the tensor

$$T^{AB} = J^{BC}g_{C\dot B}T^{A\dot B}\,. \tag{B27}$$

The representation on T^{AB} is equivalent to the representation on $T^{A\dot B}$ since they are related by the *invariant* tensors $g_{A\dot B}$ and J_{AB}. Note that

$$T^{AB}J_{BA} = g_{A\dot B}T^{A\dot B} = 0 \tag{B28}$$

by Eq. (B26). The tensors T^{AB} are, however, still not irreducible. They can be written as the sum of symmetric and antisymmetric tensors. That is,

$$T^{AB} = T^{(AB)} + T^{[AB]}\,. \tag{B29}$$

The set of tensors $T^{A\dot B}$ with the property that $T^{AB} = T^{(AB)}$ form a ten-dimensional, complex vector space. The representation of Sp(4) that they carry is denoted by 10_c. As we will shortly show, 10_c is still *reducible*. The set of tensors $T^{A\dot B}$ with the property that $T^{AB} = T^{[AB]}$ forms a five-dimensional, complex vector space. The representation of Sp(4) that they carry is denoted 5_c. It, too, is still reducible. It is clear that $T^{A\dot B}$ is the direct sum of these two kinds of tensors. Symbolically,

$$4_c \otimes \bar 4_c = 10_c \oplus 5_c \oplus 1_c\,, \tag{B30}$$

where 1_c is the one-dimensional, complex space spanned by $g_{A\dot B}$. The weight diagrams of 5_c and 10_c are given in Figs. 3 and 4, respectively. We now determine these weight vectors in terms of ξ_i^A and $\bar\xi_i^A$, the weight vectors of 4_c and $\bar 4_c$. First consider the weights of the reducible (into $10_c \oplus 5_c$) representation 15_c. Consider, for example,

$$T^{A\dot B} = a\xi_1^A\bar\xi_3^{\dot B} + b\xi_2^A\bar\xi_4^{\dot B}\,, \tag{B31}$$

where a, b are complex numbers. Then, using Eqs. (B21) and (B24) we find

$$T^{AB} = J^{BC}g_{C\dot B}T^{A\dot B} = -a\xi_1^A\xi_2^B + b\xi_1^B\xi_2^A\,. \tag{B32}$$

Obviously T^{AB} is antisymmetric (symmetric) if and only if $a=b$ $(a=-b)$. Therefore the first weight vector (normalized to unity) of the weight diagram of 5_c is

$$\psi_1^{A\dot{B}} = \frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_3^{\dot{B}} + \xi_2^A \bar{\xi}_4^{\dot{B}}) . \tag{B33}$$

The second weight vector of the weight diagram of 10_c is

$$\eta_2^{A\dot{B}} = -\frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_3^{\dot{B}} - \xi_2^A \bar{\xi}_4^{\dot{B}}) . \tag{B34}$$

In this manner all the weight vectors can be determined. For the 5_c representation they are (ordered as in Fig. 3)

$$\begin{aligned}
\psi_1^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_3^{\dot{B}} + \xi_2^A \bar{\xi}_4^{\dot{B}}) ,\\
\psi_2^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_2^A \bar{\xi}_1^{\dot{B}} + \xi_4^A \bar{\xi}_3^{\dot{B}}) ,\\
\psi_3^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_1^{\dot{B}} - \xi_2^A \bar{\xi}_2^{\dot{B}} - \xi_3^A \bar{\xi}_3^{\dot{B}} + \xi_4^A \bar{\xi}_4^{\dot{B}}) ,\\
\psi_4^{A\dot{B}} &= \frac{1}{2} (\xi_1^A \bar{\xi}_2^{\dot{B}} - \xi_3^A \bar{\xi}_4^{\dot{B}}) ,\\
\psi_5^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_3^A \bar{\xi}_1^{\dot{B}} + \xi_4^A \bar{\xi}_2^{\dot{B}}) .
\end{aligned} \tag{B35}$$

For the 10_c representation (ordered as in Fig. 4)

$$\begin{aligned}
\eta_1^{A\dot{B}} &= \xi_1^A \bar{\xi}_4^{\dot{B}} ,\\
\eta_2^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_1^A \xi_3^{\dot{B}} - \xi_2^A \bar{\xi}_4^{\dot{B}}) ,\\
\eta_3^{A\dot{B}} &= -\xi_2^A \bar{\xi}_3^{\dot{B}} ,\\
\eta_4^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_2^{\dot{B}} + \xi_3^A \bar{\xi}_4^{\dot{B}}) ,\\
\eta_5^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_1^A \bar{\xi}_1^{\dot{B}} - \xi_4^A \bar{\xi}_4^{\dot{B}}) ,\\
\eta_6^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_2^A \bar{\xi}_1^{\dot{B}} + \xi_4^A \bar{\xi}_3^{\dot{B}}) ,\\
\eta_7^{A\dot{B}} &= \xi_3^A \bar{\xi}_2^{\dot{B}} ,\\
\eta_8^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_3^A \bar{\xi}_1^{\dot{B}} - \xi_4^A \bar{\xi}_2^{\dot{B}}) ,\\
\eta_9^{A\dot{B}} &= -\xi_4^A \bar{\xi}_1^{\dot{B}} ,\\
\eta_{10}^{A\dot{B}} &= \frac{1}{\sqrt{2}} (\xi_2^A \bar{\xi}_2^{\dot{B}} - \xi_3^A \bar{\xi}_3^{\dot{B}}) .
\end{aligned} \tag{B36}$$

We have chosen $\eta_5^{A\dot{B}}$ and $\eta_{10}^{A\dot{B}}$ so that

$$E_{\pm\alpha}\eta_{10}^{A\dot{B}} = E_{\pm\gamma}\eta_5^{A\dot{B}} = 0 . \tag{B37}$$

In deriving the above we have used the fact that E_ψ acting on $\bar{4}_c$ is equal to $-E_\psi$ acting on 4_c. Any element of 5_c, denoted $T_{5c}^{A\dot{B}}$, can be written

$$T_{5c}^{A\dot{B}} = 2c_3\psi_3^{A\dot{B}} + \sqrt{2} \sum_{\substack{i=1\\ i\neq 3}}^{5} c_i \psi_i^{A\dot{B}} \tag{B38}$$

(where the c_i's are complex numbers) or, in matrix notation with respect to basis ξ_i^A,

$$[T_{5c}^{A\dot{B}}] = \begin{bmatrix} -c_3 & c_4 & c_1 & 0 \\ -c_2 & c_3 & 0 & c_1 \\ c_5 & 0 & c_3 & -c_4 \\ 0 & c_5 & c_2 & -c_3 \end{bmatrix} . \tag{B39}$$

Similarly, any element of 10_c, denoted $T_{10c}^{A\dot{B}}$, can be written

$$T_{10c}^{A\dot{B}} = \sqrt{2}\, c_5 \eta_5^{A\dot{B}} + \sqrt{2}\, c_{10} \eta_{10}^{A\dot{B}} + c_1 \eta_1^{A\dot{B}} + c_3 \eta_3^{A\dot{B}} + c_7 \eta_7^{A\dot{B}} + c_9 \eta_9^{A\dot{B}} + \sqrt{2} \sum_{i=1}^{4} c_{2i} \eta_{2i}^{A\dot{B}} , \tag{B40}$$

where the c_i's are complex numbers. In matrix notation with respect to basis ξ_i^A,

$$[T_{10c}^{A\dot{B}}] = \begin{bmatrix} -c_5 & c_4 & -c_2 & c_1 \\ c_6 & -c_{10} & -c_3 & c_2 \\ -c_8 & c_7 & c_{10} & c_4 \\ -c_9 & c_8 & c_6 & c_5 \end{bmatrix} . \tag{B41}$$

As stated earlier, both 5_c and 10_c are reducible. Consider 5_c. Then

$$T_{5c}^{A\dot{B}} = \tfrac{1}{2}(T_{5c}^{A\dot{B}} + \bar{T}_{5c}^{A\dot{B}}) + \tfrac{1}{2}(T_{5c}^{A\dot{B}} - \bar{T}_{5c}^{A\dot{B}}) . \tag{B42}$$

The first term on the right is obviously Hermitian. Such tensors form a five-dimensional, real vector space. The representation of Sp(4) that they carry is denoted by 5_r and is irreducible. The second term on the left is anti-Hermitian. Such tensors form a five-dimensional real vector space. Since any anti-Hermitian tensor is simply i times a Hermitian tensor it is clear that the representation of Sp(4) on anti-Hermitian tensors is equivalent ot 5_r. Symbolically,

$$5_c = 5_r \oplus 5_r . \tag{B43}$$

Similarly, $T_{10c}^{A\dot{B}}$ can be written as the sum of a Hermitian and an anti-Hermitian tensor. The Hermitian and anti-Hermitian tensors both form ten-dimensional real vector spaces. The representations of Sp(4) on these two vector spaces are equivalent, irreducible, and denoted by 10_r. That is,

$$10_c = 10_r \oplus 10_r\,. \tag{B44}$$

Any element of 5_r, denoted $T_{5r}^{A\dot{B}}$, can be written

$$T_{5r}^{A\dot{B}} = 2r_3 + \sqrt{2}\,c_1\psi_1^{A\dot{B}} + \sqrt{2}\,\bar{c}_1\psi_5^{A\dot{B}} + \sqrt{2}\,d_2\psi_2^{A\dot{B}} - \sqrt{2}\,\bar{d}_2\psi_4^{A\dot{B}}\,, \tag{B45}$$

where r_3 is real and c_1, d_2 complex. In matrix notation,

$$[T_{5r}^{A\dot{B}}] = \begin{bmatrix} -r_1 & -\bar{d}_2 & c_1 & 0 \\ -d_2 & r_3 & 0 & c_1 \\ \bar{c}_1 & 0 & r_3 & \bar{d}_2 \\ 0 & \bar{c}_1 & d_2 & -r_3 \end{bmatrix}. \tag{B46}$$

Any element of 10_r, denoted $T_{10r}^{A\dot{B}}$, can be written

$$\begin{aligned} T_{10r}^{A\dot{B}} = {} & \sqrt{2}\,r_5\eta_5^{A\dot{B}} + \sqrt{2}\,r_{10}\eta_{10}^{A\dot{B}} + c_1\eta_1^{A\dot{B}} - \bar{c}_1\eta_9^{A\dot{B}} \\ & + c_3\eta_3^{A\dot{B}} - \bar{c}_3\eta_7^{A\dot{B}} + \sqrt{2}\,d_2\eta_2^{A\dot{B}} + \sqrt{2}\,\bar{d}_2\eta_8^{A\dot{B}} \\ & + \sqrt{2}\,d_4\eta_4^{A\dot{B}} + \sqrt{2}\,d_4\eta_6^{A\dot{B}}\,, \end{aligned} \tag{B47}$$

where r_5, r_{10} are real and c_1, c_3, d_2, d_4 are complex. In matrix notation,

$$[T_{10r}^{A\dot{B}}] = \begin{bmatrix} -r_5 & d_4 & -d_2 & c_1 \\ d_4 & -r_{10} & -c_3 & d_2 \\ -d_2 & -c_3 & r_{10} & d_4 \\ c_1 & d_2 & d_4 & r_5 \end{bmatrix}. \tag{B48}$$

APPENDIX C: THE LIE ALGEBRA C_2

(1) We give the commutation relations of the Lie algebra C_2:

$$[T_3^\alpha, T_3^\gamma] = 0\,,$$

$$[T_3^\alpha, E_{\pm\alpha}] = \pm E_{\pm\alpha}\,, \quad [T_3^\gamma, E_{\pm\alpha}] = 0\,,$$

$$[T_3^\alpha, E_{\pm\beta}] = \pm\tfrac{1}{2}E_{\pm\beta}\,, \quad [T_3^\gamma, E_{\pm\beta}] = \pm\tfrac{1}{2}E_{\pm\beta}\,,$$

$$[T_3^\alpha, E_{\pm\gamma}] = 0\,, \quad [T_3^\gamma, E_{\pm\gamma}] = \pm E_{\pm\gamma}\,,$$

$$[T_3^\alpha, E_{\pm\xi}] = \tfrac{1}{2}E_{\pm\xi}\,, \quad [T_3^\gamma, E_{\pm\xi}] = \pm\tfrac{1}{2}E_{\pm\xi}\,,$$

$$[E_\alpha, E_{-\alpha}] = 2T_3^\alpha$$

$$[E_\beta, E_{-\beta}] = 2(T_3^\alpha + T_3^\gamma)\,,$$

$$[E_\gamma, E_{-\gamma}] = 2T_3^\gamma\,,$$

$$[E_\xi, E_{-\xi}] = 2(-T_3^\alpha + T_3^\gamma)\,,$$

$$[E_\alpha, E_\beta] = 0\,, \quad [E_\gamma, E_{-\alpha}] = 0\,,$$

$$[E_\alpha, E_\gamma] = 0\,, \quad [E_\xi, E_{-\alpha}] = 0\,,$$

$$[E_\alpha, E_{-\gamma}] = 0\,, \quad [E_{-\alpha}, E_{-\beta}] = 0\,,$$

$$[E_\alpha, E_{-\xi}] = 0\,, \quad [E_{-\alpha}, E_{-\gamma}] = 0\,,$$

$$[E_\beta, E_\gamma] = 0\,, \quad [E_{-\beta}, E_{-\gamma}] = 0\,,$$

$$[E_\gamma, E_\xi] = 0\,, \quad [E_{-\gamma}, E_{-\xi}] = 0\,,$$

$$[E_\alpha, E_\xi] = E_\beta\,, \quad [E_\beta, E_{-\gamma}] = -E_{-\xi}\,,$$

$$[E_\alpha, E_{-\beta}] = -E_{-\xi}\,, \quad [E_\beta, E_{-\xi}] = 2E_\alpha\,,$$

$$[E_\beta, E_\xi] = 2E_\gamma\,, \quad [E_\gamma, E_{-\xi}] = E_\beta\,.$$

All other commutation relations can be obtained from the above using the rule that if $[E_a, E_b] = N_{ab}E_c$ then $N_{-a-b} = -N_{ab}$.

(2) Let T_1, T_2, T_3 be the standard basis for the Lie algebra of SU(2). Then,

$$[T_i, T_j] = i\epsilon_{ijk}T_k\,.$$

The standard basis T_+, T_-, T_3 for the complexified Lie algebra of SU(2) is defined by

$$T_+ = T_1 + iT_2\,,$$

$$T_- = T_1 - iT_2\,,$$

$$T_3 = T_3\,.$$

The new commutation relations are

$$[T_3, T_+] = T_+\,,$$

$$[T_3, T_-] = -T_-\,,$$

$$[T_+, T_-] = 2T_3\,.$$

(3) The matrix representations of our basis of C_2 with respect to the weight vector basis ξ_i^A of the fundamnetal representation 4_c, are given by

$$[T^\alpha] = \begin{bmatrix} \tfrac{1}{2} & & & \\ & 0 & & \\ & & 0 & \\ & & & \tfrac{1}{2} \end{bmatrix},$$

$$[T^\gamma] = \begin{bmatrix} 0 & & & \\ & \tfrac{1}{2} & & \\ & & -\tfrac{1}{2} & \\ & & & 0 \end{bmatrix},$$

$$[E_\alpha] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [E_\beta] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[E_\gamma] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$[E_\xi] = \begin{bmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{bmatrix}.$$

The remaining matrices are simply the Hermitian conjugates of those above. For example,

$$[E_{-\alpha}]=[E_\alpha]^\dagger=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

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