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Theory of static quark forces

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Working within the framework of classical algebraic chromodynamics, I develop a theory of the equilibrium color forces acting between stationary quarks. To identify the equations appropriate to describing static forces on a fixed time slice, I assume that the "static" color fields (1) appear as the end point of a most probable tunneling process, and (2) satisfy the principle of virtual work. The resulting equations differ from the vanishing-time-derivative specialization of the Euler-Lagrange equations. The static equations imply certain compatibility conditions on the orientations of discrete quark charges relative to the local color field. Unlike the analogous static equations in Abelian electrodynamics, which take the same form on all time slices, the static equations for chromodynamics take a simple form only at the instant of tunneling through. Static forces and potentials calculated on this favored time slice describe the behavior of the system at all later times because of gluon energy conservation. When the static equations of algebraic chromodynamics for the $q\bar{q}$ color-singlet force problem are rewritten as equations for the overlying SU(2) classical Yang-Mills field, they take the form of the equations of the 't Hooft-Polyakov model, but with the Higgs field reinterpreted as the static potential and, of course, with external source charges present. I develop these equations in a perturbation expansion in the color gluon coupling g. I conjecture that the relevant zeroth-order solution is the unit-topological-charge solution of 't Hooft and Polyakov, which appears to behave as a quark-confining "bag." This solution contributes a zeroth-order orientation energy to the quark potential, which by the principle of virtual work must be extremal with respect to variations in the position and orientation of the "bag." The orientation energy gives the $q\bar{q}$ potential a repulsive central core, and also leads to a zeroth-order $\sigma^m_{(n)}\vec{Q}^{\rm eff}_{(n)}\cdot\vec{B}^m$ interaction between the quark spins and the "bag" color magnetic field. The compatibility conditions guarantee that the orientation energy is invariant under changes of gauge of the zeroth-order solution. Confinement, I believe, comes about in order g^2, where there are strong distortions of the quark color flux lines arising from the presence of the background "bag" field. A test of this hypothesis requires construction of the color gluon propagator in a Prasad-Sommerfield background field. I adapt the methods of Brown *et al.* to get an expression for the vector propagator in terms of the scalar propagator and to give an explicit contour integral formula for the latter, the detailed evaluation of which is now in progress. Since the compatibility conditions and the minimization involved in constructing the orientation energy make the source current orthogonal to all normalizable zero modes, all zeroth-order degeneracies are resolved and a consistent order-g^2 perturbation theory exists. I conclude with a brief discussion of expected limits of validity of the perturbation expansion, and of the extension of the methods developed here to nonstatic problems and to the force problem for three (or more) quarks. In the first appendix, I give technical details of the scalar propagator construction. In the second appendix, I illustrate in the case of the Abelian Higgs model the subtleties involved in finding static equations which satisfy the principle of virtual work. I also sketch an argument which shows that assumption (2) above follows from assumption (1), and give a thermodynamic interpretation for the equation of energy conservation in processes in which confining "bags" are being created.

I. THE STATIC EQUATIONS

In an earlier paper[1] (hereafter referred to as I) I constructed an extension of chromodynamics, called algebraic chromodynamics, which permits the introduction of classical noncommuting quark color charges. I now turn to the problem of developing methods, within the framework of algebraic chromodynamics, for calculating the equilibrium color forces acting between stationary quarks. In the formal developments of this section, I will assume the validity of Eq. (I.22) for color-charge algebras of arbitrary rank N. In the specific application to the $q\bar{q}$ color force given in the next section, I use only the rank $N=2$ algebras which were explicitly calculated in I.

Under the assumption of stationary quarks, the quark source current $J^{a\mu}$ of Eq. (I.17) has time component

$$J^0=\sum_n Q_{(n)}(t)\delta^3(x-x_n) \tag{1a}$$

and vanishing space components

$$J^k=0,\quad k=1,2,3, \tag{1b}$$

with the charges evolving in time according to Eq. (I.20),[2]

$$\frac{dQ_{(n)}(t)}{dt}=-igP(b_0(x_n),Q_{(n)}(t)). \tag{2}$$

Before proceeding further, I generalize Eqs. (1) and (2) in order to include quark static spin currents, by assuming that these are given by the natural covariant generalization $\partial/\partial x^l \to D_l$ of the usual Abelian expression for the spin current

$$J^k_{\text{spin}} = \sum_n \frac{-1}{2m_{(n)}} \epsilon^{klm} D_l[Q_{(n)}\delta^3(x-x_n)\sigma^m_{(n)}], \tag{3}$$

with $m_{(n)}$ and $\sigma_{(n)}$ the nth quark mass and Pauli spin matrix. In order to simply express the effect of the spin current on the time evolution of the quark charges, I introduce the color electric and magnetic fields

$$E^m = f^{0m}, \quad B^m = \tfrac{1}{2}\epsilon^{klm} f^{kl}. \tag{4}$$

Then substituting Eq. (3) into the equation of source current conservation $D_\mu J^\mu = 0$, and using Eq. (I.10) to simplify the spin current contribution, gives the modified version of Eq. (2),

$$\frac{dQ_{(n)}(t)}{dt} = iP(u_{\text{eff}}(x^n), Q_{(n)}(t)), \quad u_{\text{eff}}(x^n) = -g\left[b_0(x^n) + \frac{B^m(x^n)\sigma^m_{(n)}}{2m_{(n)}}\right]. \tag{5}$$

In everything that follows, I will treat spin only as a lowest-order perturbation, ignoring effects arising from the noncommutativity of the Pauli spin matrices (this gives correctly the interaction energy terms of the form $\sigma_{(n)}$ and $\sigma_{(n)}\sigma_{(m)}$, $m \neq n$). Then the $\sigma^m_{(n)}$ components in Eq. (5) can effectively be replaced by their expectations, making Eq. (5) identical in form to the local algebraic gauge transformation on the quark charge given in Eq. (I.18). Equation (5) specifies the time evolution of the quark charges, starting from the forms on a fixed initial time slice assumed in I.

I now turn to the key problem, which is to find the equations appropriate to describing static quark forces on a fixed time slice. In ordinary electrodynamics this is readily accomplished by setting all time derivatives equal to zero in the Euler-Lagrange equations. The situation is not so simple in chromodynamics, where the field-potential relations and Euler-Lagrange equations, obtained by expressing Eqs. (I.3′), (I.14), and (I.15) in terms of E^j and B^j, are

$$\begin{aligned} E^j &= -D_j b^0 - \frac{\partial}{\partial x^0} b^j, \\ B^j &= \epsilon^{jkl}\left[\frac{\partial}{\partial x^k} b^l - \tfrac{1}{2} igP(b^k, b^l)\right], \\ D_j E^j &= gJ^0, \\ D_j B^j &= 0, \\ \epsilon^{kjm} D_j B^m - D_0 E^k &= gJ^k_{\text{spin}}, \\ \epsilon^{kjm} D_j E^m + D_0 B^k &= 0. \end{aligned} \tag{6}$$

If we attempt to assume that time derivatives of the potentials are identically zero, then the constraint equation $D_j E^j = gJ^0$ implies an immediate conflict with Eq. (5). Hence "static" chromodynamics necessarily involves time dependence. In order to infer the equations which describe the color fields of quarks at rest, I introduces two assumptions:

(1) First, I assume that "static" color fields (and in fact all color fields of physical interest) appear as the end point of a most probable tunneling process. Taking the instant of tunneling through to be $t=0$, and requiring that the tunneling probability be extremal with respect to all variations of the color fields at $t=0$ which are consistent with the constraint $D_j E^j = gJ^0$, gives the dondition

$$0 = \delta_{\text{end point}} \int_{-\infty}^{0} dt \int d^3x \left[\tfrac{1}{2}S(E^j, E^j) - \tfrac{1}{2}S(B^j, B^j) - S(\lambda, D_j E^j - gJ^0)\right]. \tag{7}$$

I have enforced the constraint in Eq. (7) by the natural device of introducing a Lagrange multiplier λ into the action.[3] Expressing E^j and B^j in terms of potentials, integrating by parts in the time-derivative terms, and keeping only the end-point contribution, Eq. (7) becomes

$$0 = -\int d^3x\, S(\delta b^j(x, t=0), E^j + D_j\lambda), \tag{8}$$

which implies that at the instant of tunneling through we have

$$t=0: \quad E^j = -D_j\lambda = -\left[\frac{\partial}{\partial x^j}\lambda - igP(b^j, \lambda)\right]. \tag{9}$$

Equation (9) may be interpreted by noting that the general radiation gauge form of E^j is[4]

$$E^j = -D_j\lambda - \dot{b}^j, \quad D_j \dot{b}^j = 0, \tag{10}$$

and so Eq. (9) states that at the instant of tunneling through, the transverse contribution to E^j vanishes. This statement is the natural generalization of Coleman's "bounce" condition[5] to the situation where a source constraint is present.

(2) Second, I assume that the color fields associated with quarks held in equilibrium by external restraining forces satisfy the principle of virtual work. Recall that the principal of virtual work states that when a system in equilibrium is subjected to arbitrary virtual displacements, the sum of all increments of work (energy) is zero. In the present context, this requires that when the quark-gluon system is subjected to arbitrary virtual displacements, the change in quark potential (which is the net work done by the restraining forces) must

equal the change in gluon field energy; it is hard to see how the concept of a static potential can make sense without such a balance holding. To get this condition in quantitative form, I use Eq. (I.26) and Eq. (10) to write the radiation-gauge gluon field energy as

$$E_{\text{gluon}} = \int d^3x \tfrac{1}{2}[S(E^j,E^j)+S(B^j,B^j)]$$
$$= \int d^3x \tfrac{1}{2}[S(D_j\lambda,D_j\lambda)+S(\dot{b}^j,\dot{b}^j)+S(B^j,B^j)]. \quad (11)$$

Making arbitrary small variations in the quark positions and field strengths, and using

$$\delta D_j\lambda = D_j\delta\lambda - igP(\delta b^j,\lambda),$$
$$\delta B^j = \epsilon^{jlm}D_l\delta b^m \quad (12)$$

gives

$$\delta E_{\text{gluon}} = \int d^3x[S(\delta\lambda, D_jE^j)+S(\delta\dot{b}^j,\dot{b}^j) + S(\delta b^m, \epsilon^{mlj}D_lB^j - igP(\lambda, D_m\lambda))]. \quad (13a)$$

This expression must be equated to the change in the static quark potential[6]

$$\delta V_{\text{static}} = g\int d^3x[S(\delta\lambda,J^0)+S(\delta b^k,J^k_{\text{spin}})], \quad (13b)$$

where I have used the fact, apparent from Eq. (10), that λ is playing a role analogous to the scalar potential b_0. The variations in Eq. (13) are not independent on a general Cauchy surface, but rather are restricted by the condition

$$0=\delta(D_j\dot{b}^j)=D_j\delta\dot{b}^j - igP(\delta b^j,\dot{b}^j), \quad (14)$$

which relates δb^j to $\delta\dot{b}^j$ in a complicated way. Now, however, let us exploit the fact that at the instant of tunneling through ($t=0$) we have $\dot{b}^j=0$, so that Eq. (14) is a constraint only on $\delta\dot{b}^j$, which, however, no longer contributes to Eq. (13a). The remaining infinitesimals δb^j and $\delta\lambda$ are then independent, and equating the left- and right-hand sides of Eq. (13) gives us the desired equations of chromostatics

$$t=0:\quad E^j=-D_j\lambda,$$
$$D_jE^j=gJ^0,$$
$$B^j=\epsilon^{jkl}\left[\frac{\partial}{\partial x^k}b^l-\tfrac{1}{2}igP(b^k,b^l)\right], \quad D_jB^j=0, \quad (15)$$
$$\epsilon^{mlj}D_lB^j=gJ^m_{\text{spin}}+igP(\lambda,D_m\lambda),$$
$$D_jw=\frac{\partial}{\partial x^j}w-igP(b^j,w).$$

Before proceeding to applications, I pause to comment on a number of features of these equations:

(i) It is possible[7] that quantum fluctuations around the static background fields lead to a Higgs potential for λ, changing Eq. (11) to read[8]

$$E_{\text{gluon}} = \int d^3x \tfrac{1}{2}[S(D_j\lambda,D_j\lambda)+S(\dot{b}^j,\dot{b}^j) + S(B^j,B^j)+V_H(S(\lambda,\lambda))]. \quad (16)$$

Equation (16) then has an extra term

$$\Delta\delta E_{\text{gluon}} = \int d^3x\, V_H'(S(\lambda,\lambda))S(\delta\lambda,\lambda), \quad (17)$$

and the second equation in Eq. (15) is changed to read

$$D_jE^j+V_H'(S(\lambda,\lambda))\lambda=gJ^0, \quad (18)$$

but the identity $\delta E_{\text{gluon}}=\delta V_{\text{static}}$ continues to hold.

(ii) Equations (15) can be obtained as the condition that a functional $\mathfrak{F}$ of the original potentials b^0, b^j and of a Lagrange multiplier λ be extremal with respect to independent variations of λ and of the Cauchy data $b^0, b^j, \dot{b}^j$ on a fixed time slice. The functional $\mathfrak{F}$ is

$$\mathfrak{F}=\mathfrak{F}_{\text{charge}}+\mathfrak{F}_{\text{spin}},$$
$$\mathfrak{F}_{\text{charge}}=\int d^3x\{\tfrac{1}{2}[S(E^j,E^j)-S(B^j,B^j)] - S(\lambda, D_jE^j-gJ^0)\}, \quad (19)$$
$$\mathfrak{F}_{\text{spin}}=V_{\text{static,spin}}=-g\sum_n S\left(B^m(x_n),\frac{Q_{(n)}\sigma^m_{(n)}}{2m_{(n)}}\right),$$

where I have explictly separated off the spin contribution. The variation of $V_{\text{static,spin}}$ is

$$\delta V_{\text{static,spin}}=-g\int d^3x\, S\left(\epsilon^{mlk}D_l\delta b^k(x), \sum_n \frac{Q_{(n)}\sigma^m_{(n)}}{2m_{(n)}}\delta^3(x-x_n)\right)$$
$$=g\int d^3x\, S(\delta b^k(x), J^k_{\text{spin}}), \quad (20)$$

which justifies the identification of spin current made in Eq. (3). Expressing E^j and B^j in terms of potentials by Eq. (6), and varying, gives

$$\delta\mathfrak{F}_{\text{charge}}=\int d^3x[S(E^j,-D_j\delta b^0+igP(\delta b^j,b^0)-\delta\dot{b}^j)-S(B^j,\epsilon^{jlm}D_l\delta b^m)-S(\delta\lambda,D_jE^j-gJ^0) - S(\lambda,-igP(\delta b^j,E^j)-D_jD_j\delta b^0+igD_jP(\delta b^j,b^0)-D_j\delta\dot{b}^j)]. \quad (21)$$

After doing integrations by parts[9] this takes the form

$$\delta\mathcal{F}_{\text{charge}} = \int d^3x [S(\delta b^0, D_jE^j + D_jD_j\lambda) - S(\delta\dot{b}^j, E^j + D_j\lambda) - S(\delta\lambda, D_jE^j - gJ^0) + S(\delta b^m, igP(b^0, E^m + D_m\lambda) - \epsilon^{mlj}D_lB^j + igP(E^m, \lambda))], \tag{22}$$

from which it is clear by inspection that equating $\delta\mathcal{F}$ to zero gives Eqs. (15). This derivation makes it apparent that, while the argument of Eqs. (11)–(14) makes use of the radiation gauge, Eqs. (15) themselves do not involve a gauge specification.[10] (For a discussion of the connection between this derivation and the maximization of tunneling probability, see Appendix B.)

(iii) The static equations imply compatibility conditions on the orientations of discrete quark charges relative to the local color fields. These are obtained by applying D_m to the "curl B" equation in Eq. (15), and using Eq. (I.10) on both left- and right-hand sides,

$$\epsilon^{mlj}D_mD_lB^j = -igP(B^j, B^j) = 0 = gD_mJ^m_{\text{spin}} - ig^2P(\lambda, J^0) = ig^2\sum_n \delta^3(x - x_n)\left[P\left(B^m(x_n), \frac{Q_{(n)}\sigma^m_{(n)}}{2m_{(n)}}\right) - P(\lambda, Q_{(n)})\right], \tag{23}$$

from which we get

$$D_mJ^m_{\text{spin}} - igP(\lambda, J^0) = 0 \longrightarrow P\left(B^m(x_n), \frac{Q_{(n)}\sigma^m_{(n)}}{2m_{(n)}}\right) - P(\lambda(x_n), Q_{(n)}) = 0, \quad n = 1, \ldots, N. \tag{24}$$

Equation (24) will play an important role in the subsequent analysis.

(iv) Although λ plays a role analogous to the scalar potential in the above analysis, Eqs. (15) differ in a crucial respect from what one obtains by simply dropping all time derivatives in the Euler-Lagrange equations. This procedure (which I argued above is inconsistent in chromodynamics) gives

$$E^j = -D_jb^0, \quad D_jE^j = gJ^0, \quad \epsilon^{mlj}D_lB^j = gJ^m_{\text{spin}} + D_0E^m = gJ^m_{\text{spin}} - igP(b^0, D_mb^0). \tag{25}$$

The difference in sign between the $P(b^0, D_mb^0)$ term in Eq. (25) and the corresponding $P(\lambda, D_m\lambda)$ term in Eq. (15) is not a calculational error, but rather reflects the fact that Eqs. (25) are obtained by an unconstrained variation of

$$\mathcal{L} = \tfrac{1}{2}[S(E^j, E^j) - S(B^j, B^j)] + \cdots,$$

while Eqs. (15) result from either an unconstrained variation (at $t=0$) of $\frac{1}{2}[S(E^j, E^j) + S(B^j, B^j)]$, as in Eqs. (11)–(13), or a constrained variation of $\frac{1}{2}[S(E^j, E^j) - S(B^j, B^j)] + \cdots$, as in Eqs. (19)–(22). To complete this comparison, I note that Eqs. (25) (with λ replacing b^0) would again result from the variation[9] of the following constrained functional with all Cauchy data treated as independent:

$$\mathcal{F}' = \mathcal{F}'_{\text{charge}} - \mathcal{F}_{\text{spin}}, \quad \mathcal{F}'_{\text{charge}} = \int d^3x \{\tfrac{1}{2}[S(E^j, E^j) + S(B^j, B^j)] - S(\lambda, D_jE^j - gJ^0)\}. \tag{26}$$

Neglecting spin, the variational problem posed in Eq. (26) is just that of finding, on a given time slice, the field configuration of minimum energy consistent with the presence of specified quark charges. The fact that the chromostatics equations do not agree with Eqs. (25) means that static quark configurations are not absolute minima of the field energy. This is consistent with the view that the key ingredient in the analysis of static quark forces, and of quark confinement, is not the absolute minimization of energy but rather the maximization of probability. Unconfined quark states may have much lower energy than confined quark states, but if they occur with only very small probability, they will not dominate the physics. See Appendix B for a further discussion of this point.

(v) I have already stressed that the derivation of Eqs. (15) applies only at the instant of tunneling through, when $\dot{b}_j = 0$. One can still ask whether Eqs. (15) can be valid at later times, or equivalently, whether when the solution of Eqs. (15) is used as $t=0$ initial Cauchy data for the time-dependent Euler-Lagrange equations of Eq. (6), the time-evolved solution at a time $t>0$ continues to satisfy Eqs. (15). I will show below that the answer in general must be no, by considering the particular case where the $t=0$ Cauchy data are the Prasad-Sommerfield solution of Eqs. (15), for which a simple calculation gives $d^2\mathfrak{F}_{\text{charge}}/dt^2>0$ for the time-evolved solution. Since the functional $\mathfrak{F}_{\text{charge}}$ has no explicit time dependence, this means that the time-evolved solution is not extremal, and so does not satisfy Eqs. (15). Evidently, chromostatics behaves very differently from electrostatics. The equations of electrostatics take the same simple form on all time slices. The equations of chromostatics take a simple form only on a subset of time slices of measure zero.

(vi) Even though the equations of chromostatics take a simple form only on special time slices, they predict the same static forces at all times because E_{gluon} is (almost) a conserved quantity. To see this, we use Eq. (I.30),

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}{}_{\text{gluon}}=gS(f^\alpha{}_\gamma,J^\gamma)\,, \tag{27}$$

which implies that

$$\frac{d}{dt}E_{\text{gluon}}=\int d^3x\,\frac{\partial}{\partial x^\beta}T^{0\beta}{}_{\text{gluon}} = -\int d^3x\,gS(E^k,J^k_{\text{spin}})\,. \tag{28a}$$

Hence, dE_{gluon}/dt vanishes at all times in the absence of spin currents. To study the effect of spin currents, I note that on the $t=0$ time slice Eq. (28a) can be put in the form

$$\frac{d}{dt}E_{\text{gluon}}\bigg|_0=\sum_n ig^2S\left(\frac{B^m(x_n)\sigma^m_{(n)}}{2m_{(n)}},P(\lambda(x_n),Q_{(n)})\right), \tag{28b}$$

which by the compatibility condition of Eq. (24) takes the form

$$\frac{d}{dt}E_{\text{gluon}}\bigg|_0=\sum_n ig^2S\left(P\left(\frac{B^m(x_n)\sigma^m_{(n)}}{2m_{(n)}},\frac{B^k(x_n)\sigma^k_{(n)}}{2m_{(n)}}\right),Q_{(n)}\right) =0 \tag{28c}$$

when noncommutativity of Pauli spin matrices is ignored [cf. the discussion of spin approximations which followed Eq. (5) above]. However, an attempt to go beyond first derivatives at $t=0$ leads to difficulties. For example, if we try to evaluate

$$\frac{d^2E_{\text{gluon}}}{dt^2}=-\int d^3x\,g\left[S\left(\frac{\partial E^k}{\partial t},J^k_{\text{spin}}\right)+S\left(E^k,\frac{\partial J^k_{\text{spin}}}{\partial t}\right)\right] \tag{28d}$$

by using Eq. (3) to describe the spin current at arbitrary times, nonvanishing terms linear in the spins are encountered. I believe that this indicates that Eq. (3), while providing a consistent description of spin effects at $t=0$, does not correctly describe dynamical spin effects. This limitation would be consistent with the fact that Eq. (3) was not given an *a priori* justification, but was inferred from the expected form of the spin interaction energy by an application of the principle of virtual work on the $t=0$ time slice.

II. THE $q\bar{q}$ STATIC-FORCE CALCULATION

I turn now to a calculation of the $q\bar{q}$ static force in the color-singlet channel, for general underlying color group U(n). To recapitulate some results from I, in this case the diagonalized P and S tables reduce to an SU(2) algebra, over which the inner product S is a multiple $D=n/2$ of the unit matrix. Going over to the natural vector notation for the overlying SU(2) algebra by the replacements

$$\begin{aligned}
&P(u,v)\to i\vec{u}\times\vec{v}\,,\\
&S(u,v)\to D\vec{u}\cdot\vec{v}\,,\\
&Q_q\to\vec{Q}^{\text{eff}}_q=\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\frac{3}{(n^2+8)^{1/2}}\right),\\
&Q_{\bar{q}}\to\vec{Q}^{\text{eff}}_{\bar{q}}=\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},-\frac{(1+\frac{1}{2}n^2)}{(n^2+8)^{1/2}}\right),
\end{aligned} \tag{29}$$

the relevant equations from Sec. I take the following form:

source currents:

$$\begin{aligned}
&\vec{J}^0=\sum_{n=1}^{2}\vec{Q}_{(n)}\delta^3(x-x_n)\,,\\
&\vec{Q}_{(1)}=\vec{Q}^{\text{eff}}_q\,,\quad \vec{Q}_{(2)}=\vec{Q}^{\text{eff}}_{\bar{q}}\,,\\
&\vec{J}^k_{\text{spin}}=\sum_{n=1}^{2}\frac{-1}{2m_q}\epsilon^{klm}D_l[\vec{Q}_{(n)}\delta^3(x-x_n)\sigma^m_{(n)}]\,,
\end{aligned} \tag{30}$$

principle of virtual work:

$$\begin{aligned}
\delta E_{\text{gluon}}&=\delta\int d^3x\,\tfrac{1}{2}[D(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)+V_H(D\vec{\lambda}^2)]\\
&=\delta V_{\text{static}}\\
&=gD\int d^3x(\delta\vec{\lambda}\cdot\vec{J}^0+\delta\vec{b}^k\cdot\vec{J}^k_{\text{spin}})\,,
\end{aligned} \tag{31}$$

equations of chromostatics:

$$\begin{aligned}
&\vec{E}^j=-D_j\vec{\lambda}\,,\\
&D_j\vec{E}^j+V'_H(D\vec{\lambda}^2)\vec{\lambda}=g\vec{J}^0\,,\\
&\vec{B}^j=\epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l+\tfrac{1}{2}g\vec{b}^k\times\vec{b}^l\right),\quad D_j\vec{B}^j=0\,,\\
&\epsilon^{mlj}D_l\vec{B}^j=g\vec{J}^m_{\text{spin}}-g\vec{\lambda}\times D_m\vec{\lambda}\,,\\
&D_j\vec{w}=\frac{\partial}{\partial x^j}\vec{w}+g\vec{b}^j\times\vec{w}\,,\\
&[D_j,D_l]\vec{w}=g\epsilon^{jlm}\vec{B}^m\times\vec{w}\,,
\end{aligned} \tag{32}$$

compatibility conditions:

$$\vec{B}^m(x_n)\times\frac{\vec{Q}_{(n)}\sigma^m_{(n)}}{2m_q}-\vec{\lambda}(x_n)\times\vec{Q}_{(n)}=0,\quad n=1,2. \qquad (33)$$

I will assume that if radiative corrections do produce a Higgs potential for $\vec{\lambda}$, it appears in order g^4 of perturbation theory as in the analysis of Coleman and Weinberg,[7] and has the approximate form

$$V_H(x)=\frac{C_H}{2}\frac{g^4}{D^2}(x-\langle x\rangle)^2 \qquad (34)$$

with $\langle x\rangle$ a dimensional subtraction constant. Equations (32) and (34) are familiar ones—apart from the presence of the source currents, they are the static equations of the 't Hooft-Polyakov[11] model, with the static potential $\vec{\lambda}$ playing the role of the Higgs scalar field.

It will be convenient in the subsequent analysis to follow the standard practice of scaling out the coupling constant g by making the replacements $\vec{\lambda}\to\vec{\lambda}/g$, $\vec{b}^j\to\vec{b}^j/g$, $\vec{E}^j\to\vec{E}^j/g$, $\vec{B}^j\to\vec{B}^j/g$ in Eqs. (31)–(34), giving scaled equations [Eqs. (30) and (33) are unchanged]

$$\begin{aligned}\delta E_{\text{gluon}}&=\delta\int d^3x\,\tfrac{1}{2}\left[\frac{D}{g^2}(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)+\frac{C_H}{2}(\vec{\lambda}\cdot\vec{\lambda}-\kappa^2)^2\right]\\&=\delta V_{\text{static}}\\&=D\int d^3x(\delta\vec{\lambda}\cdot\vec{J}^0+\delta\vec{b}^k\cdot\vec{J}^k_{\text{spin}}),\end{aligned} \qquad (31')$$

$$\begin{aligned}&\vec{E}^j=-D_j\vec{\lambda},\\&D_j\vec{E}^j+C_H\frac{g^2}{D}(\vec{\lambda}^2-\kappa^2)\vec{\lambda}=g^2\vec{J}^0,\\&\vec{B}^j=\epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l+\tfrac{1}{2}\vec{b}^k\times\vec{b}^l\right),\quad D_j\vec{B}^j=0,\\&\epsilon^{mlj}D_l\vec{B}^j=g^2\vec{J}^m_{\text{spin}}-\vec{\lambda}\times D_m\vec{\lambda},\\&D_j\vec{w}=\frac{\partial}{\partial x^j}\vec{w}+\vec{b}^j\times\vec{w},\quad \kappa^2=\langle x\rangle g^2/D,\\&[D_j,D_l]\vec{w}=\epsilon^{jlm}\vec{B}^m\times\vec{w}.\end{aligned} \qquad (32')$$

The natural thing to do with Eqs. (31′) and (32′) is to assume that the coupling constant g^2 is small and make a perturbation expansion. Since we see that the Higgs potential [assuming Eq. (34)] is an order-g^2 correction to the leading terms, it in fact will be a good approximation to neglect it, provided (i) we always calculate V_{static} from the second line of Eq. (31′), where V_H has been eliminated by use of the equations of motion, and (ii) we enforce the boundary condition arising from the Higgs potential

$$\vec{\lambda}^2\underset{r\to\infty}{\longrightarrow}\kappa^2. \qquad (35)$$

Since whether radiative corrections produce a Higgs potential for $\vec{\lambda}$ is an open question at present, Eq. (35) should be considered as a postulate which I make in the purely classical analysis which follows.

A. Zeroth-order approximation

I begin the perturbation analysis by considering the zeroth-order approximation to Eqs. (31′) and (32′),

$$\delta V_{0\,\text{static}}=D\int d^3x(\delta\vec{\lambda}_0\cdot\vec{J}^0+\delta\vec{b}^k_0\cdot\vec{J}^k_{\text{spin}}), \qquad (36)$$

$$\begin{aligned}&\vec{E}^j_0=-D_{0j}\vec{\lambda}_0,\\&D_{0j}\vec{E}^j_0=0,\\&\vec{B}^j_0=\epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l_0+\tfrac{1}{2}\vec{b}^k_0\times\vec{b}^l_0\right),\quad D_{0j}\vec{B}^j_0=0\\&\epsilon^{mlj}D_{0l}\vec{B}^j_0=-\vec{\lambda}_0\times D_{0m}\vec{\lambda}_0,\\&D_{0j}\vec{w}_0=\frac{\partial}{\partial x^j}\vec{w}_0+\vec{b}^j_0\times\vec{w}_0,\quad \vec{\lambda}_0^2\underset{r\to\infty}{\longrightarrow}\kappa^2\\&[D_{0j},D_{0l}]\vec{w}=\epsilon^{jlm}\vec{B}^m_0\times\vec{w}.\end{aligned} \qquad (37)$$

The properties of Eqs. (37) have been discussed extensively in the literature. They are solved by the ansatz[12]

$$\vec{B}^j_0=\pm\vec{E}^j_0=\mp D_{0j}\vec{\lambda}_0, \qquad (38)$$

since this gives

$$\begin{aligned}\epsilon^{mlj}D_{0l}\vec{B}^j_0&=\mp\tfrac{1}{2}\epsilon^{mlj}[D_{0l},D_{0j}]\vec{\lambda}_0\\&=\pm\vec{\lambda}_0\times\vec{B}^m_0=-\vec{\lambda}_0\times D_{0m}\vec{\lambda}_0,\\D_{0j}D_{0j}\vec{\lambda}_0&=\mp D_{0j}\vec{B}^j_0=0.\end{aligned} \qquad (39)$$

It is at this point that the sign of the outer-product term in the "curl B" equation in Eq. (15) plays a crucial role; had we chosen Eqs. (25) as the equations of chromostatics, we would have ended up with the equation $\epsilon^{mlj}D_{0l}\vec{B}^j_0=\vec{\lambda}_0\times D_{0m}\vec{\lambda}_0$, which is solved, not by Eq. (36) but rather by $\vec{B}^j_0=\pm i\vec{E}^j_0=\mp iD_{0j}\vec{\lambda}_0$. In real space-time, such complex solutions are not permitted. The solutions of Eq. (37) are characterized by a topological index m, which takes integer values. The case $m=0$ corresponds to the trivial solution $\vec{\lambda}_0=\vec{E}^j_0=\vec{B}^j_0=0$; the solution for $m=1$, first obtained by Prasad and Sommerfield,[13] is known to be classically stable,[12] while the soltuions for $m>1$ are unstable (if a Higgs potential is included) against breakup into $m=1$ solutions. I conjecture that the $m=1$ solution of Eqs. (37), rather than the trivial $m=0$ solution, is the relevant zeroth-order solution for the $q\bar{q}$ force problem. (A plausible direction in which to look for an *a priori* justification of this conjecture would be to try to show that creation of a $q\bar{q}$

pair by vacuum tunneling is much more probable when associated with an $m=1$, rather than with an $m=0$, background field.) I will show that this solution has properties which make it plausible that it behaves as a quark-confining "bag."[14]

The explicit form of the $m=1$ solution to Eq. (37) is [indices $a,b,c,\ldots$ are SU(2)-vector indices; indices $i,j,k,\ldots$ are spatial-vector indices]

$$\lambda_0^a = \mp\frac{x^a}{r^2}(1-\kappa r\coth\kappa r)\,,\qquad b_0^{ai}=\frac{\epsilon^{aij}x^j}{r^2}\left(1-\frac{\kappa r}{\sinh\kappa r}\right), \tag{40}$$

from which the field strength is readily obtained,

$$B_0^{ak} = -\delta^{ka}\left(\frac{\kappa^2\cosh\kappa r}{\sinh^2\kappa r}-\frac{\kappa}{r\sinh\kappa r}\right) - x^k x^a\left(\frac{1}{r^4}+\frac{\kappa}{r^3\sinh\kappa r}-\frac{\kappa^2}{r^2\sinh^2\kappa r}-\frac{\kappa^2\cosh\kappa r}{r^2\sinh^2\kappa r}\right). \tag{41}$$

The zeroth-order fields make a contribution to the gluon energy

$$E_{0\ \text{gluon}} = (D/g^2)\int d^3x\,\tfrac{1}{2}(\vec{E}_0^j\cdot\vec{E}_0^j+\vec{B}_0^j\cdot\vec{B}_0^j) = (D/g^2)4\pi\kappa\,, \tag{42}$$

which I will interpret as the energy needed to create the "bag." While $E_{0\ \text{gluon}}$ does not contribute to δV_{static}, I argue below that it determines the limits of validity of the perturbation expansion. From Eqs. (38) and (42), it is also easy to evaluate the zeroth-order field contribution to the space integral of the axial anomaly of Eq. (I.55),

$$\int d^3x\,\mathcal{A} = \frac{Cg^2}{8\pi^2}D\frac{1}{g^2}\int d^3x\,\vec{E}_0^j\cdot\vec{B}_0^j = \pm CD\frac{\kappa}{2\pi}\,, \tag{43}$$

and so there may be a relation to the mechanism for resolution of the U(1) problem proposed by 't Hooft.[15]

From Eqs. (36), (20), and (29), the contribution of the zeroth-order fields to the static quark potential is

$$\delta V_{0\ \text{static}} = D\sum_{n=1}^{2}\left[\delta\vec{\lambda}_0(x_n)\cdot\vec{Q}_{(n)} - \delta\vec{B}_0^m(x_n)\cdot\frac{\vec{Q}_{(n)}\sigma_{(n)}^m}{2m_q}\right]. \tag{44}$$

This equation is valid for arbitrary variations, but there are two special classes of variations on which I wish to focus initially. Consider, first of all, the local gauge transformations[16]

$$\delta_{\text{gauge}}\vec{\lambda}_0=\vec{\lambda}_0\times\vec{\phi}\,,\qquad \delta_{\text{gauge}}\vec{b}_0^j = D_{0j}\vec{\phi}\,,\qquad \delta_{\text{gauge}}\vec{B}_0^m=\epsilon^{mlj}D_{0l}D_{0j}\vec{\phi}=\vec{B}_0^m\times\vec{\phi}\,, \tag{45}$$

which are easily verified to be an invariance of $E_{0\ \text{gluon}}$ of Eq. (42) and hence are an invariance of Eqs. (37). From Eq. (44) the corresponding change in $V_{0\ \text{static}}$ is

$$\delta_{\text{gauge}}V_{0\ \text{static}} = D\sum_{n=1}^{2} -\vec{\phi}\cdot\left[\vec{\lambda}_0(x_n)\times\vec{Q}_{(n)} - \vec{B}_0^m(x_n)\times\frac{\vec{Q}_{(n)}\sigma_{(n)}^m}{2m_q}\right], \tag{46}$$

which vanishes by virtue of the compatibility conditions of Eq. (33). Consider next the class of variations which leave the quark positions and spin orientations fixed, but translate the center of the "bag" [which was arbitrarily fixed at $r=0$ in Eq. (40)] and reorient it in coordinate space. Such variations do no work against the quark restraining forces, and hence by the principle of virtual work must give $\delta V_{0\ \text{static}}=0$. Hence, for fixed quark variables, the "bag" orientation is fixed by the condition that

$$\sum_{n=1}^{2}[\vec{\lambda}_0(x_n)\cdot\vec{Q}_{(n)} - \vec{B}_0^m(x_n)\cdot\vec{Q}_{(n)}\sigma_{(n)}^m/2m_q] \tag{47}$$

be a minimum; this then uniquely determines $V_{0\ \text{static}}$ to be

$$V_{0\ \text{static}} = \min D\sum_{n=1}^{2}\left[\vec{\lambda}_0(x_n)\cdot\vec{Q}_{(n)} - \vec{B}_0^m(x_n)\cdot\frac{\vec{Q}_{(n)}\sigma_{(n)}^m}{2m_q}\right],$$

min = minimum over changes in "bag" origin and orientation, consistent with maintaining the compatibility conditions (48)

$$0=\vec{\lambda}_0(x_n)\times\vec{Q}_{(n)} - \vec{B}_0^m(x_n)\times\frac{\vec{Q}_{(n)}\sigma_{(n)}^m}{2m_q}\,,\quad n=1,2.$$

It thus seems natural to regard $V_{0\ \text{static}}$ as a quark-bag orientation energy.

Before evaluating Eq. (48) more explicitly, let me first argue that, for purposes of evaluating V_{static}, it makes no difference which sign is chosen in Eq. (38). The point is simply that the sign of $\vec{E}$ relative to $\vec{B}$

can only affect pseudoscalar observables, such as the axial-vector anomaly of Eq. (43), but cannot change even parity observables, such as the energy. Hence to avoid a proliferation of ± signs in the formulas, I choose for definiteness the upper sign in Eqs. (38) and (40), and use henceforth

$$\vec{B}_0^j = \vec{E}_0^j = -D_{0j}\vec{\lambda}_0\,, \qquad \lambda_0^a = -(x^a/r^2)(1-\kappa r\coth\kappa r). \tag{49}$$

Having made this choice, I set up the geometry necessary for the minimization in Eq. (48) in the approximation of regarding the spin terms as a small perturbation on the charge contribution. Thus, I consider first the minimization problem

$$V_{0\,\text{static,charge}} = \min D\sum_{n=1}^{2}\vec{\lambda}_0(x_n)\cdot\vec{Q}_{(n)},$$

min = minimum over changes in bag origin and orientation, consistent with (50)

$$0=\vec{\lambda}_0(x_n)\times\vec{Q}_{(n)}\,,\quad n=1,2.$$

Once this problem has been solved, spin effects linear in $\sigma_{(n)}^m/2m_q$ can be calculated from the expression

$$V_{0\,\text{static, spin}} = -D\sum_{n=1}^{2}\vec{B}_0^m(x_n)\cdot\frac{\vec{Q}_{(n)}\sigma_{(n)}^m}{2m_q}\,, \tag{51}$$

$\vec{\lambda}_0, \vec{b}_0^j$ = bag potentials determined by Eq. (50),

since the reorientation induced by the spins will change both $V_{0\,\text{static, charge}}$, which is already extremal, and $V_{0\,\text{static,spin}}$ only to second order in the quark magnetic moments. In order to calculate bilinear spin effects of the form $\sigma_{(1)}^m\sigma_{(2)}^n/m_q^2$, which are in principle determined by the formalism through order g^2, a careful evaluation of spin-induced reorientation energies will be required.

To proceed with the geometry, let a be the separation between the q and $\bar{q}$ and let $\psi=\pi-\cos^{-1}(-1/n)$ be the complement of the angle between $\vec{Q}_q^{\text{eff}}$ and $\vec{Q}_{\bar{q}}^{\text{eff}}$ [cf. Eq. (I.44)]. It is obviously easier to orient the quarks relative to a fixed bag than the other way round, so taking bag potentials as given by Eq. (49), we must choose $q,\bar{q}$ locations $\vec{x}_q, \vec{x}_{\bar{q}}$ in the bag with $\vec{x}_q\times\vec{Q}_q^{\text{eff}}=0$, $\vec{x}_{\bar{q}}\times\vec{Q}_{\bar{q}}^{\text{eff}}=0$, $|\vec{x}_q-\vec{x}_{\bar{q}}|=a$ which minimize the orientation energy. Since the spatial variation of $\vec{\lambda}_0(x)$ is given by

$$\begin{aligned}&\vec{\lambda}_0(x)=\kappa V(\kappa r)\hat{x}\,,\\ &V(z)=\coth z-1/z=-V(-z),\\ &V(0)=0,\quad V(\infty)=1,\\ &V'(z)=\frac{1}{z^2}-\frac{1}{\sinh^2 z}\geqslant 0,\quad 0\leqslant z\leqslant\infty\end{aligned} \tag{52}$$

it is clear that to get a minimum we must insert the $\bar{q}$, which has the larger effective charge, at a location where $\vec{Q}_{\bar{q}}^{\text{eff}}$ is antiparallel to $\vec{x}_{\bar{q}}$.[17] This leads to the minimization problem illustrated in Fig. 1, and expressed analytically by

$$V_{0\,\text{static, charge}}(a)=\min_{\psi>\alpha>\psi-\pi} -\kappa D\left[V\left(\kappa a\frac{\sin\alpha}{\sin\psi}\right)+\frac{n}{2}V\left(\kappa a\frac{\sin(\psi-\alpha)}{\sin\psi}\right)\right]. \tag{53}$$

Denoting by $\alpha(a)$ the minimizing value of α for

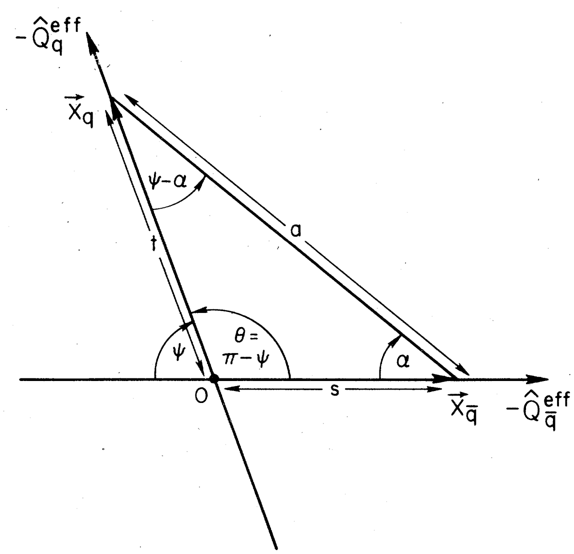

FIG. 1. Geometry for calculation of the orientation potential. The angle α is varied to minimize the orientation energy [cf. Eq. (53)]. The angle θ and the unit-normalized effective charges $\hat{Q}_{\bar{q}}^{\text{eff}}$, $\hat{Q}_q^{\text{eff}}$ are given by

$$\theta=\cos^{-1}(-1/n),$$

$$\hat{Q}_{\bar{q}}^{\text{eff}}=\frac{2}{n}\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\frac{-(1+\frac{1}{2}n^2)}{(n^2+8)^{1/2}}\right),$$

$$\hat{Q}_q^{\text{eff}}=\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\frac{3}{(n^2+8)^{1/2}}\right).$$

The sides s,t of the triangle are given by the law of sines,

$$\frac{a}{\sin(\pi-\psi)}=\frac{t}{\sin\alpha}=\frac{s}{\sin(\psi-\alpha)}\,.$$

given $q,\bar{q}$ separation a, the $q,\bar{q}$ locations are fixed to be

$$\vec{x}_q = -a\,\frac{\sin\alpha(a)}{\sin\psi}\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\ \frac{3}{(n^2+8)^{1/2}}\right),$$

$$\vec{x}_{\bar{q}} = -a\,\frac{\sin[\psi-\alpha(a)]}{\sin\psi}\,\frac{2}{n}\left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\ \frac{-(1+\frac{1}{2}n^2)}{(n^2+8)^{1/2}}\right), \tag{54}$$

which when substituted into Eq. (51) determine $V_{0\,\text{static, spin}}$. From Eqs. (52) and (53) we immediately see that

$$V_{0\,\text{static, charge}}(0) - V_{0\,\text{static, charge}}(\infty) = \kappa D(1+\tfrac{1}{2}n), \tag{55}$$

showing that the orientation potential gives the $q\bar{q}$ potential a repulsive central core.

As a final topic relating to the zeroth-order solutons, I will show that when Eq. (49) is used as $t=0$ initial Cauchy data for the time-dependent equations of motion, the functional[18]

$$\mathfrak{F}_{\text{charge}} = (D/g^2)\int d^3x\,\tfrac{1}{2}(\vec{E}^j\circ\vec{E}^j - \vec{B}^j\circ\vec{B}^j) \tag{56}$$

varies with time. I have dropped subscripts zero denoting the zeroth-order solution so as not to confuse them with Lorentz subscripts 0. Working in a general gauge $\vec{b}_0, \vec{b}^j$, Eq. (49) gives as the $t=0$ initial condition

$$\vec{B}^j = \vec{E}^j = -D_j\vec{\lambda} = -D_j\vec{b}_0 - \frac{\partial}{\partial t}\vec{b}^j\Big|_0,$$
$$\vec{\lambda} = -\frac{\vec{x}}{r^2}(1-\kappa r\coth\kappa r). \tag{57}$$

From this initial condition and the time evolution equations

$$\frac{\partial}{\partial t}\vec{B}^l = \vec{b}_0\times\vec{B}^l - \epsilon^{ljm}D_j\vec{E}^m,$$
$$\frac{\partial}{\partial t}\vec{E}^l = \vec{b}_0\times\vec{E}^l + \epsilon^{ljm}D_j\vec{B}^m, \tag{58}$$

the following results are found for the derivatives of $\vec{E}^j\cdot\vec{E}^j$ and $\vec{B}^j\cdot\vec{B}^j$ at $t=0$:

$$\frac{\partial}{\partial t}\vec{E}^j\circ\vec{E}^j\Big|_0 = \frac{\partial}{\partial t}\vec{B}^j\circ\vec{B}^j\Big|_0 = 0,$$
$$\frac{\partial^2}{\partial t^2}\vec{E}^j\cdot\vec{E}^j\Big|_0 = 4(\vec{\lambda}\times D_l\vec{\lambda})\cdot(\vec{\lambda}\times D_l\vec{\lambda}), \tag{59}$$
$$\frac{\partial^2}{\partial t^2}\vec{B}^j\circ\vec{B}^j\Big|_0 = 4(\vec{\lambda}\times D_l\vec{\lambda})\cdot(\vec{\lambda}\times D_l\vec{\lambda}) - 4\epsilon^{ljm}D_l\vec{\lambda}\cdot(D_j\vec{\lambda}\times D_m\vec{\lambda}).$$

As expected, these equations are independent of the choice of gauge, even though quantities such as $(\partial^2/\partial t^2)\vec{E}^j|_0$ which appear at intermediate stages of the calculation depend explicitly on $\vec{b}_0$. Since an integration by parts gives[19]

$$-\int d^3x\,4\epsilon^{ljm}D_l\vec{\lambda}\circ(D_j\vec{\lambda}\times D_m\vec{\lambda}) = -8\int d^3x(\vec{\lambda}\times D_l\vec{\lambda})\circ(\vec{\lambda}\times D_l\vec{\lambda}), \tag{60}$$

Eqs. (59) give

$$\frac{\partial^2}{\partial t^2}\mathfrak{F}_{\text{charge}}\Big|_0 = (D/g^2)4\int d^3x(\vec{\lambda}\times D_l\vec{\lambda})\circ(\vec{\lambda}\times D_l\vec{\lambda}) > 0, \tag{61a}$$

and also, as a consistency check

$$\frac{\partial^2}{\partial t^2}E_{0\,\text{gluon}}\Big|_0 = 0. \tag{61b}$$

This analysis implies that the time-evolved solution does not extremize $\mathfrak{F}_{\text{charge}}$, and so does not take the form of Eq. (37) at times later than $t=0$.

B. Second-order approximation

I turn now to an analysis of the order-g^2 terms of Eqs. (31′) and (32′). Writing

$$\vec{\lambda} = \vec{\lambda}_0 + g^2\vec{\lambda}_1,$$
$$\vec{b}^j = \vec{b}_0^j + g^2\vec{b}_1^j,$$
$$\vec{E}^j = \vec{E}_0^j + g^2\vec{E}_1^j, \tag{62}$$
$$\vec{B}^j = \vec{B}_0^j + g^2\vec{B}_1^j,$$
$$D_j = D_{0j} + g^2 D_{1j},\quad D_{1j}\vec{w} = \mathrm{b}_1^j\times\vec{w},$$

and replacing the Higgs potential by the boundary condition of Eq. (35), the order-g^2 terms in Eq. (32′) take the form

$$D_{0j}D_{0j}\vec{\lambda}_1 + 2\vec{b}_1^j\times D_{0j}\vec{\lambda}_0 + (D_{0j}\vec{b}_1^j)\times\vec{\lambda}_0 = -\vec{J}^0,$$
$$D_{0k}D_{0k}\vec{b}_1^j - (\vec{\lambda}_0\cdot\vec{\lambda}_0)\vec{b}_1^j + (\vec{\lambda}_0\cdot\vec{b}_1^j)\vec{\lambda}_0 - 2\vec{f}_0^{kj}\times\vec{b}_1^k - \vec{\lambda}_1\times D_{0j}\vec{\lambda}_0 - \vec{\lambda}_0\times D_{0j}\vec{\lambda}_1 - D_{0j}(D_{0k}\vec{b}_1^k) = -\vec{J}^j_{\text{spin}}, \tag{63}$$
$$\vec{f}_0^{kj} = -\epsilon^{kjm}D_{0m}\vec{\lambda}_0.$$

These equations (without the source terms) were first derived and analyzed by Polyakov,[16] and I follow his treatment in a number of important respects. I also will employ many of the techniques pioneered by Brown *et al.*[20] in their study of propagation functions in pseudoparticle (instanton) fields. The basic method for studying Eqs. (63) consists of analyzing properties of the mode functions

$$\vec{x}^{\mu} = (\vec{x}^0, \vec{x}^j), \tag{64}$$

which satisfy

$$D_{(1)}{}^2 \vec{x}^{\mu} = -\omega^2 \vec{x}^{\mu},$$

which is an abbreviated notation for

$$\begin{aligned} &D_{0j} D_{0j} \vec{x}^0 + 2\vec{x}^j \times D_{0j} \vec{x}^0 + (D_{0j} \vec{x}^j) \times \vec{x}^0 = -\omega^2 \vec{x}^0, \\ &D_{0k} D_{0k} \vec{x}^j - (\vec{\lambda}_0 \circ \vec{\lambda}_0) \vec{x}^j + (\vec{\lambda}_0 \cdot \vec{x}^j) \vec{\lambda}_0 - 2\vec{f}_0^{kj} \times \vec{x}^k - \vec{x}^0 \times D_{0j} \vec{\lambda}_0 - \vec{\lambda}_0 \times D_{0j} \vec{x}^0 - D_{0j}(D_{0k} \vec{x}^k) = -\omega^2 \vec{x}^j. \end{aligned} \tag{65}$$

It proves convenient to introduce an inner product[16] for mode functions,

$$(x_1, x_2) = \int d^3x\, \vec{x}_1^{\mu} \circ \vec{x}_2^{\mu} = \int d^3x (\vec{x}_1^0 \cdot \vec{x}_2^0 + \vec{x}_1^j \cdot \vec{x}_2^j), \tag{66}$$

where the contraction of two upper Greek indices indicates a Euclidean inner product. It is easily checked that the differential operator of Eq. (65) is Hermitian with respect to the inner product,

$$(x_1, D_{(1)}{}^2 x_2) = (x_2, D_{(1)}{}^2 x_1), \tag{67}$$

which implies that if x_1, x_2 are mode functions with distinct eigenvalues, they are orthogonal,

$$(x_1, x_2) = 0, \quad \omega_1{}^2 \neq \omega_2{}^2. \tag{68}$$

Now we already have found one set of mode functions of Eq. (65), since the gauge transformations of the zeroth-order solution given in Eq. (45) satisfy Eq. (65) with eigenvalue 0,

$$\begin{aligned} &\vec{x}^{(0)0} = \vec{\lambda}_0 \times \vec{\phi}, \quad \vec{x}^{(0)j} = D_{0j} \vec{\phi}, \\ &D_{(1)}{}^2 x^{(0)} = 0. \end{aligned} \tag{69}$$

Recall also the compatibility condition, which to leading order takes the form [cf. Eq. (24), and remember that g has now been scaled out]

$$D_{0m} \vec{J}^m_{\text{spin}} + \vec{\lambda}_0 \times \vec{J}^0 = 0, \tag{70}$$

and which implies

$$0 = \int d^3x [(\vec{\lambda}_0 \times \vec{\phi}) \circ \vec{J}^0 + D_{0m} \vec{\phi} \cdot \vec{J}^m_{\text{spin}}] = (x^{(0)}, J) \tag{71}$$

with the source four-current defined by

$$\vec{J}^{\mu} = (\vec{J}^0, \vec{J}^j_{\text{spin}}). \tag{72}$$

Thus the gauge modes are orthogonal to the source current which appears in Eq. (63), and the indeterminate components of the order-g^2 perturbation along the gauge modes can be defined to be zero,

$$\begin{aligned} &(x_1, x^{(0)}) = 0, \\ &\vec{x}_1^{\mu} = (\vec{\lambda}_1, \vec{b}_1^j). \end{aligned} \tag{73}$$

By reversing the steps leading from Eq. (70) to Eq. (72), Eq. (73) takes the form

$$D_{0m} \vec{b}_1^m + \vec{\lambda}_0 \times \vec{\lambda}_1 = 0, \tag{74}$$

a gauge condition on the perturbation x_1 which Polyakov[16] calls the "natural gauge." Using Eq. (74) to rewrite Eq. (63), the original expression

$$D_{(1)}{}^2 x_1 = -J \tag{63'}$$

can be put in the modified form

$$D_{(2)}{}^2 x_1 = -J$$

which is an abbreviated notation for

$$\begin{aligned} &D_{0j} D_{0j} \vec{\lambda}_1 - \vec{\lambda}_0 \circ \vec{\lambda}_0 \vec{\lambda}_1 + \vec{\lambda}_0 (\vec{\lambda}_0 \cdot \vec{\lambda}_1) - 2(D_{0j} \vec{\lambda}_0) \times \vec{b}_1^j = -\vec{J}^0, \\ &D_{0k} D_{0k} \vec{b}_1^j - \vec{\lambda}_0 \cdot \vec{\lambda}_0 \vec{b}_1^j + \vec{\lambda}_0 (\vec{\lambda}_0 \cdot \vec{b}_1^j) + 2(D_{0j} \vec{\lambda}_0) \times \vec{\lambda}_1 + 2\vec{f}_0^{jk} \times \vec{b}_1^k = -\vec{J}^j_{\text{spin}}. \end{aligned} \tag{75}$$

Before proceeding with the careful analysis, I pause to give a heuristic argument, based on Eqs. (35), (69), and (75), which suggests that the zeroth-order solution may behave as a quark-confining bag. An unconfined quark of charge $\vec{Q}^{\text{eff}}$ would have an asymptotic Coulombic color potential $\vec{Q}^{\text{eff}}/(4\pi r)$. Can such a potential appear asymptotically as a perturbation on the bag solution? According to Eq. (69), any small transverse perturbation on $\vec{\lambda}_0$ is just a gauge transformation of the zeroth-order solution, and hence can be rotated away, leaving only (asymptotically vanishing) longitudinal perturbations. Thus it appears that the inserted q and $\bar{q}$ charges $\vec{Q}_q^{\text{eff}}, \vec{Q}_{\bar{q}}^{\text{eff}}$ may be screened beyond recognition in the asymptotic region. This same conclusion is suggested by the structure of Eq. (75), which is the relevant equation for nongauge perturbations. If the cross coupling to $\vec{b}_1^j$ is neglected,[21] the equation for $\vec{\lambda}_1$ takes the form in the region $r \to \infty$,

$$D_{0j} D_{0j} \vec{\lambda}_1 - \kappa^2 \vec{\lambda}_1 + \kappa^2 \hat{x} \hat{x} \cdot \vec{\lambda}_1 = -\vec{J}^0. \tag{76}$$

The differential operator on the left behaves as $D_{0j} D_{0j} - \kappa^2$ for components of $\vec{\lambda}_1$ perpendicular to $\hat{x}$, and the resemblance of this operator to $(\partial/\partial x^j)^2 - \kappa^2$ again makes plausible the conjecture that the transverse components of $\vec{\lambda}_1$ arising from inserted quark charges are strongly damped at infinity. (Of course, the longitudinal components of λ_1 might still carry information about the inserted charges to infinity when studied in a global, as opposed to a local, manner; this is why the heuristic arguments are suggestive at best, and a detailed calculation is required.) In effect, I am guessing that the 't Hooft-Polyakov-Sommerfield-Prasad solution has rigidity properties resembling those of the Schwarzschild solution in general relativity, with $r\to\infty$ in the gauge theory case playing the role of the horizon and κ playing the role of the mass, and that there are decoupling theorems in the gauge theory case resembling those familiar for a black hole.[22] If such decoupling takes place, then the quark color flux lines must be strongly distorted by the background bag field, and this distortion could be expected to lead to a confining potential.[23]

The way both to test for decoupling at infinity and to calculate the order-g^2 contribution to V_{static} is to calculate the propagator for the perturbation x_1 on the bag background field. Fortunately, as I will now show, the problem of finding this propagator can be exactly solved, in closed form, by an adaptation of the methods used by Brown, Carlitz, Creamer, and Lee[20] for calculating propagators in pseudoparticle fields. It proves convenient to rewrite Eq. (75) in a Euclidean four-dimensional notation, by defining a Euclidean covariant derivative D^μ by

$$D^0\vec{w}=\vec{\lambda}_0\times\vec{w},\qquad D^j\vec{w}=D_{0j}\vec{w}=\left(\frac{\partial}{\partial x^j}+\vec{b}_0^j\times\right)\vec{w},\tag{77a}$$

which satisfies

$$\begin{aligned}(D^\mu D^\nu-D^\nu D^\mu)\vec{w}&=\vec{f}_0^{\mu\nu}\times\vec{w},\\ -\vec{f}_0^{j0}=\vec{f}_0^{0j}&=\vec{E}_0^j=-D_{0j}\vec{\lambda}_0,\\ \vec{f}_0^{kj}=\epsilon^{kjm}\vec{B}_0^m&=-\epsilon^{kjm}D_{0m}\vec{\lambda}_0\\ &=\frac{\partial}{\partial x^k}\vec{b}_0^j-\frac{\partial}{\partial x^j}\vec{b}_0^k+\vec{b}_0^k\times\vec{b}_0^j.\end{aligned}\tag{77b}$$

The two equations of Eq. (75) may then be combined into the single equation

$$(D_{(2)}{}^2x_1)^\mu=D^\sigma D^\sigma\vec{x}_1^\mu+2\vec{f}_0^{\mu\tau}\times\vec{x}_1^\tau=-\vec{J}^\mu.\tag{78}$$

Following Brown *et al.*, the vector propagator $G^{a\mu,b\nu}(x,y)$ is defined by the equation

$$[D_x^\sigma D_x^\sigma\vec{G}^{\mu,b\nu}(x,y)+2\vec{f}_0^{\mu\tau}(x)\times\vec{G}^{\tau,b\nu}(x,y)]^a=-Q^{a\mu,b\nu}(x,y),\tag{79}$$

$$Q^{a\mu,b\nu}(x,y)=\delta^{ab}\delta^{\mu\nu}\delta^3(x-y)-\sum_{\substack{\text{zero modes}\\ s}}x_{(s)}^{a\mu}(x)x_{(s)}^{b\nu}(y),$$

with Q differing from the unit operator by the deletion of the projection of unity on the subspace of normalizable zero-eigenvalue modes of the mode equation

$$D^\sigma D^\sigma\vec{x}^\mu+2\vec{f}_0^{\mu\tau}\times\vec{x}^\tau=-\omega^2\vec{x}^\mu.\tag{80}$$

The procedure used by Brown *et al.* to construct the vector propagator consists of two steps. The first step, carried out immediately below, is to relate the vector propagator to the scalar propagator $\Delta^{ab}(x,y)$ defined by

$$D_x^\sigma D_x^\sigma\Delta^{ab}(x,y)=-\delta^{ab}\delta^3(x-y),\tag{81a}$$

while the second step, carried out in Appendix A, is to explicitly construct the scalar propagator. Defining a complete set of normalized scalar mode functions $\vec{\phi}(x)$ by

$$D^\sigma D^\sigma\vec{\phi}(x)=-\omega^2\vec{\phi}(x),\qquad \int d^3x\,\vec{\phi}(x)\cdot\vec{\phi}(x)=1,\tag{81b}$$

the scalar propagator may be written as a mode sum

$$\Delta^{ab}(x,y)=\sum_\omega\frac{\phi^a(x)\phi^b(y)}{\omega^2}.\tag{81c}$$

I will now show that a complete set of vector mode functions $\vec{x}^\mu$ can be constructed from the scalar mode functions $\vec{\phi}$. Since the vector index μ takes on four values, for each eigenvalue ω^2 we must find four linearly independent mode functions $\vec{x}^{(n)\mu}$, $n=0,1,2,3$. As already suggested by my choice of notation, the $n=0$ eigenmodes are given by Eq. (69), which using Eq. (77a) takes the form

$$\vec{x}^{(0)\mu}=D^\mu\vec{\phi}.\tag{82}$$

Note that while $\vec{x}^{(0)\mu}$ is a zero eigenmode (a gauge freedom) or the original operator $D_{(1)}{}^2$, it does not satisfy the "natural gauge" condition (it is not self-orthogonal), and hence will not be a zero eigenmode of $D_{(2)}{}^2$. In fact, a simple calculation shows that

$$D^\sigma D^\sigma\vec{x}^{(0)\mu}+2\vec{f}_0^{\mu\tau}\times\vec{x}^{(0)\tau}=-\omega^2\vec{x}^{(0)\mu},\tag{83}$$

and that the modes of Eq. (82) have the normalization

$$(x^{(0)},x^{(0)})=-\int d^3x\,\vec{\phi}\cdot D^\mu D^\mu\vec{\phi}=\omega^2\int d^3x\,\vec{\phi}\cdot\vec{\phi}=\omega^2.\tag{84}$$

A straightforward but lengthy calculation shows that the remaining mode functions given by

$$\begin{aligned}\vec{x}^{(n)0}&=-D^n\vec{\phi}=-D_{0n}\vec{\phi},\\ \vec{x}^{(n)j}&=\delta^{jn}D^0\vec{\phi}+\epsilon^{jln}D^l\vec{\phi}\\ &=\delta^{jn}\vec{\lambda}_0\times\vec{\phi}+\epsilon^{jln}D_{0l}\vec{\phi},\quad n=1,2,3\end{aligned}\tag{85}$$

also satisfy

$$D^\sigma D^\sigma \vec{x}^{(n)\mu} + 2\vec{f}_0^{\mu\tau} \times \vec{x}^{(n)\tau} = -\omega^2 \vec{x}^{(n)\mu}, \quad n=1,2,3 \tag{86}$$

and that the $\vec{x}^{(\alpha)\mu}$, $\alpha = 0,1,2,3$ have inner products given by

$$(x^{(\alpha)}, x^{(\beta)}) = \omega^2 \delta^{\alpha\beta}. \tag{87}$$

Taking into account the factor of ω^2 appearing in the inner product of Eq. (87), the vector propagator can thus be expressed in terms of the modes $\vec{x}^{(\alpha)\mu}$ by the formula

$$G^{a\mu,b\nu}(x,y) = \sum_{\alpha,\omega} \frac{x^{(\alpha)a\mu}(x)x^{(\alpha)b\nu}(y)}{\omega^4}. \tag{88}$$

To express Eq. (88) in terms of the scalar propagator, we note that the mode functions of Eq. (85) are summarized by the compact formula

$$\begin{aligned} &\vec{x}^{(n)\mu} = \eta^{(+)\mu\lambda n} D^\lambda \vec{\phi}, \\ &\eta^{(+)\mu\lambda n} = -\eta^{(+)\lambda\mu n}, \\ &\eta^{(+)kln} = \epsilon^{kln}, \quad \eta^{(+)k0n} = \delta^{kn}, \end{aligned} \tag{89}$$

allowing Eq. (88) to be rewritten as[24]

$$G^{a\mu,b\nu}(x,y) = -q^{(-)\mu\nu\lambda\kappa}\, {}^a[\vec{D}_x \Delta_2(x,y) \overleftarrow{D}_y^{\kappa}]^b, \tag{90}$$

with[25]

$$\begin{aligned} &\vec{D}_x^\lambda \vec{w}(x) = -\vec{w}(x)\overleftarrow{D}_x^\lambda = D_x^\lambda \vec{w}(x), \\ &q^{(-)\mu\nu\lambda\kappa} = \delta^{\mu\lambda}\delta^{\nu\kappa} + \sum_n \eta^{(+)\mu\lambda n}\eta^{(+)\nu\kappa n} \\ &\qquad = \delta^{\mu\lambda}\delta^{\nu\kappa} + \delta^{\mu\nu}\delta^{\lambda\kappa} - \delta^{\mu\kappa}\delta^{\nu\lambda} - \epsilon^{\mu\nu\lambda\kappa}, \end{aligned} \tag{91}$$

and with $\Delta_2^{ab}(x,y)$ the convolution of two scalar propagators,

$$\Delta_2^{ab}(x,y) = \sum_\omega \frac{\phi^a(x)\phi^b(y)}{\omega^4} = \int d^3z\, \Delta^{ac}(x,z)\Delta^{cb}(z,y). \tag{92}$$

Although the above derivation is heuristic (it parallels the Appendix of Brown *et al.*), the operator argument of Sec. III of Brown *et al.* can be taken over in its entirety to the case under consideration here, and gives a direct formal proof that the construction of Eqs. (90)–(92) gives the vector propagator defined by Eq. (79).

Having found the vector propagator, I discuss next how to use it to solve for x_1 and to calculate the order-g^2 contribution to the static potential. First of all, it is evident that the propagator of Eqs. (90)–(92) does not satisfy the "natural gauge" condition, since it includes the gauge modes $\vec{x}^{(0)\mu}$ These modes can be readily deleted by dropping the term $\delta^{\mu\lambda}\delta^{\nu\kappa}$ in $q^{(-)\mu\nu\lambda\kappa}$, but in fact this is not necessary, since the orthogonality of the gauge modes to the source current [Eq. (71)] implies that they make a vanishing contribution to the expression for δV_{static} which I give below. A second potential problem is that Eq. (78) for the perturbed fields $\vec{x}_1^\mu$ can be solved only if the source current $\vec{J}^\mu$ is orthogonal to all normalizable eigenmodes with zero eigenvalue of the operator $D_{(2)}{}^2$. These eigenmodes are associated with translations and reorientations of the bag, and the orthogonality of $\vec{J}^\mu$ to these modes is in fact guaranteed by the requirement that the quarks be inserted in a manner which minimizes $V_{0\ \text{static}}$. This is immediately seen by use of Eq. (36),

$$\begin{aligned} 0 &= \delta_{\text{orientation}} V_{0\ \text{static}} \\ &= D\int d^3x (\delta_{\text{orientation}} \vec{\lambda}_0 \cdot \vec{J}^0 + \delta_{\text{orientation}} \vec{b}_0^k \cdot \vec{J}^k_{\text{spin}}). \end{aligned} \tag{93}$$

It is important to note that although there is a dilational mode with zero eigenvalue

$$\begin{aligned} &\vec{x}^\mu_{\text{dil}} = \left(\frac{\partial\vec{\lambda}_0}{\partial\kappa}, \frac{\partial \vec{b}_0^j}{\partial\kappa}\right), \\ &\frac{\partial\lambda_0^a}{\partial\kappa} = \frac{x^a}{r}\left(\coth\kappa r - \frac{\kappa r}{\sinh^2\kappa r}\right) \xrightarrow[r\to\infty]{} \hat{x}^a, \\ &\frac{\partial b_0^{ai}}{\partial\kappa} = -\frac{\epsilon^{aij}x^j}{r}\frac{1}{\sinh\kappa r}(1-\kappa r\coth\kappa r) \xrightarrow[r\to\infty]{} 0, \end{aligned} \tag{94}$$

it is not normalizable, and so does not cause problems in solving Eq. (78).[26] A third potential problem is the fact that, by analogy with the results of Brown *et al.* in the pseudoparticle case, the vector propagator of Eq. (90) is likely to have divergences arising from the convolution integral in Eq. (92). However, in the pseudoparticle case these divergences are proportional to a sum over normalizable zero modes, and if (as seems likely) this structure holds in the case considered here, the orthogonality of $\vec{J}^\mu$ to the normalizable zero modes implies that the divergences will make no contribution to the static potential. My conclusion, then, is that the conditions imposed on the quark locations in the zeroth-order perturbation theory analysis are just the conditions needed for the existence of a well-defined order-g^2 correction. Writing

$$V_{\text{static}} = V_{0\ \text{static}} + g^2 V_{1\ \text{static}}, \tag{95}$$

Eqs. (31′), (78), and (79) give for the order-g^2 contribution

$$\delta V_{1\ \text{static}} = D\int d^3x(\delta\vec{\lambda}_1 \cdot \vec{J}^0 + \delta\vec{b}_1^k \cdot \vec{J}^k_{\text{spin}}), \tag{96a}$$

with $\vec{x}_1^\mu = (\vec{\lambda}_1, \vec{b}_1^j)$ given by

$$x_1^{a\mu}(x) = \int d^3y\, G^{a\mu,b\nu}(x,y)J^{b\nu}(y). \tag{96b}$$

Equation (96) contains both a q-$\overline{q}$ interaction contribution and q-q and $\overline{q}$-$\overline{q}$ self-interactions. The latter, by virtue of the δ's (which indicate variation with respect to shifts in quark location) have the Coulombic self-energy divergences automatically deleted. The self-interactions may well play an important role in confinement, since they contain the interaction back on the $q,\bar{q}$ of the distortions or polarizations of the bag induced by the $q,\bar{q}$. One final point is that when the order-g^2 q spin-$\bar{q}$ spin interaction is calculated, one expects it to have the same leading short distance ($x_q \to x_{\bar{q}}$) singularity as in the Abelian case, and so by familiar arguments[27] the potential has to be supplemented by a contact term of the usual form

$$V_{\text{contact}} = D\frac{g^2}{4\pi}\,\frac{8\pi}{3}\,\delta^3(x_q - x_{\bar{q}}) \times \left(\frac{1}{2m_q}\sigma^m_{(q)}\vec{Q}^{\text{eff}}_q\right)\cdot\left(\frac{1}{2m_q}\sigma^m_{(\bar{q})}\vec{Q}^{\text{eff}}_{\bar{q}}\right). \tag{97}$$

III. DISCUSSION

In conclusion I make some brief remarks:

(1) Obviously, the key open question about the theory of static quark forces developed above is whether, as conjectured, it leads to a quark-confining potential in order g^2. This is a concrete computational question which, I hope, will soon be settled. If the order-g^2 propagator does give confinement, then what are the limits of validity of the perturbation expansion? One obvious criterion is that the order-g^2 static potential must be small relative to the zeroth-order gluon field energy of the bag,

$$g^2 V_{1\ \text{static}} \ll (D/g^2)4\pi\kappa, \tag{98}$$

since when $g^2 V_{1\ \text{static}}$ is of the order of the zeroth-order energy, the bag is likely to become highly distorted.[14] When $g^2 V_{1\ \text{static}} \gtrsim (D/g^2)4\pi\kappa$, the bag will become unstable against tunneling into a state with two bags and a new light-$q\bar{q}$ pair. According to this picture, since κ^{-1} is the radius of the bag, κ should be identified with the Hagedorn energy $\simeq 100$ MeV; with $g^2/4\pi \sim 0.2$ and $\langle D\rangle_1 = \frac{3}{2}$ this would give a bag zeroth-order energy of order 750 MeV, which would not be unreasonable for the light-$q\bar{q}$ 35-plet central mass. Of course, the static formalism developed here is only expected to be quantitatively reliable for heavy-quark systems, such as charmonium; to do detailed calculations for light-quark systems will require a relativistic generalization of the static equations. In nonstatic situations, the collective coordinates associated with bag motions will also begin to play a dynamical role.

(2) In Eq. (49) and the work which follows, I made an arbitrary choice of sign $\vec{B}^j_0 = +\vec{E}^j_0$, while in fact zeroth-order solutions with $\vec{B}^j_0 = \pm\vec{E}^j_0$ are allowed. The sign choice here is equivalent to a choice between topological charge +1 or −1 for the zeroth-order solution. It may be that both solutions occur with equal probability, but it is also possible that in the charge-conjugation-asymmetric nine-gluon version of the theory used here, a definite sign choice is tied at some deeper level[28] to the choice of sign $d+if$ implicit in the outer product P. It is interesting to note in this connection that Manton[29] has shown that like-topological-charge, widely separated 't Hooft-Polyakov solutions are noninteracting, a result which may play a role in giving an algebraic chromodynamic theory of strong interactions the correct cluster decomposition properties.

(3) The generalization of the $q\bar{q}$ force calculation to the case qqq will require computation of the $N=3$ color-charge algebra, as discussed in I. It will also require the SU(3) analog of the Prasad-Sommerfield solution and the associated propagators, which perhaps can be obtained from the general SU(3) pseudoparticle or instanton solution[30] and associated propagators by an extension of the contour integration tranformations which I employ in Appendix A to discuss the SU(2) case. Analogous statements apply to the computation of static forces for $N>3$ quarks.

ADDED NOTES

(1) The equations of chromostatics given in Eq. (31′) possess a full local gauge invariance given by

$$\delta_{\text{gauge}}\,\vec{b}^j = D_j\vec{\phi}\,,$$

$$\delta_{\text{gauge}}\,\vec{V} = \vec{V}\times\vec{\phi}\,,\quad \vec{V} = \vec{\lambda}, \vec{B}^j, \vec{E}^j, \vec{J}^0, \vec{J}^k_{\text{spin}}\,.$$

(2) The zeroth-order spin potential of Eq. (51) is a parity-violating $\sigma^m_{(n)}x^m_n$ type potential which changes sign when the sign of $\vec{\lambda}_0$ in the zeroth-order solution is reversed. (The sign change arises because the minimization procedure requires the relative orientation of $\vec{Q}_n$ and $\vec{x}_n$ to reverse with $\vec{\lambda}_0$.) Hence Eq. (51) averages to zero when the quantum state is an equal superposition of the two signs of $\vec{\lambda}_0$, as would be expected for the end point of a parity-conserving tunneling process.

Note added in proof. Calculations just completed by R. Gonsalves, D. Neville, and P. Cvitanović show that the ansatz for the color-charge algebra made in I does not give an algebra with the needed trace property in the $(N_q, N_{\bar{q}}) = (3, 0)$ case. Hence while providing a useful model for discussing the two-particle color problem, as done in this paper, this ansatz does not provide the basis for a full color-charge theory. It now becomes important to study more general color-charge algebras, with the aim of finding a definition of color charges, outer product P, and inner product S, for which the Jacobi identity [Eq. (I.8)] and the trace property [Eq. (I.22)] hold over the color-charge algebra. [It is not necessary for the Jacobi identity to hold as a formal identity, as in the ansatz used in I; it need hold only over the algebra for the derivations of Sec. I of paper I to remain valid. The color-charge algebra constructed by Giles and McLerran is an example of an algebra (without the trace property) which satisfies the Jacobi identity over the algebra, but not as a formal identity for arbitrary noncommuting quantities.] While it is interesting to study the general-n, general-color state case, for physical applications it is of course only necessary that Eq. (I.8) and Eq. (I.22) hold in the $n=3$ case, with the trace restricted to its expectation $\langle S\rangle_1$ in color-singlet states. It may be that this specialization will be needed to get a workable recipe. Obviously, alteration of the color-charge algebra will in general change the values of $\langle D\rangle_1$ and $\vec{\mathbb{Q}}^{\text{eff}}_{q,\bar q}$ for the $q,\bar q$ case, but in all likelihood v ill not alter the SU(2) gauge structure on which the computations of this paper are based.

Let me suggest one natural generalization of the algebra constructed in I, continuing to assume that color transforms according to the fundamental n-dimensional representation of SU(n):

Color charges:

$$Q_q=\begin{pmatrix}Q_{q+}\\ Q_{q-}\\ Q_{q0}\end{pmatrix},\quad Q_{\bar q}=\begin{pmatrix}Q_{\bar q+}\\ Q_{\bar q-}\\ Q_{\bar q0}\end{pmatrix},$$

$$Q^A_{q+}=Q^A_{q-}=\tfrac{1}{2}\lambda^A,\quad Q^A_{\bar q+}=Q^A_{\bar q-}=-\tfrac{1}{2}\lambda^{*A},$$

$$A=1,\ldots,n^2-1$$

$$Q_{q0}=\alpha 1,\quad Q_{\bar q0}=-\bar\alpha 1\,.$$

Outer product:

$$u=\begin{pmatrix}u_+\\ u_-\\ u_0\end{pmatrix},\quad v=\begin{pmatrix}v_+\\ v_-\\ v_0\end{pmatrix},\quad w=\begin{pmatrix}w_+\\ w_-\\ w_0\end{pmatrix},$$

$$w=P(u,v)=-P(v,u),$$

$$w^A_\pm=\pm\beta_\pm([u_0,v^A_\pm]+[u^A_\pm,v_0])$$

$$\pm\gamma_\pm d^{ABC}[u^B_\pm,v^C_\pm]+\tfrac{1}{2}if^{ABC}\{u^B_\pm,v^C_\pm\}\,,$$

$$w_0=\delta[u_0,v_0]+\epsilon_+[u^A_+,v^A_+]-\epsilon_-[u^A_-,v^A_-]\,.$$

Inner product:

$$S(u,v)=S(v,u)$$

$$=\eta\tfrac{1}{2}\{u_0,v_0\}+\sigma_+\tfrac{1}{2}\{u^A_+,v^A_+\}+\sigma_-\tfrac{1}{2}\{u^A_-,v^A_-\}\,.$$

When $\alpha=\bar\alpha$, $\delta=0$, $\beta_+=\beta_-=\beta$, $\gamma_+=\gamma_-=\gamma$, $\epsilon_+=\epsilon_-=\epsilon$, $\sigma_+=\sigma_-=\sigma$, with all parameters real, this ansatz gives quark and antiquark algebras which are simply related by the natural charge-conjugation operation

$$u=\begin{pmatrix}u_+\\ u_-\\ u_0\end{pmatrix}\to u^C=\begin{pmatrix}-u^*_-\\ -u^*_+\\ -u_0\end{pmatrix}.$$

The ansatz evidently allows for the possibility that each color charge may contain two n^2-1 plets, composing under opposite relative rotational senses $\pm\gamma d^{ABC}+if^{ABC}$, and with a common shared 0 component. More complicated recipes can be constructed in a similar fashion.

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APPENDIX A: SCALAR PROPAGATOR CONSTRUCTION

I construct in this appendix the scalar propagator $\Delta^{ab}(x,y)$ defined in Eq. (81) of the text, which satisfies

$$D^\mu_x D^\mu_x\Delta^{ab}(x,y)=(D^0_xD^0_x+D^j_xD^j_x)\Delta^{ab}(x,y)$$

$$=-\delta^{ab}\delta^3(x-y),$$

$$D^0_x\vec{\mathrm{w}}(x)=\vec\lambda_0(x)\times\vec{\mathrm{w}}(x),$$

$$D^j_x\vec{\mathrm{w}}(x)=\left[\frac{\partial}{\partial x^j}+\vec{\mathrm{b}}^j_0(x)\times\right]\vec{\mathrm{w}}(x)\,, \tag{A1}$$

$$\lambda_0^a(x) = \frac{x^a}{r^2}(1 - r\coth r),$$

$$b_0^{ai}(x) = \frac{\epsilon^{aij}x^j}{r^2}\left(1 - \frac{r}{\sinh r}\right),$$

where I have set $\kappa = 1$. To change to general κ one simply uses the scaling law

$$\Delta^{ab}(x,y)_{\text{general}\kappa} \equiv \Delta^{ab}(x,y;\kappa) = \kappa\Delta^{ab}(\kappa x, \kappa y; 1). \tag{A2}$$

The reversal in sign of $\vec{\lambda}_0$ as compared with Eq. (49) [which I have made because the sign in Eq. (A1) corresponds to the convention I used in my calculations] has no effect on Δ^{ab}, since $D_x^\mu D_x^\mu$ is even in $\vec{\lambda}_0$. As in the vector propagator calculation in the text, I make extensive use of the results of Brown *et al.*[20] for propagators in pseudoparticle fields. The first step of the calculation, following Manton,[31] is to make a complex gauge transformation which changes the potentials from λ_0^a, b_0^{ai} of Eq. (A1) to $\bar{\lambda}_0^a, \bar{b}_0^{ai}$, with

$$\bar{\lambda}_0^a = \lambda_0^a = \frac{x^a}{r^2}(1 - r\coth r),$$

$$\bar{b}_0^{ai} = \frac{\epsilon^{aij}x^j}{r^2}(1 - r\coth r) + i\delta^{ia}. \tag{A3}$$

Introducing a matrix $M^{a\bar{a}}(x)$ given by

$$M^{a\bar{a}}(x) = \cosh r\left(\delta^{a\bar{a}} - \frac{x^a x^{\bar{a}}}{r^2}\right) - i\sinh r\,\epsilon^{a\bar{a}l}\frac{x^l}{r} + \frac{x^a x^{\bar{a}}}{r^2}, \tag{A4}$$

$$M^{a\bar{a}}(x)M^{b\bar{a}}(x) = \delta^{ab},$$

it is straightforward to verify that

$$\frac{\partial}{\partial x^j}M^{a\bar{a}}(x) = -\epsilon^{abc}b_0^{bj}M^{c\bar{a}} + M^{a\bar{e}}\epsilon^{\bar{e}\bar{b}\bar{a}}\bar{b}_0^{\bar{b}j}, \tag{A5}$$

which implies that

$$D_x^j M^{a\bar{a}}(x) = M^{a\bar{a}}(x)\bar{D}_x^j. \tag{A6}$$

So once we have obtained the scalar propagator $\bar{\Delta}^{\bar{a}\bar{b}}(x)$ satisfying

$$\bar{D}_x^\mu \bar{D}_x^\mu \bar{\Delta}^{\bar{a}\bar{b}}(x,y) = -\delta^{\bar{a}\bar{b}}\delta^3(x-y), \tag{A7}$$

we get the desired propagator Δ^{ab} by transforming both SU(2) indices with the matrix M,

$$\Delta^{ab}(x,y) = M^{a\bar{a}}(x)M^{b\bar{b}}(y)\bar{\Delta}^{\bar{a}\bar{b}}(x,y). \tag{A8}$$

From this point on I will work exclusively with the gauge-transformed potentials of Eq. (A3). For notational convenience I will drop all bars, but it should be kept in mind that I am now constructing the propagator $\bar{\Delta}$ in the new gauge, not the final propagator Δ given by Eq. (A8). The advantage of the potentials of Eq. (A3) is that they take the form used by Brown *et al.* as the starting point for their analysis,

$$A^{a\mu}(x) = (\lambda_0^a, b_0^{ai}) = -\eta^{(-)\mu\nu a}\partial^\nu \ln\pi(x),$$

$$\pi(x) = e^{ix^0}\frac{\sinh r}{r}, \quad \partial^0 = \frac{\partial}{\partial x^0}, \quad \partial^j = \frac{\partial}{\partial x^j}, \tag{A9}$$

$$\eta^{(-)\mu\nu a} = -\eta^{(-)\nu\mu a}, \quad \eta^{(-)kla} = \epsilon^{kla}, \quad \eta^{(-)k0a} = -\delta^{ka}.$$

Note that although $\pi(x)$ depends on x^0, the potential $A^{a\mu}(x)$ depends only on the spatial components x^j of x. Hence if I define a Euclidean time-dependent propagator $\Delta^{ab}(\vec{x},\vec{y},x^0,y^0)$ by

$$D_x^\mu D_x^\mu \Delta^{ab}(\vec{x},\vec{y},x^0,y^0) = -\delta^{ab}\delta^4(x-y),$$

$$D_x^0\vec{w}(x) = \left[\frac{\partial}{\partial x^0} + \vec{\lambda}_0(x)\times\right]\vec{w}(x), \tag{A10}$$

then it will actually depend only on the time difference $\lambda = x^0 - y^0$, and the desired propagator $\Delta^{ab}(\vec{x},\vec{y})$ is obtained by integrating over the time difference,

$$\Delta^{ab}(\vec{x},\vec{y}) = \int_{-\infty}^{\infty} d\lambda\,\Delta^{ab}(\vec{x},\vec{y},\lambda). \tag{A11}$$

The final observation needed, in order to make contact with the work of Brown *et al.*, is that $\pi(x)$ can be written as a contour integral,

$$\pi(x) = \frac{e^{ix_0}\sinh r}{r} = -\frac{1}{2\pi}\int_{-\infty-iK}^{\infty-iK} ds\, e^{is}\frac{1}{r^2 + (x^0 - s)^2}, \quad K > r. \tag{A12}$$

I now will list a number of results from the analysis of Brown *et al.*, with occasional small changes in notation. Brown *et al.* construct the general scalar, isovector propagator $\Delta^{ab}(x,y,x^0,y^0)$ satisfying Eq. (A10) for potentials $A^{a\mu}(x) = -\eta^{(-)\mu\nu a}\partial^\nu \ln\pi(x)$ representing a general N-pseudoparticle (instanton) configuration

$$\pi(x) = (1) + \sum_s \frac{\rho_s^2}{x_s^2}, \quad x_s \equiv x - z_s. \tag{A13}$$

Their result takes the form of a sum of two pieces (with x, y Euclidean four-vectors)

$$\Delta^{ab}(x,y)=\Delta^{ab}(x,y)^{(1)}+\Delta^{ab}(x,y)^{(2)}. \tag{A14}$$

The first piece is constructed in terms of spin-$\frac{1}{2}$ propagators by the recipe

$$\begin{aligned}
\Delta^{ab}(x,y)^{(1)} &= \frac{U^{ab}(x,y)}{4\pi^2(x-y)^2\pi(x)\pi(y)}\,,\\
U^{ab}(x,y) &= \tfrac{1}{2}\operatorname{tr}[\tau^a F^{(+)}(x,y)\tau^b F^{(+)}(y,x)],\\
F^{(+)}(x,y) &= (1)+\sum_s \rho_s^{\,2}\,\frac{\tau\cdot x_s}{x_s^{\,2}}\,\frac{\tau^\dagger\cdot y_s}{y_s^{\,2}}\,,
\end{aligned} \tag{A15}$$

$\tau^\mu=(i,\tau^j)$, $\tau^{\dagger\mu}=(-i,\tau^j)$,

$\tau^j=\mathrm{SU}(2)$ Pauli matrices, $x_s=x-z_s$, $y_s=y-z_s$.

The second piece has the form

$$\begin{aligned}
\Delta^{ab}(x,y)^{(2)} &= \frac{C_{ab}(x,y)}{4\pi^2\pi(x)\pi(y)}\,,\\
C_{ab}(x,y) &= \sum_{r,s,u,v}\Phi^{(+)}_{rsa}(x)C_{rs,uv}\Phi^{(+)}_{uvb}(y)\,,\\
\Phi^{(+)}_{rsa}(x) &= \frac{\rho_r\rho_s}{x_r^{\,2}x_s^{\,2}}\,\eta^{(-)\mu\nu a}x_r^\mu x_s^\nu\,,\\
C_{rs,uv} &= \frac{\delta_{ru}\delta_{sv}-\delta_{rv}\delta_{su}}{(z_r-z_s)^2}-\frac{1}{(z_r-z_s)^2}\\
&\quad\times\left(\frac{\rho_r\rho_u}{\rho_s\rho_v}h_{sv}-\frac{\rho_r\rho_v}{\rho_s\rho_u}h_{su}-\frac{\rho_s\rho_u}{\rho_r\rho_v}h_{rv}+\frac{\rho_s\rho_v}{\rho_r\rho_u}h_{ru}\right)\frac{1}{(z_u-z_v)^2}\,,
\end{aligned} \tag{A16}$$

and with the numbers[32] h_{sv} determined by the matrix inversion problem

$$\sum_t \frac{g_{st}h_{tv}}{\rho_t\rho_v}=\delta_{sv}\,, \tag{A17a}$$

$$g_{st}=\left[(1)+\sum_{r\neq s}\frac{\rho_r^{\,2}}{(z_r-z_s)^2}\right]\delta_{st}-\frac{\rho_s\rho_t}{(z_s-z_t)^2}\,(1-\delta_{st})\,. \tag{A17b}$$

To make use of these rather formidable looking equations, I note that Eq. (A13) becomes identical to Eq. (A12) under the substitutions

$$\begin{aligned}
&(1)\to 0\,,\\
&z_s\to(s,0),\quad x_s^{\,2}\to(x^0-s)^2+\vec{x}^2\,,\\
&\sum_s\to\int ds,\quad \rho_s\to-(1/2\pi)e^{is}\,,
\end{aligned} \tag{A18}$$

so that the Sommerfield-Prasad solution is in effect a continuum of complex instantons. Corresponding to the substitution $(1)\to 0$, the terms (1) in Eqs. (A15) and (A17) must also be deleted. The transition from sums to integrals can be made with no ambiguity in $\Delta^{ab}(x,y)^{(1)}$, giving (recall that $\lambda=x^0-y^0$)

$$\begin{aligned}
\Delta^{ab}(\vec{x},\vec{y})^{(1)} &\equiv \int_{-\infty}^{\infty}d\lambda\,\Delta^{ab}(\vec{x},\vec{y},\lambda)^{(1)}\\
&=\int_{-\infty}^{\infty}d\lambda\,\frac{1}{4\pi^2[(\vec{x}-\vec{y})^2+\lambda^2]}\,\frac{|\vec{x}|}{\sinh|\vec{x}|}\,\frac{|\vec{y}|}{\sinh|\vec{y}|}\,e^{-ix^0-iy^0}\,\tfrac{1}{2}\operatorname{tr}[\tau^aF^{(+)}(x,y)\tau^bF^{(+)}(y,x)]\,,\\
F^{(+)}(x,y) &= -\frac{1}{2\pi}\int ds\,e^{is}\,\frac{\vec{\tau}\cdot\vec{x}+i(x^0-s)}{\vec{x}^2+(x^0-s)^2}\,\frac{\vec{\tau}\cdot\vec{y}-i(y^0-s)}{\vec{y}^2+(y^0-s)^2}\,.
\end{aligned} \tag{A19}$$

Making a shift $s\to s+x^0$ in the integration over s in $F^{(+)}(x,y)$ and a shift $t\to t+x^0$ in the corresponding integration over t in $F^{(+)}(y,x)$, gives

$$\begin{aligned}
\Delta^{ab}(\mathrm{x},\mathrm{y})^{(1)} &= \frac{1}{(2\pi)^4}\,\frac{|\vec{x}|}{\sinh|\vec{x}|}\,\frac{|\vec{y}|}{\sinh|\vec{y}|}\int ds\,e^{is}\int dt\,e^{it}\int_{-\infty}^{\infty}\frac{d\lambda\,e^{i\lambda}}{(\vec{x}-\vec{y})^2+\lambda^2}\\
&\quad\times\tfrac{1}{2}\operatorname{tr}\left[\frac{\vec{\tau}\cdot\vec{x}+it}{\vec{x}^2+t^2}\,\tau^a\,\frac{\vec{\tau}\cdot\vec{x}-is}{\vec{x}^2+s^2}\,\frac{\vec{\tau}\cdot\vec{y}+i(s+\lambda)}{\vec{y}^2+(s+\lambda)^2}\,\tau^b\,\frac{\vec{\tau}\cdot\vec{y}-i(t+\lambda)}{\vec{y}^2+(t+\lambda)^2}\right].
\end{aligned} \tag{A20}$$

Now make, in the order indicated, the following changes of variables:

$$\begin{aligned}
&\text{(i)}\quad \lambda\to z-\tfrac{1}{2}(s+t)=z-w\,,\\
&\text{(ii)}\quad w=\tfrac{1}{2}(s+t),\quad v=\tfrac{1}{2}(s-t),\quad ds\,dt=2\,dw\,dv\,.
\end{aligned} \tag{A21}$$

This gives as the final result the following symmetrical-looking formula:

$$\Delta^{ab}(x,y)^{(1)}=\frac{2}{(2\pi)^4}\frac{|\vec{x}|}{\sinh|\vec{x}|}\frac{|\vec{y}|}{\sinh|\vec{y}|}\int_{-\infty}^{\infty}dv\int_{-\infty-iK}^{\infty-iK}dw\int_{-\infty-iK}^{\infty-iK}dz\frac{e^{iw}e^{iz}}{(\vec{x}-\vec{y})^2+(z-w)^2}$$
$$\times\tfrac{1}{2}\mathrm{tr}\left[\frac{\vec{\tau}\cdot\vec{x}+i(w-v)}{\vec{x}^2+(w-v)^2}\tau^a\frac{\vec{\tau}\cdot\vec{x}-i(w+v)}{\vec{x}^2+(w+v)^2}\frac{\vec{\tau}\cdot\vec{y}+i(z+v)}{\vec{y}^2+(z+v)^2}\tau^b\frac{\vec{\tau}\cdot\vec{y}-i(z-v)}{\vec{y}^2+(z-v)^2}\right]. \tag{A22}$$

Turning next to the second piece, I note that time-translation invariance implies that $h_{sv}=h(s-v)$. Anticipating the fact that only $H=\sum_v h(s-v)$ is needed, I proceed first to extract this quantity from the matrix inversion problem of Brown *et al.* stated in Eq. (A17). Because the expressions of Eqs. (A16) and (A17) contain singular factors $(z_u-z_v)^{-2}$, etc., it is necessary to separate the various integration contours r,s,u,v by small imaginary displacements. In order to do this in a way which preserves the validity of various algebraic operations used by Brown *et al.* in getting their solution,[33] it is necessary to symmetrize over all possible "stacking orders" of the contours on the complex plane, a procedure which will eventually lead to the appearance of principal-value integrals in the answer. Summing over v in Eq. (A17a) gives

$$\left[\sum_v h(t-v)\right]\left[\sum_t \frac{g_{st}}{\rho_t}\right]=\sum_v \rho_v\delta_{sv}=\rho_s . \tag{A23}$$

Dropping the (1) in the expression for g_{st} in Eq. (A17b) gives

$$\sum_t \frac{g_{st}}{\rho_t}=\rho_s\sum_{r\neq s}\frac{\rho_r^2/\rho_s^2-1}{(z_r-z_s)^2}, \tag{A24}$$

so that

$$H=\sum_v h(t-v)=\left[\sum_{r\neq s}\frac{\rho_r^2/\rho_s^2-1}{(z_r-z_s)^2}\right]^{-1}. \tag{A25}$$

As a consistency check, note that if we multiply Eq. (A23) by ρ_s and sum, we get

$$H\sum_{s,t}\frac{g_{st}}{\rho_t}\rho_s=H\sum_{r\neq s}\frac{\rho_r^2-\rho_s^2}{(z_r-z_s)^2}=0=\sum_s\rho_s^2, \tag{A26}$$

but in the continuum limit

$$\sum_s\rho_s^2\to-\frac{1}{2\pi}\int_{-\infty-iK}^{\infty-iK}ds\,e^{is}=0, \tag{A27}$$

so that Eq. (A26) is in fact satisfied. Passing to the limit in Eq. (A25), and remembering that we must average over the cases where the r contour goes over and under s, we get

$$H=\left[\mathrm{P}\int_{-\infty-iK}^{\infty-iK}dr\frac{e^{i(r-s)}-1}{(r-s)^2}\right]^{-1}=-\frac{1}{\pi}. \tag{A28}$$

The final step in the calculation is to make the transition froms sums to integrals in Eq. (A16), bearing in mind the necessity of symmetrizing over the ordering of integration contours. Noting that

$$\eta^{(-)\mu\nu a}x_{r\mu}x_{s\nu}=-x^a(r-s), \tag{A29}$$

we get from Eq. (A16) (again with $\lambda=x^0-y^0$)

$$\Delta^{ab}(\vec{x},\vec{y})^{(2)}\equiv\int_{-\infty}^{\infty}d\lambda\,\Delta^{ab}(\vec{x},\vec{y},\lambda)^{(2)}$$
$$=\frac{1}{(2\pi)^2}\frac{|\vec{x}|}{\sinh|\vec{x}|}\frac{|\vec{y}|}{\sinh|\vec{y}|}x^ay^b$$
$$\times\int_{-\infty}^{\infty}d\lambda\,e^{-i(x^0+y^0)}\left\{\frac{2}{(2\pi)^2}\int dr\,e^{ir}\int ds\,e^{is}\frac{1}{[\vec{x}^2+(x^0-r)^2][\vec{x}^2+(x^0-s)^2][\vec{y}^2+(y^0-r)^2][\vec{y}^2+(y^0-s)^2]}\right.$$
$$-\frac{4}{(2\pi)^2}\int dr\,e^{ir}\int du\,e^{iu}\int\frac{ds\,dv\,h(s-v)}{(r-s)(u-v)}$$
$$\left.\times\frac{1}{[\vec{x}^2+(x^0-r)^2][\vec{x}^2+(x^0-s)^2][\vec{y}^2+(y^0-u)^2][\vec{y}^2+(y^0-v)^2]}\right\}. \tag{A30}$$

Again it is necessary to make, in the order indicated, the following changes of variables:

First term in { }:

(i) $r \to r + x^0, \quad s \to s + x^0$,

(ii) $\lambda \to z - \frac{1}{2}(r+s) = z - w$, (A31a)

(iii) $w = \frac{1}{2}(r+s), \quad v = \frac{1}{2}(r-s), \quad drds = 2\,dwdv$.

Second term in { }:

(i) $r \to r + x^0, \quad s \to s + x^0, \quad u \to u + x^0, \quad v \to v + x^0$,

(ii) $\lambda \to z_1 - \frac{1}{2}(u+v)$, (A31b)

(iii) $z_2 = \frac{1}{2}(r+s), \quad w = \frac{1}{2}(u+v), \quad v_2 = \frac{1}{2}(r-s), \quad v_1 = \frac{1}{2}(u-v),$

$drds = 2dz_2dv_2, \quad dudv = 2dwdv_1$.

After these transformations, the only place where w appears is in

$$\int dw\, h(z_2 - v_2 + v_1 - w) = -1/\pi \ , \tag{A32}$$

giving as the final answer

$$\begin{aligned}\Delta^{ab}(\vec{x},\vec{y})^{(2)} = &\frac{4}{(2\pi)^4}\frac{|\vec{x}|}{\sinh|\vec{x}|}\frac{|\vec{y}|}{\sinh|\vec{y}|}x^a y^b \\ &\times\Bigg\{\int_{-\infty}^{\infty} dv \int_{-\infty - iK}^{\infty - iK}\frac{dw\, e^{iw}}{[\vec{x}^2+(w-v)^2][\vec{x}^2+(w+v)^2]}\int_{-\infty-iK}^{\infty-iK}\frac{dz\, e^{iz}}{[\vec{y}^2+(z-v)^2][\vec{y}^2+(z+v)^2]} \\ &+\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} dv_2\frac{e^{iv_2}}{v_2}\int_{-\infty-iK}^{\infty-iK}\frac{dz_2\, e^{iz_2}}{[\vec{x}^2+(z_2-v_2)^2][\vec{x}^2+(z_2+v_2)^2]} \\ &\times \mathrm{P}\int_{-\infty}^{\infty} dv_1\frac{e^{iv_1}}{v_1}\int_{-\infty-iK}^{\infty-iK}\frac{dz_1\, e^{iz_1}}{[\vec{y}^2+(z_1-v_1)^2][\vec{y}^2+(z_1+v_1)^2]}\Bigg\}.\end{aligned} \tag{A33}$$

Although it took a more involved argument to extract Eq. (A33) from the work of Brown *et al.* than was needed to get Eq. (A22), the evaluation of the contour integrals appearing in Eq. (A33) is relatively easy. Writing $x = |\vec{x}|$, $y = |\vec{y}|$, the answer is

$$\begin{aligned}\Delta^{ab}(\vec{x},\vec{y})^{(2)} = &\frac{1}{4\pi}\frac{x^a}{\sinh x}\frac{y^b}{\sinh y} \\ &\times\Bigg\{\frac{1}{xy}\bigg[\cosh x\cosh y - \frac{1}{2}\left(\frac{\sinh x}{x}\cosh y + \frac{\sinh y}{y}\cosh x\right)\bigg] \\ &+\frac{1}{4}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{\sinh(x-y)}{x-y} - \frac{\sinh(x+y)}{x+y}\right) - \frac{1}{xy}\left(\cosh x - \frac{\sinh x}{x}\right)\left(\cosh y - \frac{\sinh y}{y}\right)\Bigg\} \\ = &\frac{1}{4\pi}\frac{x^a}{\sinh x}\frac{y^b}{\sinh y}\Bigg\{\frac{1}{2xy}\left(\frac{\sinh x}{x}\cosh y + \frac{\sinh y}{y}\cosh x\right) - \frac{\sinh x}{x^2}\frac{\sinh y}{y^2} \\ &+\frac{1}{4}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\bigg[\frac{\sinh(x-y)}{x-y} - \frac{\sinh(x+y)}{x+y}\bigg]\Bigg\}.\end{aligned} \tag{A34b}$$

The fact that the final term in Eq. (A34a), which comes from the product of principal-value integrals in Eq. (A33), cancels away the leading large-y asymptotic behavior of the first three terms is a check that the limiting argument leading to Eq. (A28) has been carried out correctly. The evaluation of $\Delta^{ab}(\vec{x},\vec{y})^{(1)}$, in which $\vec{x}$ and $\vec{y}$ dependences are highly correlated, involves straightforward but very lengthy computations, on which I am now working.

APPENDIX B: "STATIC" EQUATIONS IN THE ABELIAN HIGGS MODEL AND NON-ABELIAN GENERALIZATIONS

I examine in this appendix the question of what are the correct "static" equations in a simple and relatively familiar case, the classical Abelian Higgs model,[34] and then discuss non-Abelian generalizations. The Lagrangian density for this model is

$$\mathcal{L} = -\tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} + d_\mu d^{*\mu} - \tfrac{1}{2} C(|\phi|^2 - \kappa^2)^2 ,$$

$$f_{\mu\nu} = \frac{\partial b_\mu}{\partial x^\nu} - \frac{\partial b_\nu}{\partial x^\mu} , \qquad \text{(B1)}$$

$$d_\mu = \left(\frac{\partial}{\partial x^\mu} - ieb_\mu\right)\phi ,$$

with ϕ a complex scalar field. I study the model in the special case in which only static source charges are present, so that one can choose a gauge in which $b_i = 0$ and in which ϕ is real. With this specialization, the Euler-Lagrange equations and the total field energy are

$$\nabla^2 b_0 = 2e^2 b_0 \phi^2 - e \sum_n Q_{(n)} \delta^3(x - x_n) ,$$

$$\nabla^2 \phi - \ddot{\phi} = -e^2 b_0^2 \phi + C\phi(\phi^2 - \kappa^2) , \qquad \text{(B2)}$$

$$E_{\text{field}} = \int d^3x [\tfrac{1}{2}(\vec{\nabla} b_0)^2 + (\vec{\nabla}\phi)^2 + \dot{\phi}^2 + e^2 b_0^2 \phi^2 + \tfrac{1}{2} C(\phi^2 - \kappa^2)^2] , \qquad \text{(B3)}$$

and it is easily checked, using the equations of motion, that

$$dE_{\text{field}}/dt = 0 \qquad \text{(B4)}$$

for stationary, time-independent source charges $Q_{(n)}$. Note that even though no explicit coupling to the static source charges appears in Eq. (B1), the source charge term in Eq. (B2) arises from the contribution to the variation of the action of surface integrals over small spheres excluding the field singularities. To see this, let $\int' d^3x$ denote a volume integral over the region outside the small spheres; then

$$\delta \int' d^4x \tfrac{1}{2}(\vec{\nabla} b_0)^2 = \int dt \int' d^3x \vec{\nabla} b_0 \cdot \vec{\nabla} \delta b_0 = -\int' d^4x\, \delta b_0 \nabla^2 b_0 + \int dt \sum_n \int d\vec{S}_n \cdot \vec{\nabla} b_0 \delta b_0$$
$$= -\int' d^4x\, \delta b_0 \nabla^2 b_0 - \int dt \sum_n e Q_{(n)} \delta b_0(x_n)$$
$$\text{“=”} -\int d^4x\, \delta b_0 \left[\nabla^2 b_0 + \sum_n e Q_{(n)} \delta^3(x - x_n)\right] . \qquad \text{(B2')}$$

I will use the volume δ-function formulation, rather than the excluded spheres procedure, from here on. Note that the derivation of Eqs. (B5) and (B6) below can be equally well carried out by the excluded spheres procedure, while as noted in Ref. 9, the surface integrals over excluded spheres make contributions to the derivations of Eqs. (B8)–(B13) below which vanish, by virtue of the variational equations obtained by equating the volume integrals of the variations to zero. Let me now examine the question of what conditions are imposed by requiring that the principle of virtual work be satisfied. Making small virtual displacements in the charges and the fields, we have

$$\delta E_{\text{field}} = \int d^3x [\vec{\nabla} b_0 \cdot \vec{\nabla} \delta b_0 + 2\vec{\nabla}\phi \cdot \vec{\nabla}\delta\phi + 2\dot{\phi}\delta\dot{\phi} + 2e^2 b_0 \delta b_0 \phi^2 + 2e^2 b_0^2 \phi \delta\phi + 2C\phi\delta\phi(\phi^2 - \kappa^2)]$$
$$= \int d^3x \{\delta b_0(-\nabla^2 b_0 + 2e^2 b_0 \phi^2) + 2\delta\dot{\phi}\dot{\phi} + \delta\phi[-2\nabla^2\phi + 2e^2 b_0^2 \phi + 2C\phi(\phi^2 - \kappa^2)]\} , \qquad \text{(B5a)}$$

which must be equal to

$$\delta V_{\text{static}} = \sum_n e \delta b_0(x_n) Q_{(n)} = \int d^3x\, \delta b_0(x) \left[e \sum_n Q_{(n)} \delta^3(x - x_n)\right] \qquad \text{(B5b)}$$

if the principle of virtual work is to be satisfied. Taking the particular time slice $t = 0$ to be the one on which ϕ and $\dot{\phi}$ are independent Cauchy data, equating Eq. (B5) to Eq. (B6) for arbitrary variations $\delta\phi$, δb_0, $\delta\dot{\phi}$ gives

$$t=0:\ \nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n),\quad \nabla^2\phi = e^2 b_0^{\,2}\phi + C\phi(\phi^2-\kappa^2)\ ,\quad \dot{\phi}=0 \text{ (virtual work).} \tag{B6}$$

These equations are not the same as the specialization of the Euler-Lagrange equations to the case of vanishing time derivatives, which gives for all times

$$\nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n),\quad \nabla^2\phi = -e^2 b_0^{\,2}\phi + C\phi(\phi^2-\kappa^2),\quad \mathring{\phi}=0 \text{ (Euler-Lagrange).} \tag{B7}$$

The difference between Eqs. (B6) and Eqs. (B7) is just a change in sign in the interaction term in the ϕ equation, analogous to the change in sign of the $P(\lambda, D_j\lambda)$ term in the "curl B" equation found in the text in the case of chromodynamics.

It is easy to show that Eqs. (B6) and (B7) are obtained from variational principles on a fixed time slice, with all Cauchy data treated as independent. Consider first

$$\begin{aligned} 0 &= \delta L_{\text{constrained}} \\ &= \delta \int d^3x \left\{ \tfrac{1}{2}(\vec{\nabla} b_0)^2 - (\vec{\nabla}\phi)^2 + \mathring{\phi}^2 + e b_0^{\,2}\phi^2 - \tfrac{1}{2}C(\phi^2-\kappa^2)^2 + \lambda\left[\nabla^2 b_0 - 2e^2 b_0\phi^2 + e\sum_n Q_{(n)}\delta^3(x-x_n)\right]\right\}\ , \end{aligned} \tag{B8}$$

which implies the equations

$$\begin{aligned} &\dot{\phi} = 0\ , \\ &\nabla^2\lambda - 2e^2\lambda\phi^2 = \nabla^2 b_0 - 2e^2 b_0\phi^2\ , \\ &\nabla^2\phi = -e^2 b_0^{\,2}\phi + 2e^2 b_0\phi\lambda + C\phi(\phi^2-\kappa^2)\ , \\ &\nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n)\ . \end{aligned} \tag{B9}$$

Since the second equation implies $b_0=\lambda$, the final two equations become

$$\begin{aligned} &\nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n)\ , \\ &\nabla^2\phi = e^2 b_0^{\,2}\phi^2 + C\phi(\phi^2-\kappa^2)\ , \end{aligned} \tag{B10}$$

which are the equations of Eq. (B6) which satisfy the principle of virtual work. Consider next

$$\begin{aligned} 0 &= \delta E_{\text{field, constrained}} \\ &= \delta \int d^3x \left\{ \tfrac{1}{2}(\vec{\nabla} b_0)^2 + (\vec{\nabla}\phi)^2 + \mathring{\phi}^2 + e^2 b_0^{\,2}\phi^2 + \tfrac{1}{2}C(\phi^2-\kappa^2)^2 + \lambda\left[\nabla^2 b_0 - 2e^2 b_0\phi^2 + e\sum_n Q_{(n)}\delta^3(x-x_n)\right]\right\}\ , \end{aligned} \tag{B11}$$

which implies the equations

$$\begin{aligned} &\dot{\phi} = 0\ , \\ &\nabla^2\lambda - 2e^2\lambda\phi^2 = \nabla^2 b_0 - 2e^2 b_0\phi^2\ , \\ &\nabla^2\phi = e^2 b_0^{\,2}\phi - 2e^2 b_0\phi\lambda + C\phi(\phi^2-\kappa^2)\ , \\ &\nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n)\ . \end{aligned} \tag{B12}$$

Again the second equation implies $b_0=\lambda$, and so the final two equations give

$$\begin{aligned} &\nabla^2 b_0 = 2e^2 b_0\phi^2 - e\sum_n Q_{(n)}\delta^3(x-x_n)\ , \\ &\nabla^2\phi = -e^2 b_0^{\,2}\phi + C\phi(\phi^2-\kappa^2)\ , \end{aligned} \tag{B13}$$

which are Eqs. (B17), the specialization of the Euler-Lagrange equations to vanishing time derivatives. Thus, the solution to Eqs. (B7) gives $t=0$ Cauchy data which yield an absolute minimum of E_{field} for a specified source charge distribution, which is why this set of Cauchy data propagates without change in time. The equations which satisfy the principle of virtual work do not give an absolute minimum of the field energy for a given charge distribution.

Let me next give a simple argument which shows that the virtual work equations of Eq. (B6) always have a solution. To do this, I explicitly separate off the Coulombic piece of the potential by writing

$$b_0=\sum_n \frac{eQ_{(n)}}{4\pi|x-x_n|}+c_0 \tag{B14}$$

and define a subtracted energy $E_{\text{field,sub}}$ in which the infinite Coulomb self-energies of the charges have been removed, giving

$$E_{\text{field,sub}}=\tfrac{1}{2}\sum_{m\neq n}\frac{e^2Q_{(n)}Q_{(m)}}{4\pi|x_n-x_m|}+\sum_n eQ_{(n)}c_0(x_n)+\int d^3x[\tfrac{1}{2}(\vec{\nabla}c_0)^2+(\vec{\nabla}\phi)^2+\mathring{\phi}^2+e^2b_0^{\,2}\phi^2+\tfrac{1}{2}C(\phi^2-\kappa^2)^2]\,. \tag{B15}$$

Now define a new functional $\tilde{\mathfrak{F}}$ by

$$\tilde{\mathfrak{F}}=E_{\text{field,sub}}-\tfrac{1}{2}\sum_{m\neq n}\frac{e^2Q_{(n)}Q_{(m)}}{4\pi|x_n-x_m|}-\sum_n eQ_{(n)}c_0(x_n)=E_{\text{field,sub}}-V_{\text{static}}$$

$$=\int d^3x[\tfrac{1}{2}(\vec{\nabla}c_0)^2+(\vec{\nabla}\phi)^2+\mathring{\phi}^2+e^2b_0^{\,2}\phi^2+\tfrac{1}{2}C(\phi^2-\kappa^2)^2]\,. \tag{B16}$$

This new functional is of course no longer the field energy, but it is positive-definite, and there is a well-defined class $\mathfrak{C}$ of functions $c_0, \phi, \mathring{\phi}$ for which $\tilde{\mathfrak{F}}$ is finite. Hence the minimum of $\tilde{\mathfrak{F}}$ over the class $\mathfrak{C}$ exists, which implies that the equations of virtual work (which are the variational equations for $\tilde{\mathfrak{F}}$) have a solution.

In order to get further insight into the behavior of the two systems of equations, Eq. (B6) and Eq. (B7), I consider now the case when only one source charge Q is present, located at $x=0$. Making the separation of Eq. (B14) and assuming spherical symmetry, the equations become

$$\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_0}{dr}\right)=2e^2\left(c_0+\frac{eQ}{4\pi r}\right)\phi^2\,,\\&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)=e^2\left(c_0+\frac{eQ}{4\pi r}\right)^2\phi+C\phi(\phi^2-\kappa^2)\quad\text{(virtual work)},\end{aligned} \tag{B6$'$}$$

$$\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_0}{dr}\right)=2e^2\left(c_0+\frac{eQ}{4\pi r}\right)\phi^2\,,\\&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)=-e^2\left(c_0+\frac{eQ}{4\pi r}\right)^2\phi+C\phi(\phi^2-\kappa^2)\quad\text{(Euler-Lagrange)}\,.\end{aligned} \tag{B7$'$}$$

Analyzing the indicial equations for c_0 and ϕ around $r=0$ gives

$$c_0\sim\text{const}\quad(\text{for }\lambda>-\tfrac{1}{2})\,,\qquad \phi\sim\hat{\phi}r^\lambda,\quad \lambda=\lambda_\pm\,, \tag{B17a}$$

$$\text{Virtual work:}\quad \lambda_+=-\tfrac{1}{2}+[\tfrac{1}{4}+(e^2Q/4\pi)^2]^{1/2}>0,\quad \lambda_-<-\tfrac{1}{2}, \tag{B17b}$$

$$\text{Euler-Lagrange:}\quad \lambda_+=-\tfrac{1}{2}+[\tfrac{1}{4}-(e^2Q/4\pi)^2]^{-1/2},\quad -\tfrac{1}{2}<\lambda_+<0\quad [\text{for }\tfrac{1}{4}>(e^2Q/4\pi)^2]\quad \lambda_-<-\tfrac{1}{2}. \tag{B17c}$$

Assuming that $E_{\text{field,sub}}$ or equivalently $\tilde{\mathfrak{F}}$ is bounded, then the solutions with $\lambda=\lambda_-$ are excluded. Boundedness of these functionals also implies that $\phi\to\kappa$ at infinity and that b_0 is bounded, giving as $r\to\infty$ the asymptotic behavior

$$\begin{aligned}&\phi\approx\kappa+\frac{\hat{\phi}}{r}\exp[-r(2C\kappa^2)^{1/2}]\,,\\&b_0=c_0+\frac{eQ}{4\pi r}\approx\frac{\hat{b}_0}{r}\exp[-r(2e^2\kappa^2)^{1/2}]\,.\end{aligned} \tag{B18}$$

The argument of Eqs. (B14)–(B16) implies that a solution of Eqs. (B6) exists which interpolates between the $r=0$ behavior of Eqs. (B17a) and (B17b) with $\lambda=\lambda_+$, and the $r=\infty$ behavior of Eq. (B18). I do not at this point have an analogous existence proof for Eqs. (B7).

To summarize the above analysis, in the Abelian Higgs model with an inserted charge, just as in the non-Abelian case, the principle of virtual work gives static equations which differ by a sign from the zero-time-derivative specialization of the Euler-Lagrange equations. The solution to the virtual work equations has a ϕ which approaches zero smoothly as $r \to 0$ [$\lambda_+ > 0$ in Eq. (B17b)], whereas the solution to the Euler-Lagrange equations has a ϕ which is singular at $r=0$ [$\lambda_+ < 0$ in Eq. (B17c)]. The Euler-Lagrange solution remains constant when propagated forward in time, while the virtual work equations evolve in time in a complicated fashion. Which set of equations should one use? One's first impulse is to assume that a static situation demands equations with a time-independent solution, but one only has to remember the *ac* Josephson effect [where the response to a constant applied voltage V is an oscillating tunneling current] to realize that this impulse is wrong.

How then does one decide? My answer is that the system decides, but its behavior in tunneling[35] from a classically inaccessible configuration of equal energy, such as the vacuum with $\phi \to -\kappa$ as $r \to \infty$, to the vacuum with $\phi \to \kappa$ as $r \to \infty$. The correct equations are those which describe the state of the system (i.e., the Cauchy data for the subsequent classical time evolution) at the instant of tunneling through to the vacuum of interest along a most probable tunneling path. I argued in the text that the requirement of a most probable tunneling path requires vanishing generalized velocities at the instant of tunneling through; in the present context this just gives Coleman's original "bounce" condition $\dot\phi = 0$, and does not distinguish between Eqs. (B6) and (B7). However, there is one additional requirement that I did not impose in the text: the requirement that the tunneling probability be extremal with respect to variations in the time of tunneling through (which was treated as fixed) as well as of the fields. There are two ways of imposing this requirement. One way is to consider the phase function used in the text

$$\begin{aligned} \phi &= \int_{-\infty}^{t} du\, \mathfrak{F}, \\ \mathfrak{F} &= L_{\text{constrained}} \\ &= \int d^3x\, [\mathcal{L} - \lambda(D_j E^j - eJ^0)], \end{aligned} \tag{B19}$$

and to note that the requirement $\partial\Phi/\partial t = 0$ implies

$$\mathfrak{F}|_t = L_{\text{constrained}}|_t = 0, \tag{B20}$$

which can be satisfied trivially, without any constraint on the Cauchy data at t, by an appropriate choice of an irrelevant constant in the action. Hence the desired condition appears at the level of a second variation $\delta_{\text{field}}(\partial\Phi/\partial t) = 0$, which gives

$$\delta\mathfrak{F} = \delta L_{\text{constrained}} = 0 \tag{B21}$$

and yields Eqs. (B6) as the correct choice.

A better way to make the argument is to rephrase it as the statement that it is only variations in $\mathfrak{F}$ relative to its end-point value which are physically significant, so that we can replace Φ by

$$\begin{aligned} \tilde\Phi &= \int_{-\infty}^{t} du\, \mathfrak{F}(u) - (t - t_0)\mathfrak{F}(t), \\ \partial\tilde\Phi/\partial t &= 0. \end{aligned} \tag{B22}$$

Equation (B22), when substituted in a functional integral, gives transition amplitudes which differ only by a phase from those given by the original phase function Φ, and hence yields the same transition probabilities. Varying $\tilde\Phi$, and noting that the final state can have no dependence on the arbitrary choice t_0 of time origin, gives[35]

$$\delta\int_{-\infty}^{t} du\, \mathfrak{F}(u) = 0 \;\Rightarrow\; \text{(i) "bounce condition,"} \quad \text{(ii) Euler-Lagrange equations (satisfied along an imaginary-time tunneling path),} \tag{B23a}$$

and

$$\delta\mathfrak{F}(t) = 0 \;\Rightarrow\; \text{(iii) Eqs. (B6).} \tag{B23b}$$

I conclude with the following remarks:

(1) The fact that Eqs. (B6) emerge as the most probable tunneling end-point configuration is consistent with the fact that Eqs. (B6) predict a smooth scalar field ϕ, whereas the specialization of the Euler-Lagrange equations predicts a ϕ which is singular at $r=0$. Tunneling to such a singular field configuration should be very improbable.

(2) The argument of Eqs. (B22) and (B23) applies to the non-Abelian case discussed in the text just as to the Abelian Higgs model.[36] Its relativistic generalization[37] should apply even in the case where the quarks are not heavy, and hence where the static approximation cannot be used.

(3) The fact that the initial Cauchy data are not preserved in form under time evolution means that after tunneling through, the system point in phase space moves away from the exit of the path of maximum probability, providing great stability to the new field configuration once it has formed.

(4) All of this has a strong resemblance to thermodynamics. As in thermodynamics, equilibrium states are determined, not by minimization of energy, but by maximization of probability.

Thinking in terms of a thermodynamic analogy raises the interesting question of whether a distinction exists between reversible and irreversible processes, and whether an entropy can be identified.[38] Let me give some speculations along these lines. As noted in Sec. III, when the two quarks are pulled apart the bag in which they are embedded is deformed. As long as the bag does not fission, the process is clearly reversible— the quarks can be allowed to move back together doing work against the restraining forces. On the other hand, if the quarks are pulled too far apart, the bag fissions--i.e., the topological quantum number $n=1$ solution tunnels into a topological quantum number $n=2$ solution (and an extra light $q\bar{q}$ pair is created). According to point (3) above, the fissioned configuration evolves in time away from the tunneling path of maximum probability, so this transformation is essentially irreversible. Hence if an analog of entropy exists, it must increase in the process; this argument makes plausible the identification of topological quantum number n with entropy. The statement of energy conservation then reads [for fission processes involving only SU(2) bags]

$$\delta E = 4\pi\kappa (D/g^2)\delta n + \delta V_{\text{static}} + \sum m_q (\delta N_q + \delta N_{\bar{q}}) , \tag{B24}$$

with κ (as already noted) playing the role of the temperature, n the role of entropy, V_{static} the role of the mechanical energy, and m_q the role of chemical potential. When bags in the SU(j) overlying Yang-Mills theories with $j>2$ are involved, the first term in Eq. (B24) should generalize (with color-singlet expectations of $D_{(j)}$ understood) to

$$T\delta S = \kappa \frac{4\pi}{g^2}\sum_j D_{(j)} C_{(j)} \delta n_{(j)} , \tag{B25}$$

with $C_{(j)}$ numerical constants which are determined by the structure of the SU(j) analogs of the Prasad-Sommerfield solution. The above arguments make plausible the following conjecture: In purely strong-interaction processes, probable transitions between initial and final asymptotic states are characterized by $\delta S \geq 0$.

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[1]S. L. Adler, Phys. Rev. D 18, 3212 (1978).

[2]I use the notations x^0 and t interchangeably.

[3]Equation (7) is the structure obtained by replacing an unrestricted path integral

$$\int \mathfrak{D}b \exp\left(i\int d^4x\,\mathcal{L}\right)$$

by a path integral restricted to field configurations with specified source charges,

$$\int \mathfrak{D}b\, \delta(D_j E^j - gJ^0)\exp\left(i\int d^4x\mathcal{L}\right) = (2\pi)^{-1}\int \mathfrak{D}b\mathfrak{D}\lambda \exp\left\{i\int d^4x\left[\mathcal{L} - \lambda(D_jE^j - gJ^0)\right]\right\}.$$

[4]A. Hanson *et al.*, *Constrained Hamiltonian Systems* (Accademia Nazionale Dei Lincei, Rome 1976); R. N. Mohapatra, Phys. Rev. D 4, 378 (1971); J. Schwinger, Phys. Rev. 125, 1043 (1962).

[5]S. Coleman, Phys. Rev. D 15, 2929 (1977); see also K. Bitar and S. Chang, *ibid.* 17, 486 (1978) and Ref. 35, which gives a complete bibliography. See also the papers of Callan *et al.* and Jackiw and Rebbi cited in Ref. 3 of I.

[6]Note that $g\,S(\delta b^k, J^k_{\text{spin}})$ is opposite in sign to the spatial contribution $-gS(\delta b^k, J^k)$ to $\delta\mathcal{H}_{\text{int}} = -\delta\mathcal{L}_{\text{int}} = gS(\delta b_\mu, J^\mu)$ of Eq. (I. 25). This is because the current J^μ appearing in Eq. (I. 25) is the convective current of Eq. (I. 17), and for convective currents, variations at constant current require an external battery to maintain the current, and the work done by the battery figures in the energy balance. When a spin moment is rotated to a position parallel to the magnetic field, work is done on the constraining forces and the field energy is lowered. When a current loop of identical magnetic moment is similarly rotated, the same work is done on the constraining forces, but the field energy increases, and the battery supplies twice the work. This subtlety of course has nothing to do with the generalization from the Abelian theory to chromodynamics. For a good discussion, see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1964), Vol. II, Chap. 15. In their language, the total energy U_{total}, not the mechanical energy U_{mech}, is always equal (in statics) to the field energy integral. For a convective current, $U_{\text{mech}} = -U_{\text{total}}$; for a spin, $U_{\text{mech}} = U_{\text{total}}$.

[7]A mechanism for this has been discussed by S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[8]In Eq. (16), V_H should really be written $V_H(S(\lambda,\lambda), S(z_{(j)}, z_{(j)}))$, since it could be a different function in each SU(j) diagonal bloc of the overlying Lie algebra.

[9]The surface terms on small spheres surrounding the charges are

$$\int dS^j_{(n)}\, S(\lambda, \delta E^j) \propto \int dS^j_{(n)}\, S(\lambda, \delta Q_{(n)})$$

and

$$\int dS^j_{(n)}\, S(E^j + D_j\lambda, \delta b_0),$$

both of which vanish by virtue of the variational equations obtained by equating the volume integrals of the variations to zero.

[10]This is important because of the problems connected with gauge specification in non-Abelian theories. See V. N. Gribov (unpublished); J. M. Singer (unpublished).

[11]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[12]S. Coleman *et al.*, Phys. Rev. D 15, 544 (1977); E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].

[13]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1965). See also M. A. Lohe, Phys. Lett. 70B, 325 (1977).

[14]The physical picture which emerges (although not the detailed equations) is essentially that of the MIT "bag" model; see A. Chodos *et al.*, Phys. Rev. D 9, 3471 (1974).

[15]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976).

[16]A. M. Polyakov, Ref. 11.

[17]To see this, note that if the $q, \bar{q}$ are originally positioned at $\vec{x}_q = t(-\hat{Q}_q^{\text{eff}})$, $\vec{x}_{\bar{q}} = s(\hat{Q}_{\bar{q}}^{\text{eff}})$, $s,t > 0$, then there will be another allowed position $\vec{x}_q = s(\hat{Q}_q^{\text{eff}})$, $\vec{x}_{\bar{q}} = t(-\hat{Q}_{\bar{q}}^{\text{eff}})$ with lower orientation energy.

[18]Since $D_j E^j = gJ^0 = 0$ at $t = 0$, and since the equations of motion guarantee preservation of the constraint $D_j E^j - gJ^0 = 0$ once it is valid on the initial Cauchy surface, the term involving λ makes no contribution to Eq. (56).

[19]Since $D_j \vec{\lambda} D_m \vec{\lambda} \sim r^{-4}$ at infinity [cf. Eq. (41)], no contribution appears from the sphere at infinity.

[20]L. S. Brown, R. D. Carlitz, D. B. Creamer and C. Lee, Phys. Rev. D 17, 1583 (1978).

[21]This is not a satisfactory approximation; there is no substitute for an exact calculation of the propagator.

[22]J. Bekenstein, Phys. Rev. D 5, 1239 (1972); 5, 2403 (1972); C. Teitelboim, *ibid.* 5, 2941 (1972).

[23]For a somewhat related discussion of confinement mechanisms in a gas of instantons, see C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D 17, 2717 (1978); A. Duncan, *ibid.* (to be published).

[24]It is easy to check that if I had chosen the lower sign in Eqs. (38) and (40), $q^{(-)\mu\nu\lambda\kappa}$ in Eq. (90) would have changed to

$$q^{(+)\mu\nu\lambda\kappa} = \delta^{\mu\lambda}\delta^{\nu\kappa} + \sum_n \eta^{(-)\mu\lambda n}\eta^{(-)\nu\kappa n}$$

$$= \delta^{\mu\lambda}\delta^{\nu\kappa} + \delta^{\mu\nu}\delta^{\lambda\kappa} - \delta^{\mu\kappa}\delta^{\nu\lambda} + \epsilon^{\mu\nu\lambda\kappa},$$

with $\eta^{(-)}$ given by Eq. (A9).

[25]My sign convention for ϵ, $\epsilon^{0123} = -1$, agrees with the convention $\epsilon_{1234} (= \epsilon^{1234}) = 1$ used by Brown *et al.*

[26]This is consistent with the fact that in applying the principle of virtual work, variations which violate the boundary condition of Eq. (35) are not allowed, since they make the energy integral diverge.

[27]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, New York, 1957), pp. 107 and 108.

[28]Perhaps through an understanding of the tunneling process which creates the "bag" field.

[29]N. S. Manton, Nucl. Phys. B126, 525 (1977).

[30]See M. F. Atiyah and R. S. Ward, Commun. Math. Phys. 55, 117 (1977); M. F. Atiyah *et al.* (unpublished); C. W. Bernard *et al.*, Phys. Rev. D 16, 2967 (1977); E. F. Corrigan *et al.* (unpublished); N. Christ *et al.*, (unpublished), and Ref. 20. See also Sec. VI B3 and Ref. 43 of W. Marciano and H. Pagels, Phys. Rep. 36C, 137 (1978).

[31]N. S. Manton (unpublished).

[32]My h_{sv} are related to the f_{sv} of Brown *et al.* by $h_{sv} = \rho_s \rho_v f_{sv}$.

[33]In particular, they have made use of interchanges of summation orders and of antisymmetry properties of summands.

[34]The results contained in this appendix were, in part, an outgrowth of work I did in collaboration with R. B. Pearson on "no-hair" theorems for Abelian Higgs black holes (unpublished).

[35]For a survey and a complete bibliography, see S. Coleman, in the Proceedings of the 1977 International School of Physics "Ettore Majorana" (unpublished).

[36]As discussed by S. Coleman in Refs. 5 and 35, the Abelian case has infinite-volume divergences which are absent from the physically interesting non-Abelian case.

[37]In talking about an "instant of tunneling through" I am neglecting possible retardation effects. I believe the formulation which I have used is adequate to describe what happens in the vicinity of massive source charges. A relativistic generalization should assume a general surface of tunneling through, not a time slice. (It may be that this will make no difference—a calculation is needed.)

[38]I am again pursuing the analogy, suggested in the text, between 't Hooft-Polyakov-Prasad-Sommerfield "bag" solutions in non-Abelian gauge theories and black holes in general relativity. The thermodynamic interpretation of black-hole processes, discovered by Bekenstein and Hawking, is of course well known.