New photon-nucleon dispersion relation for evaluating the Thomson limit using rising total cross sections

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New data showing that the photon-nucleon total cross section increases with energy for $\nu \geq 50$ GeV invalidate earlier comparisons with dispersion relations. Parametrization of the data are presented and used in a new formulation of the dispersion relations, in which an assumed asymptotic behavior avoids the need for subtraction. With this form the fitted amplitude can be compared directly with the Thomson limit. The experimental uncertainties are shown to have a significant effect upon such a comparison.

I. INTRODUCTION

Recent measurements at Fermilab have shown that the photon-proton total cross section increases in the energy range from 30 to 180 GeV.[1] These new data, in behaving just like those for purely hadronic interactions, fulfill the predictions of models such as geometrical scaling[2] or vector-meson dominance,[3] which assumed such a similar behavior. The implications of the increase for photoproduction processes involving charmed quarks or higher-mass quarks have already been studied by several authors.[1,4] The consequences for more "classical" topics, such as the evaluation via dispersion relations of the real part of the spin-averaged Compton amplitude, have not yet been as thoroughly explored.[2]

Damashek and Gilman,[5] in 1970, carried out the first dispersion relation calculations of the real part of $f_1(\nu)$, the spin-averaged forward γp amplitude, using the equation

$$\mathrm{Re} f_1(\nu) = f_1(0) + \frac{\nu^2}{2\pi^2} \mathrm{P} \int_{\nu_0}^{\infty} \frac{\sigma_T(\nu')}{\nu'^2 - \nu^2} d\nu' . \qquad (1)$$

The total-cross-section data available at that time, ranging from the low-energy region up to 20 GeV, decreased with energy in the fashion traditionally parametrized by simple Regge-pole dominance. The real part they calculated, by assuming a Regge-pole asymptotic form, showed a corresponding decreasing behavior. However, this calculated real part was not identified with the real part predicted by the Regge-pole amplitude; the difference between the two was consistent with a constant value of about -3 μb GeV, in agreement with the Thomson limit $f_1(0) = \alpha/M_N$. The presence of such a constant real term in $f_1(\nu)$ could be interpreted in terms of a fixed pole at angular momentum $J = 0$. Subsequent calculations using finite-energy sum rules (FESR)[6] or continuous-moment sum rules[7] (CMSR) either confirmed the presence of this effect or reported an uncertainty in determining the residue because of lack of precision in the data. Moffat and Snell[8] suggested a Regge-cut mechanism in place of the fixed-pole term, since the latter violates unitarity. They were able to show that the cut term was an acceptable alternative, even though it produced a total cross section which asymptotically *rose* to a constant (although their results continued to decrease to beyond $\nu = 100$ GeV/c, in contradiction to current data).

The new Fermilab data show clearly that the Regge-pole form assumed by Damashek and Gilman is incorrect. Neither their evaluation of the dispersion integral nor the theoretical real part of the Regge amplitude with which they compared their result is valid for the Compton amplitude. The dispersion integral can be calculated using whatever energy dependence one wishes to assume for $\sigma_T(\nu) = 4\pi \,\mathrm{Im}\, f_1(\nu)/\nu$; but in order to evaluate the possible contribution of the Thomson limit, one must know the real part of the corresponding amplitude. As Eden[9] has shown, there is a simple relation, based on analyticity and crossing symmetry, between the phase of the scattering amplitude and its energy dependence. From this relationship, it follows that if the total cross section rises asymptotically, the real part of the amplitude must also rise, rather than decrease to zero as in Damashek and Gilman's Regge-pole form. For example, the real part of $f_1(\nu)$ obtained in Ref. 2 using geometrical-scaling arguments reaches a minimum with a negative value, then turns toward zero and increases through positive values. Similar features for $\mathrm{Re} f_1(\nu)$ were obtained by Weise,[10] using an assumed γp total cross section which saturates the Froissart bound (but lies significantly higher than the Fermilab data).

This paper is intended to study γp dispersion relations in the light of the new data and the need for a new assumption about the asymptotic form of the amplitude. Since the results depend to some extent on the exact form assumed for the asymptotic energy dependence of $f_1(\nu)$, we devote the next section to a comparative evaluation of possible fits to the total cross section. In Sec. III we derive a modified form of the dispersion relation which allows us to evaluate directly the magnitude of any constant real term present in $f_1(\nu)$, using any assumed asymptotic form for the amplitude consistent with Eden's phase-energy relation.

II. FITS TO THE DATA

The behavior of the total-cross-section data[11] is indicated in Fig. 1. It is clear that $\sigma_T(\nu)$ is increasing for ν values greater than about 40 GeV, although it is certainly not possible to say what form that increase will take as $\nu \to \infty$. We have considered three fundamental possibilities:

(a) rising to saturate the Froissart bound, $\sigma_T \sim \ln^2\nu$;

(b) rising as in the geometrical-scaling model, $\sigma_T \sim \ln\nu$;

(c) rising toward a constant as the effects of an absorptive cut disappear, $\sigma_T \sim \sigma_\infty - \sigma_C/\ln\nu$.

These basic functional forms have been fitted to the data for $\nu > 2.0$ GeV. Because the lower end of this energy range includes a region in which $\sigma_T(\nu)$ is decreasing, we have also included a $\nu^{-1/2}$ term in each of our fits. The precise forms we have used are

$$\sigma_G(\nu) = \sigma_1 + \sigma_2\left(\frac{\nu}{\nu_0}\right)^{-1/2} + \sigma_4 \ln\left(\frac{\nu}{\nu_0}\right), \tag{2}$$

$$\sigma_F(\nu) = \sigma_1 + \sigma_2\left(\frac{\nu}{\nu_0}\right)^{-1/2} + \sigma_3\left(\ln^2\left(\frac{\nu}{\nu_0}\right) - \pi^2/4\right), \tag{3}$$

$$\sigma_C(\nu) = \sigma_1 + \sigma_2\left(\frac{\nu}{\nu_0}\right)^{-1/2} + \sigma_5 \ln\left(\frac{\nu}{\nu_0}\right)\Big/\left(\ln^2\left(\frac{\nu}{\nu_0}\right) + \pi^2/4\right), \tag{4}$$

where $\nu_0 = 0.015$ GeV is the energy of the first inelastic threshold.

Each of these forms produces an excellent fit to the data, as shown in Table I. The confidence levels are high, and the results are essentially indistinguishable, although $\sigma_C(x)$ is very slightly less satisfactory than the other two. In Fig. 1 we show the comparison of these three fits with the data, on a logarithmic energy scale. The differences between the three parametrizations do not become measurable until the energy reaches $\nu \approx 1000$ GeV.

The results of Barger *et al.*[12] for geometrical scaling in hadronic processes would correspond to $\sigma_4/\sigma_1 \approx 0.077$, whereas our best fit has $\sigma_4/\sigma_1 = 0.128$; we can obtain almost equally satisfactory fits with their value, however, and ascribe no great significance to the difference.

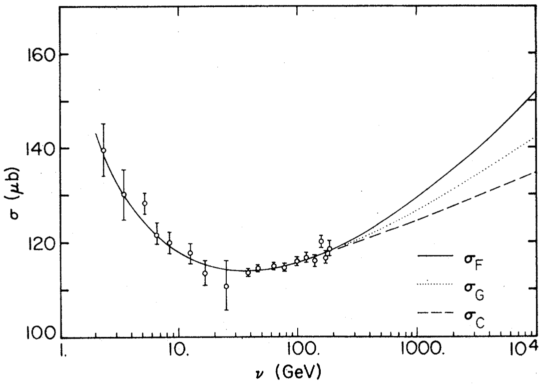

FIG. 1. Comparison of the three fitted parametrizations of Eqs. (2), (3), and (4) with each other and with experimental data. The data points shown are a representative sample of the 88 points used, the sources of which are listed in Ref. 1 and Ref. 11.

III. A SUM RULE FOR $f(0)$

We next explore the consequences of these fits for the photon-nucleon dispersion relations. For all three models, the fact that $\sigma_T(\nu)$ will exceed Damashek and Gilman's estimate for high energies implies that the value of the integral in (1) will be larger, i.e., that the value of $[\mathrm{Re} f_1(\nu) - f_1(0)]$ will be larger, than they found. Since this dispersion relation has a subtraction at $\nu = 0$, however, it is necessary to assume the value of $f_1(0)$ in order to evaluate it. As a consequence it is difficult to establish more than consistency for the value $f_1(0) = -\alpha/M_p$; to accomplish even that much requires knowing on theoretical grounds the real part corresponding to the asymptotic amplitude. Damashek and Gilman's conclusion is based on the observation that the contribution of the integral in (1) differs by about -3 μb GeV from the real part predicted by the Regge-pole amplitude they used.

To reach analogous conclusions from the analysis presented in the preceding section, we must know the real parts of the amplitudes which yield σ_G, σ_F, and σ_C. These amplitudes, unlike Damashek and Gilman's Regge poles, are not the results of any firmly based theory. It is known,[9] however, that an asymptotic amplitude satisfying the requirements of analyticity and crossing symmetry can be obtained by using a real analytic function of the variable ν/i. By this technique, we could obtain theoretical real parts corresponding to σ_F, σ_G, and σ_C, and proceed in the same way as Damashek and Gilman.

If one assumes such an amplitude, however, it is possible to do much better than merely show consistency. Instead, as we show in this section, one can derive a modified dispersion relation which

TABLE I. Best fits to γp total cross sections.

Form [Eqs. (2)–(4)]	σ_1 (μb)	σ_2 (μb)	$\sigma_{3,4,5}$ (μb)	χ^2	CL (%)
σ_G	58.42	233.45	7.446	76.4	74
σ_F	87.98	185.6	0.5201	76.9	72
σ_C	188.6	510.9	−636.2	78.1	69

yields a direct evaluation of the value of $f_1(0)$ for any assumed form of the asymptotic amplitude. Since this is possible, and since the fixed-pole question raises possible doubts about the validity of using the classical Thomson limit in (1), we do not carry out such a direct evaluation. Instead, we calculate $f_1(0)$ from each of these fits, and then compare it to a $-\alpha/M_p$.

Our technique is based upon subtracting the asymptotic behavior of $f_1(\nu)$, rather than its value at $\nu=0$, in writing the dispersion relation. It should be recalled that subtraction is necessary because $f_1(\nu)$ does not vanish as $|\nu|\to\infty$. Let us define a function $g(\nu)$ which has the same analytic structure as $f_1(\nu)$ (i.e., cuts from $\pm\nu_0=0.15$ to $\pm\infty$) and for which $|f_1(\nu)-g(\nu)|\to 0$ as $|\nu|\to\infty$. Then we can write a dispersion relation

$$f_1(\nu)-g(\nu)=\frac{1}{2\pi i}\oint\frac{f_1(\nu')-g(\nu')}{\nu'-\nu}\,d\nu' \tag{5}$$

in the traditional way, and deform the contour so that it surrounds the two cuts and closes at infinity, where the integrand vanishes by definition. Thus we obtain, on combining the integrals over the integrals over the two cuts,

$$f_1(\nu)-g(\nu)=1/\pi\int_{\nu_0}^{\infty}\frac{\nu'\,\mathrm{Im}[f_1(\nu')-g(\nu')]\,d\nu'}{\nu'^2-\nu^2} \tag{6}$$

$$=\frac{1}{2\pi^2}\int_{\nu_0}^{\infty}\frac{\nu'^2[\sigma(\nu')-\sigma_g(\nu')]}{\nu'^2-\nu^2}\,d\nu'\,, \tag{7}$$

where $\sigma_g(\nu')=4\pi\,\mathrm{Im}\,g(\nu'+i\epsilon)/\nu'$. For $\nu=0$, Eq. (7) becomes

$$f_1(0)=g(0)+\frac{1}{2\pi^2}\int_{\nu_0}^{\infty}[\sigma(\nu')-\sigma_g(\nu')]\,d\nu'\,. \tag{8}$$

Now let us assume that $g(\nu)$ has been chosen such that $\sigma_g(\nu)$ is a good parametrization of $\sigma(\nu)$ above some energy $\nu_{\max}$, i.e., that $\int_{\nu_{\max}}^{\infty}(\sigma-\sigma_g)d\nu'$ is negligible. Then it follows that

$$f_1(0)=g(0)+\frac{1}{2\pi^2}\int_{\nu_0}^{\nu_{\max}}[\sigma_T(\nu')-\sigma_g(\nu')]\,d\nu'\,, \tag{9}$$

and, by taking $\nu_{\max}$ as the highest measured energy, we may directly evaluate $f_1(0)$ for any assumed $g(\nu)$.

We have considered five different functional forms for $g(\nu)$, corresponding to the five different terms making up σ_F, σ_G, and σ_C, i.e., to cross sections which asymptotically (1) become constant, (2) rise like $\ln\nu$, and (3) rise like $\ln^2\nu$, plus correction terms behaving like (4) a Regge-pole term yielding $\nu^{-1/2}$, and (5) a Regge cut correction yielding $-1/\ln\nu$. Specifically, the functional forms we have studied, which yield respectively these total cross sections, are

$$g_1(\nu)=-\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}, \tag{10}$$

$$g_2(\nu)=-\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/4}, \tag{11}$$

$$g_3(\nu)=-\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}\ln^2\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}, \tag{12}$$

$$g_4(\nu)=-\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}\ln\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}, \tag{13}$$

$$g_5(\nu)=-\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}\left\{\left[\ln\left(\frac{\nu_0^2-\nu^2}{N^2}\right)^{1/2}\right]^{-1}+\frac{2N^2}{N^2+\nu^2-\nu_0^2}\right\}. \tag{14}$$

All five of these functions have precisely the same analytic structure as $f(\nu)$. They correspond, respectively, to the total-cross-section forms used in Sec. II, except for a factor $(1-\nu_0^2/\nu^2)^{1/2}$ which has no effect on the fits since $\nu^2/\nu_0^2>100$ is the range used. [For $g_5(\nu)$, poles in the logarithmic terms at $\nu^2=-N^2+\nu_0^2$ have been explicitly removed.] We shall use them in various combinations to evaluate the sum rule (9).

A. One- and two-parameter representations

The simplest approach to (9) is to assume that the total cross section for $\nu>183$ GeV is a multiple of a single one of the $g_i(\nu)$, e.g. $g(\nu)=\gamma_4 g_4(\nu)$. Then the sum rule can be used to determine γ_4, and its consistency with the data can be determined. For simplicity we have taken $N^2=\nu_0^2$. Using the data, we obtain

$$\frac{1}{2\pi^2}\int_{0.15}^{183.0}\sigma(\nu')d\nu'=1079.3\ \mu\mathrm{b\,GeV}\,; \tag{15}$$

the integral of σ_g is

$$\frac{1}{2\pi^2}\int_{0.15}^{183.0}\gamma_4\left(\frac{\nu^2-\nu_0^2}{\nu_0^2}\right)^{1/2}\ln\left(\frac{\nu^2-\nu_0^2}{\nu_0^2}\right)^{1/2}=4743.3\gamma_4\,, \tag{16}$$

and $g_4(0)=0$. Consequently (9) yields, for this parametrization,

$$f_1(0)=(1079.3-4743.3\gamma_4)\ \mu\text{b GeV}. \quad (17)$$

If we take $\gamma_4=0.202$, corresponding to the experimental cross sections ($\sigma\sim 117\ \mu$b at $\nu\sim 150$ GeV), then it follows that $f_1(0)=121\ \mu$b GeV, in drastic disagreement with the Thomson limit. Conversely, if we require $f_1(0)=-3\ \mu$b GeV, then $\gamma_4=0.228$, corresponding to a total cross section which is 13% larger than the experimental data.

It follows that $g_4(\nu)$ alone does not provide a satisfactory representation of the photon-nucleon asymptotic amplitude. Similar results are obtained using $g_1(\nu)$ or $g_3(\nu)$ instead of $g_4(\nu)$. This result is not surprising. To neglect $|\sigma-\sigma_g|$ for $\nu>183$ GeV it is necessary that $|\sigma-\sigma_g|<\epsilon/\nu$ where ϵ is small; otherwise the integral will diverge. Consequently $g(\nu)$ must yield the correct asymptotic total cross section down to order $1/\nu$. We know, however, that the Regge-pole terms yield a contribution of order $\nu^{-1/2}$, and this contribution cannot be neglected.

We therefore consider two-parameter forms such as $g(\nu)=\gamma_2 g_2(\nu)+\gamma_4 g_4(\nu)$, i.e., a logarithmically rising total cross section plus Regge-pole corrections. In this case one can use the sum rule (9), plus the requirement that σ_g agree with the data at the highest available energy, to determine γ_2 and γ_4, obtaining $\gamma_4=0.177$ and $\gamma_2=7.66$. While this form for $g(\nu)$ is not unacceptable, it is generally quite low throughout the range $2<\nu<100$ GeV, and it would certainly not be chosen as a good parametrization of the data. As before, replacing g_4 by g_1 or g_3 yields a similar result. Thus we conclude that a reasonable form for $g(\nu)$ must involve at least three of the forms listed above.

B. Fits using three terms for $g(\nu)$

Since the results of the preceding section indicate that no simpler parametrization will suffice, we use fits corresponding to those in Sec. II, involving three of the functional forms $g_i(\nu)$. For each of these parametrizations we have found the best fit to the γp total-cross-section data for $2<\nu\leq 183$ GeV. Using this fit, we can calculate directly from (10) the value of $f_1(0)$ implied by the data. The results are summarized in Table II.

For the fits which have an increasing total cross section, the best-fit predicted value of $f_1(0)$ is around $-7\ \mu$b GeV, substantially more negative than the Thomson limit. Since numerical integration of both experimental data and theoretical curves is involved, the uncertainties on this number are difficult to estimate. Considering that they involve the difference of the two integrated cross sections in (16), both of which are of order 1000 μb GeV, one would expect considerable uncertainty. On the other hand, the choice of parameters is predicated on minimizing $|\sigma-\sigma_g|$ for $\nu>2$ GeV, so the contributions can differ significantly only in the resonance region, where each is less than 20 μb GeV.

In any case, the best test of the accuracy of our determination of $f_1(0)$ is to see how much the χ^2 of the fit changes if we require different values of $f_1(0)$. For this purpose, we repeated the fits with the restriction that the coefficient γ_2 be determined from (9) with $f_1(0)=-3\ \mu$b GeV. The results are also included in Table I. For the rising total cross sections, imposing this requirement caused an increase of only about 2 in χ^2, with a confidence level decline of less than 5%. For the fit in which a constant asymptotic cross section is approached as subtractive cut effects disappear, the predicted value of $f_1(0)$ for the best fit is $-3.65\ \mu$b GeV, much closer to α/M_p, and the effect of restricting $f_1(0)$ to the Thomson-limit value is correspondingly less important. In general, the quality of the fits is not sensitive to variations in $f_1(0)$ of as much as 10 μb GeV.

In all of these fits we maintained the values $N=\nu_0$ for the normalization energy and $\alpha=0.5$ for the Regge-pole intercept. The parametrizations are generally insensitive to reasonable variations from these values. Indeed, in a number of cases the parametrization is degenerate, in the sense that a change in ν_0 or α can be absorbed into a

TABLE II. Values of $f_1(0)$ obtained using best-fit parameters.

Form of σ, $\nu\to\infty$	γ_1 (μb GeV)	γ_2 (μb GeV)	i	γ_i (μb GeV)	χ^2	CL (%)	$f_1(0)$ (μb GeV)
σ_G	0.697 ± 0.050	3.95 ± 0.21	4	0.0889 ± 0.0071	78.86	72.4	−7.26
	0.740 ± 0.071	3.70 ± 0.26	4	0.0825 ± 0.0077	78.95	66.4	−3 (fixed)
σ_F	1.050 ± 0.022	3.133 ± 0.150	3	0.00621 ± 0.00050	76.38	73.7	−7.53
	1.066 ± 0.028	2.964 ± 0.163	3	0.00579 ± 0.00054	78.25	68.4	−3 (fixed)
σ_C	2.25 ± 0.12	8.62 ± 0.90	5	-7.59 ± 0.95	78.10	68.9	−3.65
	2.26 ± 0.12	8.58 ± 0.90	5	-7.71 ± 0.96	78.82	66.8	−3 (fixed)

redefinition of the other parameters. For example, if $g(\nu)$ includes g_1, g_2, g_4, and g_5, any change in N simply changes the coefficients of the four terms, without any change in the value of the function, that is, $a+b\ln(\nu/N)\equiv a'+b'\ln(\nu/N')$, with $a'=a+b\ln(N'/N)$, $b'=b$, and so on. Only for g_5 is there a specific dependence on ν_0 and α, and in fits using g_5 we have tested that dependence explicitly. We find that the fits are not strongly sensitive to variations of α within the range $0.4 \lesssim \alpha \lesssim 0.6$, or to the choice of N provided N is less than about 0.5 GeV. Variations within these ranges will change the predicted value of $f_1(0)$ to an extent comparable to those considered above.

IV. CONCLUSIONS

One can scarcely doubt the validity of the photon-nucleon dispersion relations. Their utility in calculating the real part of the spin-averaged forward Compton amplitude depends, however, on (a) the asymptotic behavior assumed for the total cross section, and (b) the confidence one has in the use of the classical Thomson limit for the required subtraction.

We have reformulated the dispersion relation in a way that substitutes better knowledge of the asymptotic cross section for the assumption of the Thomson limit. The resulting sum rule makes it possible to *calculate* $f_1(0)$ and *compare* it with the predicted value. Certain very simple forms of the asymptotic amplitude are clearly inconsistent with $f_1(0)=-\alpha/M_p$; however, any reasonably flexible form can simultaneously fit the total-cross-section data for $\nu > 2$ GeV/c and a wide range of values for $f_1(0)$.

Indeed, the existing cross-section data cannot distinguish among a wide range of asymptotic forms. Correspondingly, the dispersion relations are insensitive to significant variation in the asymptotic real/imaginary ratio. Furthermore, they cannot distinguish the value of $f_1(0)$ even within several hundred percent, and the real/imaginary ratio at low energies is correspondingly uncertain. It appears, therefore, that the dispersion relation is not significantly better calculationally than techniques relying on local properties, such as derivative analyticity relations.[13]

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[1]D. O. Caldwell *et al.*, Phys. Rev. Lett. 40, 1222 (1978).

[2]V. Chang, N. W. Dean, and K. E. Lassila, Phys. Rev. D 17, 2934 (1978).

[3]B. Margolis, Phys. Rev. D17, 1310 (1978).

[4]V. Chang and K. E. Lassila, Iowa State University report (unpublished).

[5]M. Damashek and F. J. Gilman, Phys. Rev. D 1, 1319 (1970).

[6]N. R. S. Tait and J. N. J. White, Nucl. Phys. B43, 27 (1972).

[7]C. A. Dominguez, C. F. Fontan, and R. Suaya, Phys. Lett. 31B, 365 (1970).

[8]J. W. Moffat and V. G. Snell, Phys. Rev. D 6, 859 (1972).

[9]R. J. Eden, Phys. Lett. 19, 695 (1965).

[10]W. Wiese, Phys. Rep. 13C, 53 (1974).

[11]D. O. Caldwell *et al.*, Phys. Rev. Lett. 25, 613 (1970) and Phys. Rev. D 7, 1362 (1973); A. S. Belousov *et al.*, Yad Fiz. 21, 556 (1975) [Sov. J. Nucl. Phys. 21, 289 (1975)]; T. A. Armstrong *et al.*, Phys. Rev. D 5, 1640 (1972).

[12]V. Barger and R. J. N. Phillips, Nucl. Phys. B97, 452 (1975); V. Barger, J. Luthe, and R. J. N. Phillips, *ibid.* B88, 237 (1975).

[13]J. B. Bronzan, G. L. Kane, and U. P. Sukhatme, Phys. Lett. 49B, 272 (1974).