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Can the $\pi\pi$ scattering amplitude be represented by any Veneziano model?

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We propose a general definition of a Veneziano-model amplitude in terms of the support property of its double inverse Laplace transform. A practical test of this property is then formulated and applied to $\pi\pi$ scattering. It is satisfied by the nonexotic $\pi\pi$ amplitudes but not by the exotic $\pi^+\pi^+$ scattering amplitude. However there are significant deviations from the explicit Lovelace-Shapiro model and the isovector-exchange amplitude is better represented by the symmetric-group model of Frampton.

I. INTRODUCTION

The introduction of the Veneziano model[1] into high-energy physics has led to the development of highly sophisticated dual resonance models.[2] In these theories, the Veneziano amplitude provides the Born term in, it is hoped, a weak-coupling expansion of hadronic scattering amplitudes. Unfortunately, it has, till now, not been possible to find a crucial practical test of this basic assumption underlying dual models. There have been a number of phenomenological applications of the Veneziano model[3] but, due to the freedom in the choice of satellite terms and the necessary unitarization procedure, the resulting fits are rather inconclusive. The simpler four-point processes, particularly meson-meson scattering, provide the cleanest situation for testing the Veneziano model. It was quickly recognized by Lovelace and Shapiro[4] that $\pi\pi$ scattering is an especially good case for study. However the absence experimentally of odd $\pi\pi$ daughter trajectories has cast doubt on the significance of the Lovelace-Shapiro-Veneziano (LSV) model. It is, of course, possible to circumvent this difficulty by the addition of satellite terms and Frampton[5] has suggested a specific prescription based on the symmetric group.

In this paper we suggest a method for testing the essence of the Veneziano model, without the necessity of making detailed Veneziano-model fits to experimental data and varying a large number of parameters. Also it is not necessary to consider explicit unitarity corrections, since our test only involves the absorptive part of the scattering amplitude. It is based on the observation that the most general (s,t) Veneziano form $A(s,t)$ (without loop or other unitarity corrections) has a two-dimensional inverse Laplace transform $F(\xi_s,\xi_t)$ with a very characteristic support property. Namely, this transform is only nonvanishing on the curve

$$1-e^{-\xi_s/\alpha'}-e^{-\xi_t/\alpha'}=0\,, \tag{1.1}$$

in the (ξ_s,ξ_t) plane, where α' is the universal Regge-slope parameter. We take the fact that the double inverse Laplace transform has support on the curve (1.1) to be the defining property of a general Veneziano model without loops. It then becomes possible, using general amplitude analysis, to test in a very broad sense whether the experiment agrees with any Veneziano model. Of course, dual-loop corrections could give contributions anywhere in the (ξ_s,ξ_t) plane. However, there should be strong peaking along the support curve (1.1) if the Veneziano amplitude is to provide a sensible first-order term in a dual-loop perturbation series. We therefore take the attitude that the double inverse Laplace transform of a scattering amplitude must be weighted strongly along the curve (1.1), in order that the general Veneziano model be a viable description of the process.

The advantages of our method over previous attempts to compare phenomenological amplitudes with duality ideas are: (a) our criterion is satisfied by any dual-model amplitude and is therefore not dependent on details of the model; (b) as will be shown in Sec. III, our analysis only employs the imaginary part of amplitudes, and therefore we are presumably rather insensitive to unitarity corrections.

In the past the study of zero trajectories (as advocated in particular by Odorico) has constituted a popular way of emphasizing the similarity between dual models and physical amplitudes. However, physical zero trajectories have considerable deviations from the simple straight-line pattern characteristic of some dual models. This is due in part to their sensitivity to the real part of the amplitude (unitarity corrections), and therefore it is not possible to make any very precise test using them.

The particle spectrum is of course a central property to derive for any phase-shift analysis. As far as the daughter states are concerned, however, conclusions are difficult to obtain. Also, the particle spectrum changes drastically from one Veneziano model to another. For these reasons our technique appears to us to provide a cleaner way of testing the basic Veneziano properties of the amplitude than does a cumbersome and ambiguous comparison at the level of the particle spectrum.

In Sec. II, we derive Eq. (1.1) for the universal support curve and discuss the form of the weight function of the double Laplace transform along its support. We illustrate our results using the explicit LSV[4] model of $\pi\pi$ scattering. Our test is expressed in a more practical form in Sec. III. The test is then applied to $\pi\pi$ scattering in Sec. IV, using recent $\pi\pi$ phase-shift analyses as input. A different approach to the standard ρ-exchange finite-energy sum rule (FESR) is also briefly discussed. We find that the general support property required of a Veneziano model is quite well satisfied by the nonexotic $\pi\pi$ amplitudes. Significant deviations from the LSV predictions are found. These results are discussed in Sec. V. Similar calculations have been made by Lyberg for πN and KN scattering amplitudes.[6]

II. VENEZIANO AMPLITUDE AS A DOUBLE LAPLACE TRANSFORM

In order to motivate our definition of a Veneziano model, let us consider the very general (s,t) Born term

$$A(s,t)=\sum_n a_n(s,t)B(-\alpha(s)+c_n,-\alpha(t)+d_n), \tag{2.1}$$

where $B(X,Y)$ is the Euler beta function and the $a_n(s,t)$ are polynomials. The constants c_n and d_n may be interpreted as displacements of the intercept α_0 of the Regge-trajectory function

$$\alpha(s)=\alpha_0+\alpha' s. \tag{2.2}$$

The sum in Eq. (2.1) may have an infinity of terms provided that the orders of all the polynomials have a common finite upper limit N. All Veneziano-model amplitudes that have been suggested can be expressed in the form (2.1).

From the integral representation

$$B(-\alpha(s),-\alpha(t))=\int_0^1 du\int_0^1 dv\,\delta(1-u-v)u^{-\alpha(s)-1}v^{-\alpha(t)-1}, \tag{2.3}$$

it is easily shown that, for a polynomial $P(X,Y)$,

$$P(-\alpha(s)-1,\ -\alpha(t)-1)B(-\alpha(s),\ -\alpha(t))=\int_0^1 du\int_0^1 dv\,\delta(1-u-v)P\left(u\frac{\partial}{\partial u},v\frac{\partial}{\partial v}\right)u^{-\alpha(s)-1}v^{-\alpha(t)-1} \tag{2.4}$$

$$=\int_0^1\int_0^1 du\,dv\sum_{m=0}^{N}W_m(u,v)\delta^m(1-u-v)u^{-\alpha(s)-1}v^{-\alpha(t)-1}. \tag{2.5}$$

Here N is the order of the polynomial $P(X,Y)$ and $W_m(u,v)$ is a polynomial of order m. It follows that the nth term on the right-hand side of Eq. (2.1) can be rewritten as

$$A_n(s,t)=\int_0^1\int_0^1 du\,dv\sum_{m=0}^{N}\tilde{M}_m\left(\frac{u}{v}\right)\delta^m(1-u-v)u^{-\alpha(s)-1+c_n}v^{-\alpha(t)-1+d_n}. \tag{2.6}$$

Using this result, it is possible to write the general expression (2.1) in the form

$$A(s,t)=\int_0^1\int_0^1 du\,dv\,u^{-\alpha' s-1}v^{-\alpha' t-1}\sum_{m=0}^{N}f_m\left(\frac{u}{v}\right)\delta^m(1-u-v). \tag{2.7}$$

It is convenient to make the change of variables $u=e^{-\xi_s/\alpha'}$ and $v=e^{-\xi_t/\alpha'}$ to obtain

$$A(s,t)=\frac{1}{\alpha'^2}\int_0^\infty\int_0^\infty d\xi_s d\xi_t e^{s\xi_s+t\xi_t}\sum_{m=0}^{N}f_m(\xi_s-\xi_t)\delta^m(1-e^{-\xi_s/\alpha'}-e^{-\xi_t/\alpha'}). \tag{2.8}$$

Hence we see that the very general Veneziano Born term (2.1) can be rewritten as a double Laplace transform

$$A(s,t)=\int_0^\infty\int_0^\infty e^{s\xi_s+t\xi_t}F(\xi_s,\xi_t)d\xi_s d\xi_t, \tag{2.9}$$

where $F(\xi_s,\xi_t)$ is a distribution

$$F(\xi_s,\xi_t)=\frac{1}{\alpha'^2}\sum_{m=0}^{N}f_m(\xi_s-\xi_t)\delta^m(1-e^{-\xi_s/\alpha'}-e^{-\xi_t/\alpha'}), \tag{2.10}$$

with support on the curve

$$1 - e^{-\xi_s/\alpha'} - e^{-\xi_t/\alpha'} = 0\,. \tag{2.11}$$

We take the fact that its double inverse Laplace transform has support on the universal curve (2.11) to be the defining property of a Veneziano-model amplitude. Equivalently, we require the double inverse Mellin transform, see Eq. (2.7), to be of the form

$$\tilde{F}(u,v) = \sum_{m=0}^{N} \tilde{f}_m\left(\frac{u}{v}\right) \delta^m(1-u-v)\,. \tag{2.12}$$

In the following we shall refer to both the Laplace and Mellin transforms, using whichever is the most convenient. The above definition allows an infinite number of satellite terms with Regge trajectories of arbitrary intercept. It is only the Regge-slope parameter α' which must be kept constant.

Adopting such a definition, it becomes possible, using phenomenological amplitudes, to find out whether the Veneziano model is ruled out by experiment. If there is no peaking of the double inverse Laplace transform of the scattering amplitude along the universal support curve (2.11), then it can be concluded that the Veneziano model is invalid. Dual-loop corrections will in general, of course, give a double inverse Laplace transform which is nonzero everywhere. [The support of planar loops is restricted to (ξ_s, ξ_t) values greater than or equal to those on the curve (2.11). For an amplitude satisfying the Mandelstam representation, the double inverse Laplace transform can be expressed in terms of its double spectral function,

$$F(\xi_s, \xi_t) = \frac{1}{\pi^2}\int \rho(s,t) e^{-s\xi_s - t\xi_t} ds\, dt\,.]$$

However, if the dual perturbative approach is sensible, these corrections should not dominate the Born term. In fact, it might even become possible to separate the Born term from its unitarity corrections using this approach.

The above discussion applied to the (s,t) Veneziano amplitude and is appropriate to a reaction such as $\pi^+\pi^- \to \pi^+\pi^-$ with an exotic u channel. [Note that in this paper we generally take the variable u to be $e^{-\xi_s/\alpha'}$ and it is only used to denote the third Mandelstam invariant energy variable when explicit reference is made to a crossed channel. Also note that for a crossing-symmetric Veneziano amplitude $A(s,t) = A(t,s)$, the weight functions $f_m(\xi_s - \xi_t)$ are even.] In general, it is necessary to add three Born terms in order to treat all three channels correctly. We then obtain three branches of the universal support curve as shown in Fig. 1, where the homogeneous coordinates ξ_s/α', $-\xi_t/\alpha'$, and $(\xi_t - \xi_s)/\alpha'$ are used. For simplicity, we shall only discuss the (s,t) term explicitly

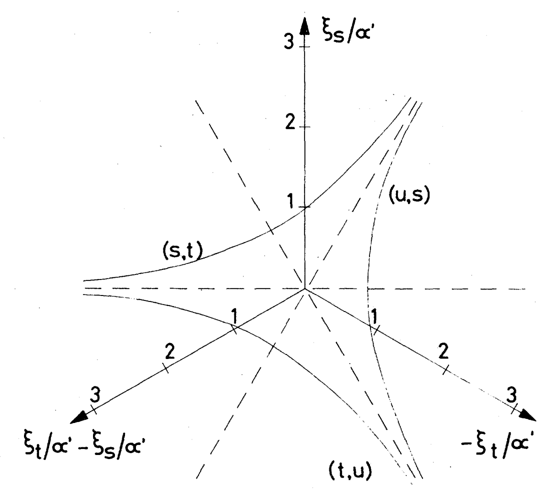

FIG. 1. Universal support curve for the double inverse Laplace transform of a general Veneziano-model amplitude.

in the following. [Here we use a normalization such that the value of the amplitude at threshold is given by the s-wave scattering length in pion mass units. Then f in Eq. (2.13) becomes the $\rho\pi\pi$ coupling constant. For $M_\rho = 770$ MeV and $\Gamma_\rho = 150$ MeV, $f^2/16\pi = 0.72$.]

The LSV formula for $\pi^+\pi^-$ scattering

$$V^{+-}(s,t) = -\frac{f^2}{16\pi}\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}\,, \tag{2.13}$$

where $\alpha(s) = \frac{1}{2} + \alpha' s$ is the ρ Regge trajectory, provides a good illustration of the general discussion above. It is readily expressed as a double Laplace transform of the general form (2.10) with

$$f_m(\xi_s - \xi_t) = \begin{cases} \left(-\frac{f^2}{16\pi}\right)\frac{1}{2}\mathrm{sech}\left(\frac{\xi_s - \xi_t}{2}\right), & \text{for } m = 0 \text{ and } 1 \\ 0, & \text{for } m > 1\,, \end{cases} \tag{2.14}$$

or, equivalently, as a Mellin representation (2.12) with

$$\tilde{f}_m(r) = \begin{cases} \left(-\frac{f^2}{16\pi}\right)\frac{\sqrt{r}}{1+r}\,, & \text{for } m = 0 \text{ and } 1 \\ 0, & \text{for } m > 1\,, \end{cases} \tag{2.15}$$

where $r = u/v$.

For later use, we also give here the Mellin representation of the Frampton pion-pion amplitude.[5] The Frampton model is constructed so as to have no odd daughters in the four-point function, a property enjoyed by the generalized Veneziano model with $\alpha_0 = 1$ and the Neveu-Schwarz-Ramond model.[7] The $\pi\pi$ amplitude is of the general form (2.1) with an infinite number of terms having linear polynomials $a_n(s,t)$ and integer constants c_n and d_n. Consequently, only the first two terms in Eq. (2.12) are nonzero and their ratio is a rational function.

$$\tilde{f}_0(r) = -\frac{f^2}{16\pi}\,\frac{r^{1/2}(5+4r+5r^2)}{(1+r)^{3/2}(1+r+r^2)^{3/4}}\,, \tag{2.16}$$

$$\tilde{f}_1(r) = -\frac{f^2}{16\pi}\,\frac{r^{1/2}(1+r+r^2)^{1/4}}{(1+r)^{3/2}}\,. \tag{2.17}$$

The amplitude is normalized so that the leading trajectory contribution agrees with the LSV model.

III. INVERSE LAPLACE TRANSFORM OF A SCATTERING AMPLITUDE

In this section we consider the practical application of Eqs. (2.10)–(2.12) to test whether an empirical amplitude has a strong component of the Veneziano form. Amplitude analysis of experimental data can provide scattering amplitudes at fixed momentum-transfer $t\in[-1\ \text{GeV}^2, 0]$ as a function of the invariant energy s, but amplitudes are not available as analytic functions of s and t.

In this section we demonstrate that the fundamental support property (2.10), (2.11) is equivalent to a very characteristic behavior of the *single* inverse Laplace transforms of the fixed-t amplitudes. These can be evaluated accurately from our data.

Using Eq. (2.10), we define $G(\xi_s, t)$ by

$$A(s,t) = \int_0^\infty d\xi_s e^{s\xi_s} G(\xi_s, t)\,, \tag{3.1}$$

so that

$$G(\xi_s,t) = \int_0^\infty d\xi_t e^{t\xi_t}\frac{1}{\alpha'^2}\sum_{m=0}^{N} f_m(\xi_s-\xi_t) \times \delta^{(m)}(1-e^{-\xi_s/\alpha'}-e^{-\xi_t/\alpha'})\,.$$

Partial integrations immediately demonstrate that this expression is of the form

$$G(\xi_s,t) = \exp(t\hat{\xi}_t)P_N(\xi_s,t)\,. \tag{3.2}$$

where the *t dependence* in P_N is a polynomial of degree N and where $\hat{\xi}_t$ is given by the fundamental duality condition expressed by the δ function as

$$\hat{\xi}_t = -\alpha'\ln[1-e^{-\xi_s/\alpha'}]\,. \tag{3.3}$$

At this point we emphasize strongly that although we are dealing here with only one-dimensional transforms, the requirement that $A(s,t)$ is of the form expressed by Eqs. (3.1)–(3.3) is completely equivalent to the full requirement expressed in the double-transform version, Eqs. (2.9) and (2.10). In principle therefore a study just of the one-dimensional transforms can give us a complete answer to the question we ask in this paper.

Since the fixed-momentum-transfer amplitudes are available for a range of t (typically $0\lesssim -t\lesssim 1$ GeV^2), it is possible to check whether Eq. (3.2) provides a good parametrization of their t dependence with the Veneziano value (3.3) for the exponent $\hat{\xi}_t$. In practice, of course, it will be necessary to restrict the order N of the polynomial (we will take linear polynomials) to make such a test sensible over a finite range of t.

Let us now consider how the single inverse Laplace transform of a scattering amplitude $A(s,t)$ may be calculated. Since $A(s,t)$ is not analytic in the left- or right-half s plane, it is necessary to use a two-sided Laplace transform. The standard Mellin inversion formula then gives

$$G(\xi_s,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{-s\xi_s}A(s,t)\,, \tag{3.4}$$

provided of course that this integral makes sense. If, however, the amplitude has an asymptotic Regge behavior

$$A(s,t) \propto s^{\alpha(t)}\,, \tag{3.5}$$

the Mellin formula (3.4) will not converge unless $\alpha(t)<-1$ and this inequality is in general not satisfied for $|t|\lesssim 1$ GeV^2. In fact, for $\alpha(t)<-1$, it is possible to deform the integration contour in Eq. (3.4) along the right-hand s channel (left-hand u channel) branch cut for positive (negative) ξ_s to obtain

$$G(\xi_s,t) = \frac{1}{\pi}\int_{\text{cut}} ds\, e^{-s\xi_s}\text{Im}A(s,t)\,. \tag{3.6}$$

For any reasonable amplitude, this integral exists for $\text{Re}\,\xi_s \neq 0$. Interpreting $G(\xi_s,t)$ as a distribution allows us to take care of any divergence at $\xi_s=0$ and Eq. (3.6) is then the correct inversion formula for any value of $\alpha(t)$. (This can be seen by remarking that the Laplace transform of a function, which is zero everywhere except in an ϵ neighborhood of $\xi_s=0$, behaves like a polynomial and then has no branch cuts for $s\ll 1/\epsilon$. So the amplitude can be split into two parts, one being the Laplace transform contribution from an ϵ neighborhood around zero and the other being the Laplace transform from the rest.) Similarly we can define the single inverse Mellin transform

$$\tilde{G}(u,t) = \frac{\alpha'}{\pi}\int_{\text{cut}} ds\, u^{\alpha' s}\text{Im}A(s,t)\,, \tag{3.7}$$

where, for a Veneziano Born term, we expect

$$\tilde{G}(u,t) = \alpha'\tilde{P}_N(u,t)e^{t\hat{\xi}_t}\,. \tag{3.8}$$

It should be noted that the calculation of the single inverse Mellin transform only requires the value of the absorptive part of the scattering amplitude $\text{Im}A(s,t)$ along the branch cuts.

The proposed test consists of expressing the t dependence of $G(\xi_s,t)$ at fixed ξ_s in the form of a polynomial times an exponential $Q(t)e^{\lambda t}$, in order to obtain the exponential slope factor λ. The Veneziano form demands $\lambda = \hat{\xi}_t = \alpha' \ln(1-u)$. In order to make a practical test over a finite t range we must restrict the order of the polynomial which henceforth we shall take to be linear, i.e., $N=1$.

By explicit integration of Eq. (2.8) over ξ_t, the polynomial $P_1(\xi_s, t)$ can be expressed in terms of the functions f_0 and f_1 of Eq. (2.10). In fact, it is more convenient to use the variable u and introduce the functions g_0 and g_1 defined by

$$g_m(u) = \tilde{f}_m\left(\frac{u}{1-u}\right) = f_m(\xi_s - \hat{\xi}_t). \tag{3.9}$$

Then we find the polynomial to be

$$\tilde{P}_1(u,t) = \frac{1}{\alpha'(1-u)^2}\left\{(1-u)g_0(u) - g_1(u) - u(1-u)\frac{d[\ln g_1(u)]}{du} - \alpha' t g_1(u)\right\}. \tag{3.10}$$

Instead of characterizing $\tilde{P}_1(u,t)$ by $g_0(u)$ and $g_1(u)$, it is more convenient to use $g_1(u)$, which is simply related to the coefficient of t, and the position of the polynomial zero $t_0(u)$ given by

$$\alpha' t_0(u) = \frac{g_0(u)}{g_1(u)}(1-u) - u(1-u)\frac{d}{du}\ln g_1(u) - 1. \tag{3.11}$$

So our general Veneziano ansatz for the inverse Mellin transform at fixed u becomes

$$\tilde{G}(u,t) = \alpha' A (t_0 - t) e^{\lambda t}, \tag{3.12}$$

with the important condition

$$\lambda = \hat{\xi}_t = -\alpha' \ln(1-u) \tag{3.13}$$

and where

$$A = g_1(u)/(1-u)^2. \tag{3.14}$$

Again we give, as examples, the results for the LSV model

$$g_1(u) = g_0(u) = -\frac{f^2}{16\pi}[u(1-u)]^{1/2}, \tag{3.15}$$

$$\alpha' t_0(u) = -0.5, \tag{3.16}$$

and the Frampton model

$$g_1(u) = -\frac{f^2}{16\pi}[u(1-u)]^{1/2}(1-u+u^2)^{1/4}, \tag{3.17}$$

$$\alpha' t_0(u) = -0.5 + \frac{(1-u)^2}{4(1-u+u^2)}. \tag{3.18}$$

The zero trajectories $t_0(u)$ for the two models differ significantly from each other at small u. However the values for $g_1(u)$ differ from each other by less than 7% at all u. We remark here that (s,t)-crossing-symmetric Veneziano amplitudes have weight functions $g_m(u)$ symmetric about $u = \frac{1}{2}$.

IV. APPLICATION TO $\pi\pi$ SCATTERING

High-statistics data on the reaction $\pi^- p \to \pi^+\pi^- n$[8] have made possible detailed studies[9-12] of the $\pi\pi$ scattering amplitude from threshold to 1.8 GeV. These recent phase-shift analyses have greatly improved our understanding of $\pi\pi$ spectroscopy. It has been known for some time that the prominent resonances on the leading ρ-f trajectory agree well with the LSV model. However, the structure of the daughter states has now been determined and it disagrees with the LSV model. In particular, the first daughter trajectory seems to be completely absent. The only possible exception seems to be the $\epsilon(600)$, which may well be just the tail of the $\epsilon(1200)$.[13] Phenomenologically, the $\epsilon(1200)$ couples strongly to $\pi\pi$ and there is no $\rho'(1250) \to 2\pi$ decay, which is completely opposite to the LSV prediction. Also there is a strong $\rho'(1600) \to 2\pi$ signal and a second daughter trajectory seems to be established. The observed resonance structure therefore more closely resembles that of the Frampton model than that of the LSV model. However, even for the Frampton model, the widths of the resonances are not in detailed agreement with experiment. In particular, the predicted $\epsilon(1200)$ is far too narrow to account for the observed behavior of the $\pi\pi$, $I=0$, s wave.

The amplitude analysis of Ref. 12 provides an analytic representation of the $\pi^+\pi^-$ elastic-scattering amplitude for fixed values of $t = 0.0$, -0.2, -0.4, -0.6, -0.8, and -1.0 GeV^2. These amplitudes are essentially uniquely determined by data, analyticity, and phase-shift analysis (unitarity) from threshold up to a dipion mass of 1.8 GeV. They correspond to an $I=0$ s-wave scattering length $a_0^0 = 0.3\ m_\pi^{-1}$ in agreement with recent K_{e4} experiments[14] and smoothly tend to a Regge asymptotic behavior above 1.8 GeV. The behavior below 1.8 GeV is rather insensitive to the precise Regge parameters, and of course the amplitudes are unreliable above this energy. For negative values of the invariant energy s the amplitudes describe $\pi^+\pi^+$ scattering, which was used as fixed input (see Ref. 12). As well as having fixed-t analyticity, the amplitudes have a high numerical consistency with fixed-s analyticity. For our purposes then, these amplitudes are the best available input.

We have evaluated the inverse Laplace transform for $\pi^+\pi^-$ and $\pi^+\pi^+$ elastic scattering at the above t values using Eq. (3.6). For clarity of discussion, we take ξ_s to be positive and explicitly refer to the direct channel process described by the amplitude

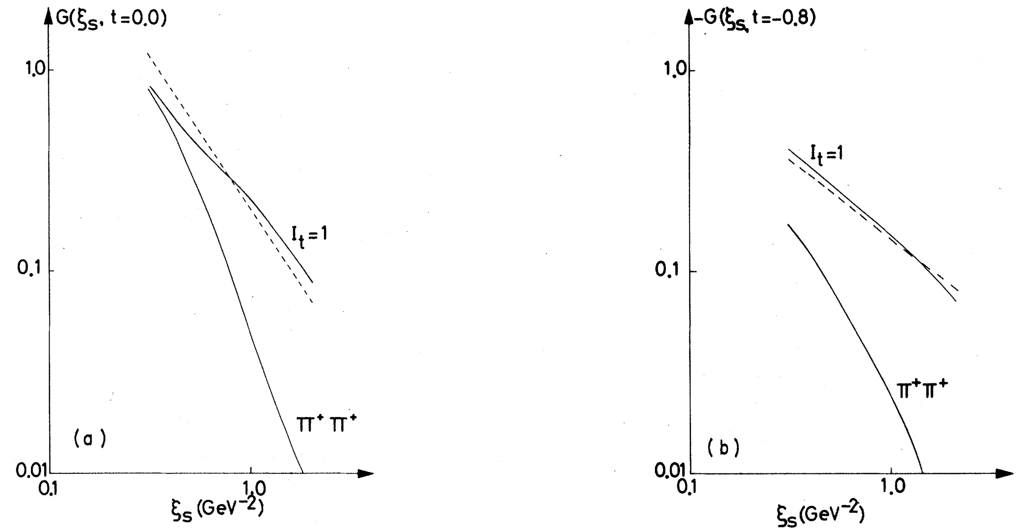

FIG. 2. The inverse Laplace transform of the $I_t = 1$ and $\pi^+\pi^+$ scattering amplitudes for (a) $t = 0.0$ and (b) $t = -0.8$ GeV^2. The dashed line is the transform of the ρ Regge amplitude corresponding to the residue function of Fig. 3.

$A(s,t)$ in Eq. (3.6). For small $\xi_s < (1.8\ \text{GeV})^{-2}$, the integral is dominated by the high-energy behavior of the $\pi\pi$ scattering amplitude and hence $G(\xi_s, t)$ is unreliable. Similarly, for large ξ_s the integral is dominated by the threshold region, where unitarity corrections to the Veneziano-Born term must be important. Therefore it is only reasonable to study the amplitudes $G(\xi_s, t)$ or $\tilde{G}(u,t)$ in a restricted region, which we shall take to be $0.1 \lesssim u \lesssim 0.7$.

The exotic amplitude $G_{\pi^+\pi^+}(\xi_s, t)$ is nonzero and gives a measure of the loop modifications required to correct the Veneziano $\pi\pi$ Born term. In fact the exotic amplitude $G_{\pi^+\pi^+}$ is typically of the order of 20% of the nonexotic amplitude $G_{\pi^+\pi^-}$, except near $t = -0.4$ GeV^2, where $G_{\pi^+\pi^-}$ has a zero. So it seems reasonable to hope that the dominant part of the nonexotic amplitude $G_{\pi^+\pi^-}$ is described by a Veneziano Born term. Since the loop corrections to Im$A_{\pi^+\pi^+}$ are presumably mostly due to the Pomeron, one might expect the isospin $I_t = 1$ exchange amplitude

$$G_{I_t=1}(\xi_s, t) = G_{\pi^+\pi^-}(\xi_s, t) - G_{\pi^+\pi^+}(\xi_s, t) \qquad (4.1)$$

to provide the best example of a dual amplitude. The behavior of the amplitudes $G_{I_t=1}$ and $G_{\pi^+\pi^+}$ are shown in Fig. 2 for $t = 0.0$ and -0.8 GeV^2 over the reliable range of ξ_s values. In the following we will give our results for both $G_{\pi^+\pi^-}$ and $G_{I_t=1}$.

The imaginary part of the $I_t = 1$ exchange amplitude is expected to be dominated at large s by the ρ Regge pole

$$\text{Im}A_{I_t=1}(s,t) \simeq \gamma_\rho(t) s^{\alpha_\rho(t)}. \qquad (4.2)$$

The corresponding small-ξ_s behavior of its inverse Laplace transform should be given by

$$G_{I_t=1}(\xi_s, t) \simeq \frac{1}{\pi} \frac{\gamma_\rho(t)\Gamma(1+\alpha_\rho(t))}{\xi_s^{\,1+\alpha_\rho(t)}}. \qquad (4.3)$$

[For a process with a leading threshold at $s = s_0$, it is more correct to consider the function $e^{s_0 \xi_s} \times G(\xi_s, t)$ on the left-hand side of Eq. (4.3).] At larger ξ_s values, the comparison of the two sides of Eq. (4.2) provides a good way of illustrating the phenomenological duality properties of the scattering amplitude. The ρ Regge residue function $\gamma_0(t)$ extracted from a standard FESR analysis of the $\pi\pi$ amplitudes from Ref. 12 is shown in Fig. 3, where $\alpha_\rho(t) = 0.5 + 0.9\,t$ and a cutoff at $\sqrt{s} = 1.8$ GeV were used. The right-hand side of Eq. (4.3) constructed from these Regge parameters is shown as a dashed line in Fig. 2. Encouraged by the

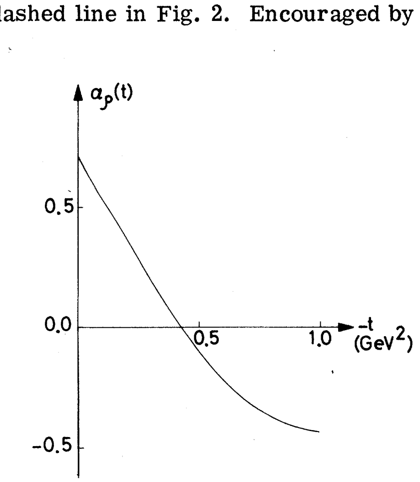

FIG. 3. The ρ residue function calculated from standard FESR integrals for the amplitudes of Ref. 12.

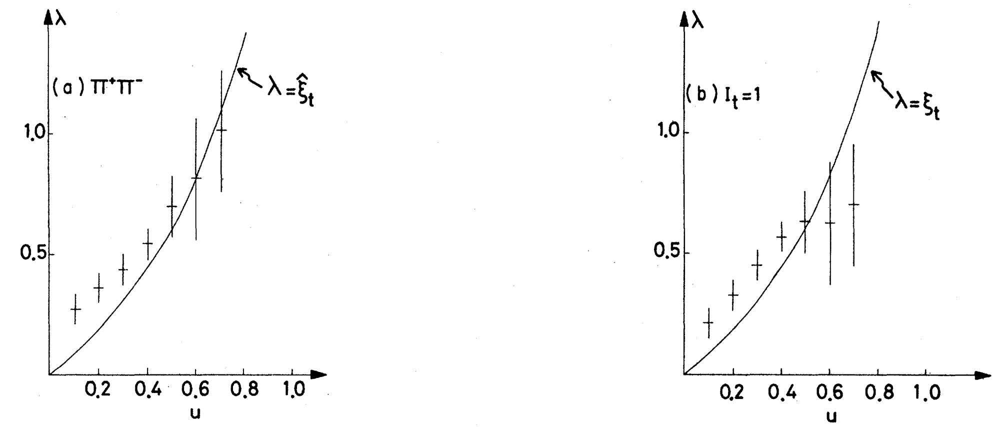

FIG. 4. Phenomenological exponential slope factor compared to the Veneziano-model prediction $\lambda = \hat{\xi}_t$ for (a) $\pi^+\pi^-$ scattering and (b) $I_t = 1$ scattering.

results, we suggest an alternative method for determining the Regge residue function $\gamma(t)$ [and the trajectory $\alpha(t)$]. The procedure is simply to fit the inverse Laplace transform $G(\xi_s, t)$ in the phase-shift region to the Regge form on the right-hand side of Eq. (4.3). It has the advantage over normal FESR's of having a smooth cutoff at large s values. Of course, it is necessary to supplement the phase-shift amplitude with a reasonable smooth high-energy imaginary part but the results are insensitive to its exact form.

Let us now return to the main point at issue in this paper. The crucial test of the Veneziano model is the requirement that, for each value of $u = e^{-\xi_s/\alpha'}$, the amplitude $G(u, t)$ should conform to the parametrization given in Eqs. (3.12)–(3.13) of the pre-

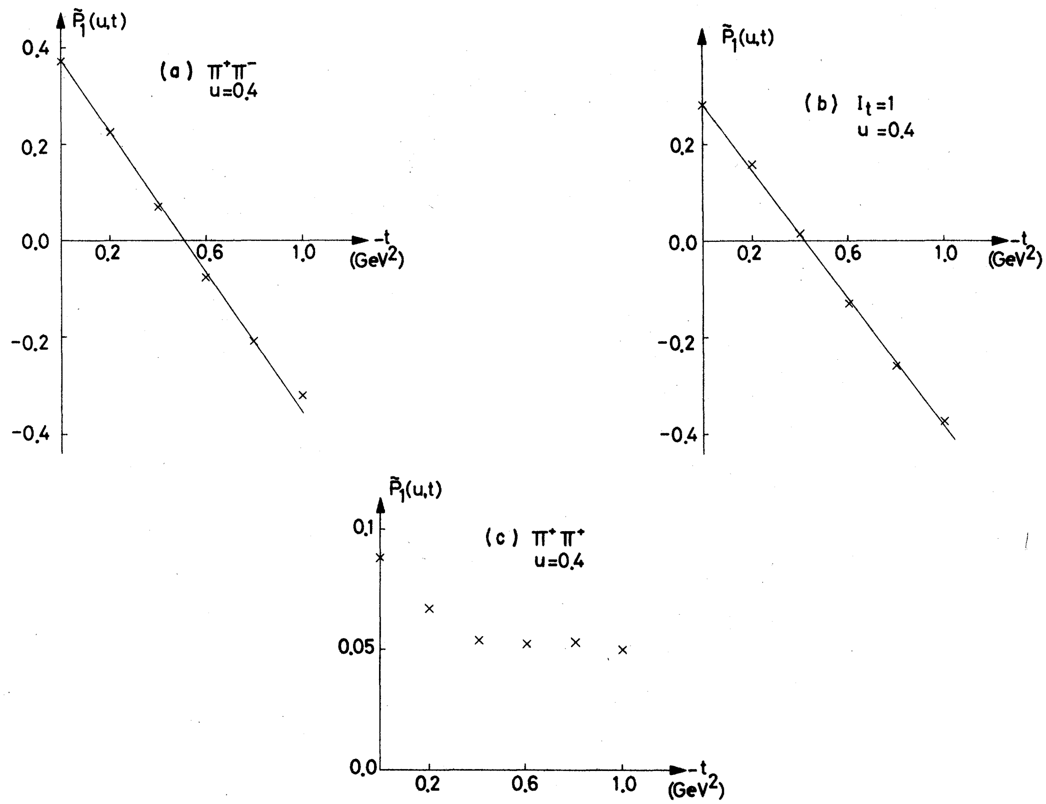

FIG. 5. Function $\tilde{P}_1$ ($u = 0.4$, t) constructed from the phenomenological amplitudes of Ref. 12 for (a) $\pi^+\pi^-$ scattering, (b) $I_t = 1$ scattering, and (c) $\pi^+\pi^+$ scattering. Linear-polynomial fits are shown for (a) and (b).

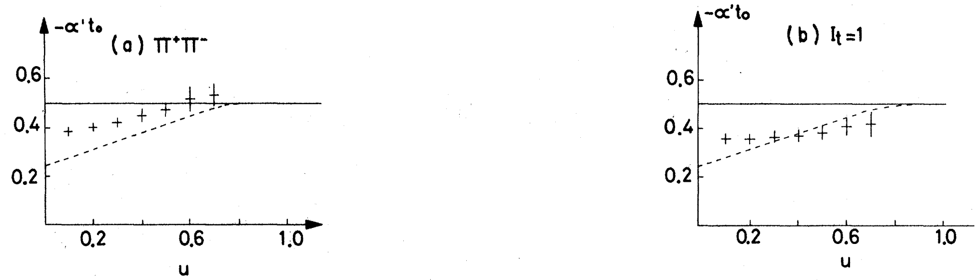

FIG. 6. Phenomenological zero trajectory $-\alpha' t_0(u)$ for (a) $\pi^+\pi^-$ scattering and (b) $I_t = 1$ scattering compared to the predictions of the LSV model (solid curve) and the Frampton model (dashed curve).

vious section. We take the value $\alpha' = 0.9$ GeV^{-2} for the universal Regge slope parameter. The amplitudes $\tilde{G}_{\pi^+\pi^-}$ and $\tilde{G}_{I_t=1}$ are well fitted by expressions of the type (3.12) and we show the values of λ obtained in Fig. 4. The error bars given in this and subsequent figures have no real statistical significance but indicate the spread of results obtained by using various fixed-t amplitudes generated by the calculations of Refs. 9 and 12. As can be seen from Fig. 4, both $G_{\pi^+\pi^-}$ and $G_{I_t=1}$ are compatible with the general Veneziano requirement $\lambda = \hat{\xi}_t$, which is shown as a full curve. In order to illustrate how well the $\pi\pi$ amplitudes conform to the general Veneziano behavior, we fix $\lambda = \hat{\xi}_t$ and plot $\tilde{P}_1(u,t)$ of Eq. (3.8) in Fig. 5 for the typical value of $u = 0.4$. The results for the $\pi^+\pi^-$ and $I_t = 1$ amplitudes are both well fitted by a linear polynomial $A(t_0 - t)$. However, the relatively small exotic $\pi^+\pi^+$ amplitude is clearly not of the Veneziano form (3.12)–(3.13). We feel that these results are positive evidence in favor of the general Veneziano model, and we, therefore, now turn to a more detailed study of its properties for $\pi\pi$ scattering.

The following discussion will be model dependent, just like that of the particle spectrum or the Odorico zero structure, Ref. 15, and should not be confused with the above fundamental test $\lambda = \hat{\xi}(t)$ of the general Veneziano support property, Eq. (2.11).

V. DISCUSSION AND CONCLUSION

The structure of the general Veneziano amplitude is characterized by the weight functions $g_m(u)$ of Eq. (3.9). In the previous section we have shown that two terms $g_0(u)$ and $g_1(u)$ are sufficient to reproduce the dominant behavior of the nonexotic $\pi\pi$ scattering amplitudes. However, as discussed in Sec. II, such two-term expansions embrace a very large class of Veneziano amplitudes and, for example, allow an infinite number of satellite terms. It is rather difficult to obtain $g_0(u)$ and it is phenomenologically more convenient to use the position of the polynomial zero $t_0(u)$ and $g_1(u)$ to specify the Veneziano amplitude. The polynomial zero $t_0(u)$ is readily obtained from our phenomenological fits of $\tilde{G}$ to the dual form (3.12)–(3.13). Our results for the $\pi^+\pi^-$ and $I_t = 1$ amplitudes are shown in Fig. 6. For comparison we have plotted the zero position predicted by the LSV model [solid curve—Eq. (3.16)] and the Frampton model [dashed curve—Eq. (3.18)]. There are significant deviations from the LSV model and,

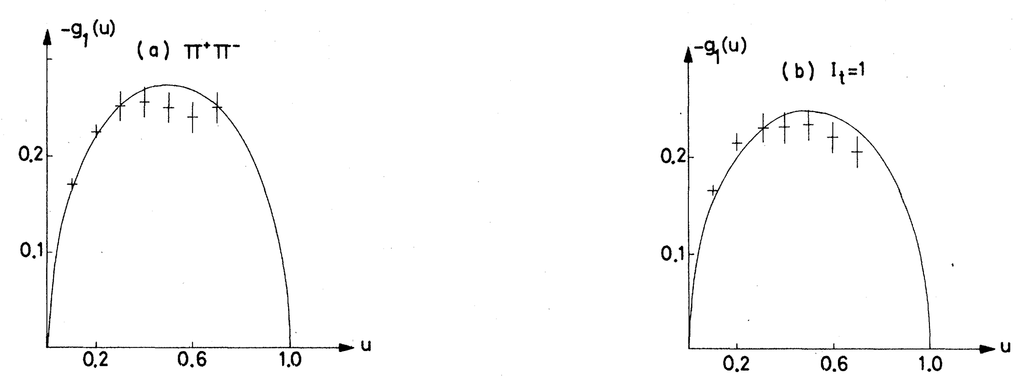

FIG. 7. Phenomenological weight function $-g_1(u)$ compared to the LSV model prediction (3.15) for (a) $\pi^+\pi^-$ scattering with $f^2/16\pi = 0.55$ and (b) $I_t = 1$ scattering with $f^2/16\pi = 0.5$.

in particular, the $I_t = 1$ exchange amplitude more closely resembles the predictions of the Frampton model.

The weight function $g_1(u)$ is also readily extracted from our fits using Eq. (3.14) and the results are shown in Fig. 7. The predictions of the LSV and Frampton models for $g_1(u)$ agree within 7% for all u values and therefore we just give the LSV curve. Our results for both the $\pi^+\pi^-$ and $I_t = 1$ amplitudes are in qualitative agreement with the behavior predicted by the two models.

In conclusion, recent $\pi\pi$ phase-shift analyses seem to uphold the general structure of the Veneziano model. The nonexotic amplitudes $G_{\pi^+\pi^-}$ and $G_{I_t=1}$ both conform well to the expected behavior (3.12) with the dual value ξ_t (3.13) for the exponent λ, while the relatively small exotic amplitude $G_{\pi^+\pi^+}$ does not. There are significant deviations from the LSV model. The $I_t = 1$ exchange amplitudes have no Pomeron contamination and favor the Frampton model over the LSV model, which presumably reflects the more realistic spectrum of the former.

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