Comment on scalar-metric-torsion gravitational theories

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The process of minimal extension of a scalar-metric theory to a scalar-metric-torsion gravitational theory, by replacing the Levi-Civita connection by a general affine connection with torsion, is shown to be nonunique. Additional parameters, in addition to the usual Brans-Dicke parameter ω, may enter the theory unless physical principals are introduced to constrain their appearance.

Aldersley has introduced[1] a generalization of general relativity with torsion (Einstein-Cartan theory)[2] which incorporates a scalar field. In a specific example, he assumes the Larangian for the scalar-metric-torsion theory to be

$$\mathcal{L} = \sqrt{-g}\phi R(\Gamma_{ij}^{k}) + \frac{a\sqrt{-g}}{\phi}\,\phi_{,i}\,\phi^{,i}\,, \tag{1}$$

where g is the determinant of the metric g_{ij}, ϕ is the scalar field, a is an arbitrary parameter, and R is the scalar curvature constructed from the affine connection with torsion

$$\Gamma_{i\,j}^{\,k} = \left\{ {k \atop ij} \right\} + S_{ij}{}^{k} - S_{j}{}^{k}{}_{i} + S^{k}{}_{ij}\,, \tag{2}$$

which contains the effect of the torsion

$$S_{ij}{}^{k} = \Gamma^{k}_{[ij]}\,. \tag{3}$$

The Lagrangian given by Eq. (1) is obtained by a minimal extension of the usual Brans-Dicke Lagrangian[3] in which the Levi-Civita connection given by $\left\{ {k \atop ij} \right\}$ is simply replaced by the general affine connection Γ^{k}_{ij}. In addition, he assumes that the parameter a is a constant independent of the scalar field. Although this assumption is not necessary, we will retain it in our discussion here.

When matter is independent of torsion, the Lagrangian given by Eq. (1) yields[1] an equation for the torsion in terms of the scalar field[4]

$$S_{ijk} = \frac{1}{4\phi}\,(\phi_{,j}\,g_{ik} - \phi_{,i}\,g_{jk})\,, \tag{4}$$

and the field equations

$$G^{ij} - \frac{1}{\phi}\left(g^{lj}\,\nabla^{\{\}}_{l}\,\phi^{,i} - g^{ij}\,\nabla^{\{\}}_{l}\,\phi^{,l}\right) + \left(a+\tfrac{3}{2}\right)\left(\frac{\phi^{,i}\,\phi^{,j}}{\phi^{2}} - \tfrac{1}{2}\,g^{ij}\,\frac{\phi_{,l}\,\phi^{,l}}{\phi^{2}}\right) = \frac{1}{\phi}\,T^{ij}\,, \tag{5}$$

$$R + \left(a+\tfrac{3}{2}\right)\left(\frac{\phi_{,i}\,\phi^{,i}}{\phi^{2}} - 2\,\frac{\nabla^{\{\}}_{i}\,\phi^{,i}}{\phi}\right) = 0.$$

The scalar field equation becomes

$$-2a\nabla^{\{\}}_{i}\phi^{,i} = T. \tag{6}$$

The Brans-Dicke scalar-tensor theory results in the identification $\omega = -\left(a+\tfrac{3}{2}\right)$.

The purpose of this paper is to show that the minimal-extension principle, which was used by Aldersley (and also by others[2]), is not unique. However, the minimal extension of the Einsteinian Lagrangian $\sqrt{-g}\,g^{ij}R_{ij}$ does seem to be unique. The addition of the scalar field term in (1) allows a much broader class of extensions than that utilized by Aldersley.

Focusing our attention on the scalar field density term, let us consider the following sequence of events:

$$\sqrt{-g}\;a\,\frac{\phi_{,i}\,\phi^{,i}}{\phi} = \sqrt{-g}\left[\nabla^{\{\}}_{j}\left(\frac{a\phi^{,j}\,\phi}{\phi}\right) - \phi\nabla^{\{\}}_{j}\left(\frac{a\phi^{,j}}{\phi}\right)\right]. \tag{7}$$

The first term on the right-hand side is in the form of an ordinary divergence and can be eliminated under the integral, so that (7) becomes

$$-\sqrt{-g}a\phi\;\nabla^{\{\}}_{j}\left(\frac{\phi^{,j}}{\phi}\right) = \sqrt{-g}\left(a\,\frac{\phi_{,j}\,\phi^{,j}}{\phi} - a\nabla^{\{\}}_{j}\;\phi^{,j}\right). \tag{8}$$

The minimal extension $\{\} \to \Gamma$ gives

$$\sqrt{-g}\left(a\,\frac{\phi^{,j}\,\phi_{,j}}{\phi} - a\,\nabla^{\Gamma}_{j}\;\phi^{,j}\right). \tag{9}$$

After expanding the covariant derivative and again eliminating ordinary divergences, we find

$$\sqrt{-g}\left(a\,\frac{\phi_{,j}\,\phi^{,j}}{\phi} + 2a\phi^{,j}\,S_{jk}{}^{k}\right). \tag{10}$$

That (1) and (10) are not the same demonstrates the nonuniqueness of the minimal-extension principle. However, the general treatment of scalar-metric-torsion theories by Aldersley allows such additional terms.

The variation of the additional term yields (under the integral)

$$2\delta(\sqrt{-g}ag^{jk}\phi_{,k}S_{jl}{}^{l}) \rightarrow 2\sqrt{-g}\,[(\tfrac{1}{2}ag^{jk}\phi^{,i}S_{il}{}^{l} - a\phi^{,j}S^{kl}{}_{l})\delta g_{jk} + a\phi^{,[j}\delta_l^{k]}\delta S_{jk}{}^{l} - a\nabla_j^{\{\}}S^{jk}{}_{k}\,\delta\phi]. \quad (11)$$

In the case of matter independent of torsion, the field equations become

$$S_{jk}{}^{l} = \frac{1}{4\phi}(a-2)\,\phi_{,[j}\delta_{k]}^{l},$$

$$G^{jk} + \frac{1}{8\phi^2}(3a^2 - 4a + 12)(\phi^{,j}\phi^{,k} - \tfrac{1}{2}g^{jk}\phi_{,l}\phi^{,l}) + \frac{1}{\phi}(g^{jk}\nabla_l^{\{\}}\phi^{,l} - \phi^{,j;k}) = \frac{T^{jk}}{\phi}, \quad (12)$$

$$-\tfrac{1}{4}(3a^2 - 4a)\nabla_j^{\{\}}\phi^{,j} = T.$$

Except for the torsion equation, the identification

$$2\omega + 3 = -\tfrac{1}{4}(3a^2 - 4a) \quad (13)$$

reproduces the Brans-Dicke equations. In the process, the single parameter a has now become double valued, although the final field equations are the same as Eqs. (4)–(6) with the identifications (13).

Lest we think that the nonunique extension of the Lagrangian density always reproduces the same field equations, the scalar field part of the Brans-Dicke Lagrangian

$$\mathcal{L}_S = \sqrt{-g}\,a\frac{\phi_{,j}\phi^{,j}}{\phi} \quad (14)$$

gives the same contribution to the Euler-Lagrange equations as[5]

$$\mathcal{L}_S' = \mathcal{L}_S - [\sqrt{-g}\,b(\phi)\phi^{,j}]_{,j} = \mathcal{L}_S - \sqrt{-g}\,b'\,\phi_{,j}\phi^{,j} - \sqrt{-g}\,b\nabla_j^{\{\}}\phi^{,j}, \quad (15)$$

where $b(\phi)$ is an arbitrary function of ϕ. The minimal extension $\{\} \rightarrow \Gamma$ leads to

$$\mathcal{L}_S'' = \mathcal{L}_S - \sqrt{-g}\,b'\,\phi_{,j}\phi^{,j} - \sqrt{-g}\,b\nabla_j^{\Gamma}\phi^{,j} = \mathcal{L}_S - (\sqrt{-g}\,b\,\phi^{,j})_{,j} + \sqrt{-g}\,b\phi^{,j}S_{jk}{}^{k}, \quad (16)$$

which has the same contributions to the Euler-Lagrange equations as

$$\mathcal{L}_S''' = \mathcal{L}_S + 2\sqrt{-g}\,b(\phi)\,\phi^{,j}S_{jk}{}^{k}. \quad (17)$$

The variation of the second term on the right-hand side gives contributions similar to Eq. (11) with a replaced by b. In the case of matter independent of torsion, the field equations now become

$$S_{jk}{}^{l} = \frac{1}{4\phi}(b-2)\phi_{,[j}\delta_{k]}^{l},$$

$$G^{jk} + \frac{1}{\phi}(g^{jk}\nabla_l^{\{\}}\phi^{,l} - \phi^{,j;k}) + \frac{1}{\phi^2}[\tfrac{3}{8}(b-2)^2 + a] \times (\phi^{,j}\phi^{,k} - \tfrac{1}{2}g^{jk}\phi_{,l}\phi^{,l}) = \frac{T^{jk}}{\phi}, \quad (18)$$

$$-(2a + \tfrac{3}{4}b^2 - 3b)\nabla_j^{\{\}}\phi^{,j} - \tfrac{3}{4}(b-2)\,b'\,\phi_{,j}\phi^{,j} = T.$$

Although the formal identification

$$-(2a + \tfrac{3}{4}b^2 - 3b) = 2\omega + 3 \quad (19)$$

reproduces the appearance of Nordtvedt scalar-tensor theory[6] [or a Brans-Dicke theory when $b \neq b(\phi)$], the analogy is meaningless since both a and b are arbitrary parameters. It is interesting to note that for the choice $b = 2$ the torsion vanishes and the field equations (18) reduce identically to the Brans-Dicke theory. Thus if the minimal extension should reproduce the Brans-Dicke equations for matter independent of torsion, then the choice of $b = 2$ in Eq. (17) gives the "correct" minimally extended Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(\phi R(\Gamma) + \frac{a}{\phi}\phi_{,j}\phi^{,j} + 4\phi^{,j}S_{jk}{}^{k}\right). \quad (20)$$

Without such additional physical insight, there does not seem to be any way to avoid the additional b parameter in the theory, in which case the first of Eqs. (18) indicates that b must be associated with a propagating torsion field.

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[1]S. J. Aldersley, Phys. Rev. D 15, 3507 (1977); 17, 1674 (E) (1978).

[2]F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976) and references therein.

[3]C. H. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[4]The notation is that of Ref. 1, except that we denote covariant derivatives with respect to the affine connection by ∇^{Γ} and those with respect to the Christoffel symbol by $\nabla^{\{\}}$ or by a semicolon. Commas denote ordinary partial derivatives.

[5]S. J. Aldersley (private communications).

[6]K. Nordtvedt, Jr., Astrophys. J. 161, 1059 (1970).