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Color-charge algebras in Adler's chromodynamics

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We show that the color-charge algebra in the three-quark sector generated by the matrices of the fundamental representation of U(n) does not have the trace properties required in Adler's extension of chromodynamics. We also discuss a diagrammatic representation of algebras generated by quark and antiquark charges in general, and an embedding of the N-quark algebra in the symmetric group $\mathcal{S}_{N+1}$.

Let $Q_1^a, \ldots, Q_N^a$, $a=1,\ldots,n^2-1$, be a system of N SU(n) charges which satisfy the commutation relations

$$[Q_m^a, Q_{m'}^b] = i f^{abc} Q_m^c \delta_{m,m'}, \tag{1}$$

where f^{abc} are the structure constants of SU(n). Such a system has been considered in connection with the problem of determining the potential due to N static quarks and antiquarks which act as sources for colored fields.[1,2]

Adler[1] introduces the following representation of the commutation relations (1):

$$\begin{aligned} Q_m^a &= \underline{1}_1 \otimes \cdots \otimes \tfrac{1}{2}\lambda_m^a \otimes \cdots \otimes \underline{1}_{\bar{m}} \otimes \cdots \otimes \underline{1}_N, \\ Q_{\bar{m}}^a &= \underline{1}_1 \otimes \cdots \otimes \underline{1}_m \otimes \cdots \otimes \tfrac{1}{2}(-\lambda_{\bar{m}}^{*a}) \otimes \cdots \otimes \underline{1}_N, \end{aligned} \tag{2}$$

where the mth particle is a quark and the $\bar{m}$th particle is an antiquark. λ^a are the $n \times n$ matrices of the fundamental representation of U(n) obtained by adjoining $\lambda^0 \equiv (2/n)^{1/2}\underline{1}$ to the standard SU(n) λ matrices. They obey

$$\begin{aligned} &\mathrm{tr}\lambda^a\lambda^b = 2\delta^{ab}, \\ &[\lambda^a, \lambda^b] = 2 i f^{abc}\lambda^c, \\ &\{\lambda^a, \lambda^b\} = 2d^{abc}\lambda^c. \end{aligned} \tag{3}$$

Now, let us define[3]

$$Q_m \equiv \lambda_0^a \otimes Q_m^a, \quad m=1,\ldots,N. \tag{4}$$

The N-particle color-charge algebra, as defined by Adler,[1] is the minimal algebra which contains the matrices Q_m and is closed under commutation. This is a Lie algebra, and Adler's outer- or P-product is simply a commutator

$$\begin{aligned} P(u,v) &= \tfrac{1}{2}[u,v], \\ P^a(u,v) &= \tfrac{1}{4}\mathrm{tr}_0(\lambda_0^a \otimes \underline{1}_1 \otimes \cdots \otimes \underline{1}_N \circ [u,v]), \end{aligned} \tag{5}$$

where u and v are any two elements of the algebra and tr_0 denotes the trace with respect to the "zeroth" matrix in the direct product. Adler further introduces an inner- or S-product

$$S(u,v) = \tfrac{1}{4}\mathrm{tr}_0\{u,v\}, \tag{6}$$

and demands that the condition

$$S(u, P(v,w)) = S(P(u,v), w) \tag{7}$$

hold over the algebra. He found[1] that (7) was indeed satisfied by the two-quark algebra. In the quark-antiquark and two-antiquark sectors he found that (7) could be made to hold by rescaling the zeroth component of the antiquark charges $Q^0 \to (1-\frac{1}{2}n^2)Q^0$.

We find, by explicitly constructing the algebra, that (7) is not satisfied in the three-quark sector. This construction, which is described in the next section, is most efficiently carried out by exploiting an isomorphism of the N-quark algebra with a subalgebra of the group algebra of the symmetric group $\mathcal{S}_{N+1}$. This isomorphism is, however, special to quark charges. In the final section we describe a useful diagrammatic representation of the algebras which has a much wider range of applicability.

$\mathcal{S}_{N+1}$ Isomorphism and results for three quarks

Consider N quark charges represented by the matrices (4). Using the completeness relation

$$(\lambda^a)_j^i(\lambda^a)_l^k = 2\delta_l^i\delta_j^k, \tag{8}$$

we have

$$\begin{aligned} (Q_m)^{i_0 i_1 \cdots i_N}_{j_0 j_1 \cdots j_N} &= (\lambda_0^a)^{i_0}_{j_0}(\underline{1}_1)^{i_1}_{j_1} \cdots (\tfrac{1}{2}\lambda_m^a)^{i_m}_{j_m} \cdots (\underline{1}_N)^{i_N}_{j_N} \\ &= \delta^{i_0}_{j_m}\delta^{i_m}_{j_0}\prod_{\nu \neq 0,m}\delta^{i_\nu}_{j_\nu}. \end{aligned} \tag{9}$$

This implies that we can represent the matrix Q_m by the permutation $(0m)$ (in cycle notation) on the numbers $0,1,\ldots,N$, and that there is an isomorphism between matrix multiplication and the usual composition[4] of permutations. For example, the product Q_1Q_2 is represented by the permutation $(02)(01)=(012)$. Thus Adler's N-quark color charge algebra is the minimal

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subalgebra of the group algebra of $\mathcal{S}_{N+1}$ generated by commuting the transpositions $(01), \ldots, (0N)$ with one another.

To construct the three-quark algebra we decompose the group algebra of $\mathcal{S}_4$ into mutually orthogonal subalgebras corresponding to each of the irreducible representations of $\mathcal{S}_4$. This decomposition is a standard exercise in Young symmetrizers[4]; it yields five subspaces with dimensions 1, 3^2, 2^2, 3^2, and 1.

The projections of the charges Q_m on the 2^2-dimensional subspace give rise to an SU(2) algebra. Perhaps the easiest way to see this is to use Table 7-3 of Ref. 4 which gives explicit irreducible matrix representations of $\mathcal{S}_4$; thus

$$\begin{aligned}
&(01) \to \sigma_3\,,\\
&(02) \to -\tfrac{1}{2}\sigma_3 - \frac{\sqrt{3}}{2}\sigma_1\,, \qquad (10)\\
&(03) \to -\tfrac{1}{2}\sigma_3 + \frac{\sqrt{3}}{2}\sigma_1\,,
\end{aligned}$$

where σ_1 and σ_3 are Pauli matrices which generate SU(2) under commutation.

$\mathcal{S}_4$ has two irreducible representations by 3×3 matrices. It turns out, however, that the minimal subalgebra generated within the corresponding subspaces is a *single* SU(3) algebra. This is a consequence of the fact that each charge can be represented in the two subspaces by matrices that are conjugate [i.e., that belong to $\underline{3}$ and $\underline{\bar{3}}$ of U(3)]. Since conjugate matrices have identical commutation relations, the sums of the projected charges in the two subspaces generate a single SU(3).[5]

The center of the group algebra is spanned by five Abelian elements which are simply the sums of permutations within each of the five classes of $\mathcal{S}_4$. Since these elements cannot be generated by commutators, the only one which belongs to the minimal algebra is the sum of transpositions (which contains the charges one started out with).

Thus the structure of the three-quark algebra is $\mathrm{SU}(3)\oplus\mathrm{SU}(2)\oplus\mathrm{U}(1)$. Tables I and II give a basis which diagonalizes the minimal algebra, and the S-products are listed in Tables III and IV. From our construction, it is obvious that S-products between elements of the SU(2) and elements of the SU(3) must vanish.[6] Furthermore, the Abelian element x has zero projection on the 2^2-dimensional subspace[7] and hence the S-products between it and elements of the SU(2) vanish.

We note that the trace condition (7) is violated by nonvanishing S-products between x and the elements of the SU(3) algebra. It is also systematically violated within the SU(3) itself since Table IV shows that

TABLE I. The 12 elements of $\mathcal{S}_4$ that belong to the three-quark algebra.

$Q_1=(01)$;	$s_0=(321)-(123)$
$Q_2=(02)$;	$s_1=(023)-(320)$
$Q_3=(03)$;	$s_2=(031)-(130)$
	$s_3=(012)-(210)$
$r_1=(23)$;	
$r_2=(31)$;	$t_1=(0312)+(2130)-(0123)-(3210)$
$r_3=(12)$;	$t_2=(0123)+(3210)-(0231)-(1320)$

$$S(z_a, P(z_b, z_c)) - S(P(z_a, z_b), z_c) = \frac{i}{4} d_{(3)}^{ace} f_{(3)}^{edb} y_d\,. \qquad (11)$$

The matrix elements of S-products in the color-singlet channel for $n=3$ can also be read off Tables III and IV. We note that the trace condition does not hold with this restriction to color singlets except in certain SU(2) subalgebras. We have also verified that these violations cannot be removed by rescaling the zeroth components of the charges.

We conclude that the ansatz (2) for the color charges is unsatisfactory, at the three-particle level, for the purposes of Ref. 1.

Isomorphism to diagrams

The color charges (9) can be conveniently

TABLE II. A basis which diagonalizes the three-quark algebra.

U(1):	$x = Q_1+Q_2+Q_3+r_1+r_2+r_3$
SU(2):	$y_1 = \frac{1}{8\sqrt{3}}(Q_1-Q_2+r_1-r_2+t_1)$
	$y_2 = \frac{1}{8\sqrt{3}}(s_0+s_1+s_2+s_3)$
	$y_3 = \frac{1}{24}(Q_1+Q_2-2Q_3+r_1+r_2-2r_3+t_1+2t_2)$
	$[y_a, y_b] = i\epsilon_{abc}y_c$
SU(3):	$z_1 = -\frac{1}{4}(Q_3-r_3)$
	$z_2 = \frac{i}{8}(s_0-s_1-s_2+s_3)$
	$z_3 = \frac{1}{8}(Q_1-Q_2+r_1-r_2-t_1)$
	$z_4 = \frac{1}{4}(Q_1-r_1)$
	$z_5 = \frac{i}{8}(s_0+s_1-s_2-s_3)$
	$z_6 = -\frac{1}{4}(Q_2-r_2)$
	$z_7 = \frac{i}{8}(s_0-s_1+s_2-s_3)$
	$z_8 = \frac{1}{8\sqrt{3}}(-Q_1-Q_2+2Q_3-r_1-r_2+2r_3+t_1+2t_2)$
	$[z_a, z_b] = if_{(3)}^{abc}z_c$

TABLE III. Operators in the three-quark Hilbert space which provides a basis for the S-product table. For $n=3$, their expectation values in the color-singlet state are $\langle\alpha_i\rangle=1$ and $\langle\beta_i\rangle=0$.

$$\alpha_1=1$$
$$\alpha_2=-\tfrac{1}{3}[(12)+(23)+(31)]$$
$$\alpha_3=\tfrac{1}{2}[(123)+(321)]$$
$$\beta_1=\frac{1}{2\sqrt3}[(23)-(31)]$$
$$\beta_2=\frac{i}{2\sqrt3}[(321)-(123)]$$
$$\beta_3=\tfrac{1}{6}[(23)+(31)]-\tfrac{1}{3}(12)$$
$$[\alpha_i,\alpha_j]=[\alpha_i,\beta_j]=0$$
$$[\beta_i,\beta_j]=i\epsilon_{ijk}\beta_k$$

drawn as "duality diagrams" by representing each Kronecker δ by a directed line. Let us illustrate this by writing all indices in (4) for the $N=2$ quark-antiquark algebra:

$$(Q_1)^{i_0i_1i_2}_{j_0j_1j_2}=(\lambda^a)^{i_0}_{j_0}(\tfrac{1}{2}\lambda^a)^{i_1}_{j_1}\delta^{i_2}_{j_2}=\delta^{i_0}_{j_1}\delta^{i_1}_{j_0}\delta^{i_2}_{j_2}, \tag{12}$$

$$(Q_{\bar 2})^{i_0i_1i_2}_{j_0j_1j_2}=(\lambda^a)^{i_0}_{j_0}\delta^{i_1}_{j_1}(-\tfrac{1}{2}\lambda^{*a})^{i_2}_{j_2}=-\delta^{i_0}_{i_2}\delta^{i_1}_{j_1}\delta^{j_2}_{j_0}. \tag{13}$$

TABLE IV. S-product table for the diagonalized three-quark algebra.

$$S(x,x)=3n(\alpha_1+\alpha_3)-12\alpha_2$$
$$S(x,y_a)=0$$
$$S(x,z_a)=-2\gamma_a$$
$$S(y_a,y_b)=\frac{n}{48}(\alpha_1-\alpha_3)\delta_{ab}$$
$$S(y_a,z_b)=0$$
$$S(z_a,z_b)=\tfrac{1}{2}d^{abc}_{(3)}\gamma_c$$
$$\gamma_0=\tfrac{1}{8}(\tfrac{3}{2})^{1/2}(n+\alpha_2)$$
$$\gamma_1=\tfrac{1}{8}(-\alpha_2-n\alpha_3+4\beta_3)$$
$$\gamma_2=\gamma_5=\gamma_7=-\frac{\sqrt3}{4}\beta_2$$
$$\gamma_3=-\frac{\sqrt3}{4}\beta_1$$
$$\gamma_4=\tfrac{1}{8}(\alpha_2+n\alpha_3+2\sqrt3\beta_1+2\beta_3)$$
$$\gamma_6=\tfrac{1}{8}(-\alpha_2-n\alpha_3+2\sqrt3\beta_1-2\beta_3)$$
$$\gamma_8=\frac{\sqrt3}{4}\beta_3$$

One way of drawing the duality diagram for Q_1 is given in Fig. 1(a). Another way, more convenient for carrying out multiplications, is given in Fig. 1(b). The multiplication uv is carried out by placing the diagram for u to the left of the diagram for v and joining the corresponding lines. Each closed loop gives a factor $\delta^i_i=n$. For example, from Fig. 1(c), $Q_{\bar 2}Q_{\bar 2}=-nQ_{\bar 2}$.

P- and S-products can be quickly computed this way.[8] Tr_0 in the S-product (6) is accomplished by looping the outgoing zeroth line back into the ingoing zeroth line.

The generalization of the color-charge algebras proposed by Adler[9] can similarly be cast in diagrammatic form. We describe two special cases of this generalization briefly using diagrams. In each instance, one has two charges $Q^a_\pm$ for each particle, and two types of outer product,

$$P=\tfrac{1}{2}[u,v],\ {}^*P=\tfrac{1}{2}[{}^*u,{}^*v], \tag{14}$$

where $u\equiv\lambda^a u^a$ and ${}^*u\equiv(-\lambda^{*a})u^a$.

In the first alternative, the P product (5) is replaced by a weighted average

$$P(u_\pm,v_\pm)\to(\tfrac{1}{2}\pm\gamma_\pm)P(u_\pm,v_\pm)+(\tfrac{1}{2}\mp\gamma_\pm)[{}^*P({}^*u_\pm,{}^*v_\pm)], \tag{15}$$

where $\gamma_\pm$ are constants. [In Adler's notation,[9] this is the case $\delta=0$, $\epsilon_\pm=\beta_\pm=(2/n)^{1/2}\gamma_\pm$.] The outer product (15) is reasonably easy to represent diagrammatically. A diagram representing ${}^*u$ differs from one representing u insofar as its zeroth line is an antiquark line.

One can go further and weight the zero components, P^0, u^0, v^0, etc, differently from the other n^2-1 components. To do this diagrammatically,

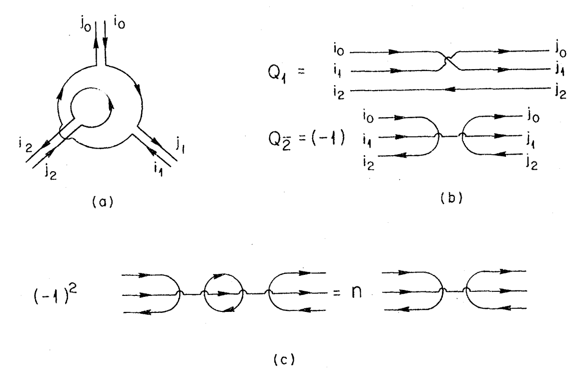

FIG. 1. (a) A representation of Eq. (12) as a duality diagram. (b) Another way of drawing the duality diagrams representing Eqs. (12) and (13). (c) Diagrammatic computation of the product $Q_{\bar 2}Q_{\bar 2}$.

say for the P-product, construct a "contracted" version of $P(u,v)$ by replacing $P(u,v)$ by $\underline{1}_0 \otimes \mathrm{tr}\,_0 P(u,v)$. Then replace $P(u,v)$ in Eq. (15) by a weighted sum of $P(u,v)$ and its contracted version. Do the same for ${}^*P({}^*u, {}^*v)$. The new outer product now has independently normalized zero components and, in Adler's notation,[9] independent parameters $\epsilon_\pm$ and $\gamma_\pm$.

Modifications of the S-product can be dealt with in similar fashion.

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[1]S. L. Adler, Phys. Rev. D 17, 3212 (1978); 18, 411 (1978).

[2]R. Giles and L. McLerran, M.I.T. Report No. 711, 1978 (unpublished).

[3]These Q_m's differ by a factor of 2 from the charges used to construct the two-particle algebras in Ref. 1.

[4]A good discussion of the symmetric group can be found in M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962).

[5]Thus, if e_3 and $e_{\bar{3}}$ are projectors onto the two subspaces ($e_3{}^2 = e_3$, $e_{\bar{3}}{}^2 = e_{\bar{3}}$, $e_3 e_{\bar{3}} = e_{\bar{3}} e_3 = 0$), and λ^a are U(3) matrices, we have $[e_3\lambda^a + e_{\bar{3}}(-\lambda^{*a}),\ e_3\lambda^b + e_{\bar{3}}(-\lambda^{*b})] = 2if^{abc}_{(3)}[e_3\lambda^c + e_{\bar{3}}(-\lambda^{*c})]$.

[6]Note however that anticommutators take one out of the minimal algebra: e.g., for the SU(2), $\{e_2\sigma_i,\ e_2\sigma_j\} = 2e_2\delta_{ij}$; and for the SU(3), $\{e_3\lambda^a + e_{\bar{3}}(-\lambda^{*a}),\ e_3\lambda^b + e_{\bar{3}}(-\lambda^{*b})\} = 2\, d^{abc}_{(3)}[e_3\lambda^c - e_{\bar{3}}(-\lambda^{*c})]$.

[7]The projections of the Abelian elements on the five subspaces can be read off the character tables (Ref. 9) of $\mathcal{S}_4$.

[8]Diagrammatic methods for computing P-products in terms of f and d symbols can be found in P. Cvitanović, Phys. Rev. D 14, 1536 (1976).

[9]See the second paper of Ref. 1, second note added in proof.