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# Some solutions for relativistic vortices interacting through a scalar field

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Attempts are made to obtain solutions for the nonlinear coupled equations obtained by Lund and Regge in connection with dynamics of relativistic vortices (or strings) interacting through a scalar field.

## I. INTRODUCTION

A recent study of the dynamics of relativistic vortices (equivalently strings) interacting through a scalar field has led to a set of two coupled Lorentz-invariant, nonlinear partial differential equations in two independent variables:[1]

$$\frac{\partial^2\theta}{\partial\tau^2}-\frac{\partial^2\theta}{\partial\sigma^2}+c\sin\theta\cos\theta+\frac{\cos\theta}{\sin^3\theta}\left|\left[\frac{\partial\lambda}{\partial\tau}\right]^2-\left[\frac{\partial\lambda}{\partial\sigma}\right]^2\right|=0, \tag{1.1}$$

$$\frac{\partial}{\partial\tau}\left(\cot^2\theta\frac{\partial\lambda}{\partial\tau}\right)=\frac{\partial}{\partial\sigma}\left(\cot^2\theta\frac{\partial\lambda}{\partial\sigma}\right). \tag{1.2}$$

Equations (1.1) and (1.2) can be rewritten as

$$\frac{\partial^2\theta}{\partial\tau^2}-\frac{\partial^2\theta}{\partial\sigma^2}+c\sin\theta\cos\theta+\frac{\sin\theta}{\cos^3\theta}\left[\frac{\partial\chi}{\partial\sigma}\right]^2-\left[\frac{\partial\chi}{\partial\tau}\right]^2=0, \tag{1.3}$$

$$\frac{\partial}{\partial\tau}\left(\tan^2\theta\frac{\partial\chi}{\partial\tau}\right)=\frac{\partial}{\partial\sigma}\left(\tan^2\theta\frac{\partial\chi}{\partial\sigma}\right), \tag{1.4}$$

where $\lambda$ and $\chi$ are connected by relations

$$\cot^2\theta\frac{\partial\lambda}{\partial\tau}=\frac{\partial\chi}{\partial\sigma},\qquad \cot^2\theta\frac{\partial\lambda}{\partial\sigma}=\frac{\partial\chi}{\partial\tau}. \tag{1.5}$$

Solutions of (1.1) and (1.2) for the case when $\theta$ and $\lambda$ are both functions of $\sigma$ have been discussed by Lund and Regge.[1] In the present paper, we seek solutions of the same equations under a less stringent condition that $\theta$ is a function of $\lambda$. It gives rise to two different cases: In one case the equations can be completely solved and in the other case the equations have been reduced to a single ordinary differential equation, which could be solved numerically by a computer. The same applies to the case when $\theta$ is a function of $\chi$.

## II. SOLUTIONS FOR $\theta=\theta(\lambda)$

Assume that

$$\theta=\theta(\lambda). \tag{2.1}$$

Set

$$\gamma=\int\cot^2\theta\,d\lambda,$$

i.e., $\theta$ and $\lambda$ are functions of $\gamma$ such that

$$\frac{d\lambda}{d\gamma}=\tan^2\theta. \tag{2.2}$$

Equation (1.2) is then solved as

$$\gamma=U(u)+V(v), \tag{2.3}$$

where

$$u=\tau-\sigma,\qquad v=\tau+\sigma, \tag{2.4}$$

and $U$ and $V$ are some functions.

Using (2.1), (2.2), and (2.3), Eq. (1.1) reduces to

$$4Y_uV_v\left(\theta_{\gamma\gamma}+\frac{\sin\theta}{\cos^3\theta}\right)+c\sin\theta\cos\theta=0, \tag{2.5}$$

where $U_u\equiv dU/du$, $V_v\equiv dV/dv$, $\theta_\gamma\equiv d\theta/d\gamma$, and so on.

From (2.5) and the fact that $\theta$ is a function of $\gamma$, we get that $U_uV_v$ is a function of $\gamma$ and hence

$$\frac{(U_uV_v)_u}{(U_uV_v)_v}=\frac{\gamma_u}{\gamma_v}=\frac{U_u}{V_v},$$

i.e.,

$$\frac{U_{uu}}{V_{U_u^2}}=\frac{V_{vv}}{V_v^2}=\text{constant}=-m\quad(\text{say}). \tag{2.6}$$

### Case I. $m=0$

Here using (2.3), (2.5), and (2.6), we get

$$U=Au,\qquad V=Bv,$$

$A, B$ being constants. Therefore,

$$\gamma=(A+B)\tau+(B-A)\sigma. \tag{2.7}$$

Equation (2.5) reduces to

$$4AB\left(\theta_{\gamma\gamma}+\frac{\sin\theta}{\cos^3\theta}\right)+c\sin\theta\cos\theta=0. \tag{2.8}$$

Equation (2.8) means that we must have $A\neq0$, $B\neq0$. Therefore, owing to (2.7), we can, by suitable transformation of coordinates, set $\gamma=\tau$ or $\gamma=\sigma$ without loss of generality. ($\gamma=\tau$ means $A=B=\frac{1}{2}$; $\gamma=\sigma$ means $A=-B=-\frac{1}{2}$.) Thus we get the following two equations:

$$\int \frac{d\theta}{(K - 1/\cos^2\theta - c\cos 2\theta)^{1/2}} = \tau ,$$
$$\lambda = \int \tan^2\theta \, d\tau , \tag{2.9}$$

and from (1.5), $\chi = \sigma$, or

$$\int \frac{d\theta}{[K - 1/(\cos^2\theta)^{1/2} + c\cos 2\theta]^{1/2}} = \sigma ,$$
$$\lambda = \int \tan^2\theta \, d\sigma , \tag{2.10}$$

and from (1.5), $\chi = \tau$, where $K$ is a constant.

### Case II. $m \neq 0$

Here from (2.3), (2.5), and (2.6), we get

$$\gamma = \frac{1}{m} \ln(uv) , \tag{2.11}$$

$$\frac{4e^{-\gamma}}{m^2}\left(\theta_{\gamma\gamma} + \frac{\sin\theta}{\cos^3\theta}\right) + c \sin\theta\cos\theta = 0 . \tag{2.12}$$

Equation (2.11) gives $\gamma$. Equation (2.12) is an ordinary differential equation with one unknown function $\theta$ to be determined, which could be done numerically by a computer. Once $\theta$ is determined in terms of $\gamma$, $\lambda$ could be determined by using (2.2) and $\chi$ is given by

$$\chi = \frac{1}{m} \mathrm{lu} \frac{v}{u} . \tag{2.13}$$

## III. SOLUTIONS FOR $\theta = \theta(\chi)$

Assume that

$$\theta = \theta(\chi) . \tag{3.1}$$

As before, set

$$\delta = \int \tan^2\theta \, d\chi ,$$

i.e., $\theta$ and $\chi$ are functions of $\delta$ such that

$$\frac{d\chi}{d\delta} = \cot^2\theta . \tag{3.2}$$

Treating (1.3) and (1.4) the same way that (1.1) and (1.2) were treated in Sec. II, we see that the solutions can be put into one of the three following forms:

case A:

$$\int \frac{d\theta}{(K' + 1/\sin^2\theta - c\cos 2\theta)^{1/2}} = \tau ,$$
$$\chi = \int \tan^2\theta \, d\tau , \tag{3.3}$$
$$\lambda = \sigma ;$$

case B:

$$\int \frac{d\theta}{(K' + 1/\sin^2\theta + c\cos 2\theta)^{1/2}} = \sigma ,$$
$$\chi = \int \tan^2\theta \, d\sigma , \tag{3.4}$$
$$\lambda = \tau ,$$

where $K'$ is a constant;

case C:

$\theta$ is given by the coupled equation,

$$\delta = \frac{1}{m} \ln(uv) , \tag{3.5a}$$

$$\frac{4e^{-\delta}}{m^2}\left(\theta_{\delta\delta} + \frac{\sin\theta}{\cos^3\theta}\right) + c \sin\theta\cos\theta = 0 . \tag{3.5b}$$

Once $\theta$ and $\delta$ are determined from (3.5), $\chi$ is given by (3.2) and $\lambda$ is given by

$$\lambda = \frac{1}{m} \ln \frac{u}{v} . \tag{3.6}$$

## IV. CONCLUSION

Thus, solutions of (1.1) and (1.2) when $\theta$ is a function of $\lambda$ or $\chi$ are of two types:[2] (a) $\theta$ is a function of linear combination of $\tau$ and $\sigma$ (b) $\theta$ is a function of $\tau^2 - \sigma^2$. Type (a) solutions are (2.9), (2.10), (3.3), and (3.4). On the other hand, (2.11), (2.12), and (2.13) together give one type (b) solution and the other type (b) solution is given by (3.5) and (3.6).

Also for type (b) solutions, i.e., for solutions for which $\theta$ is a function of $\tau^2 - \sigma^2$, it is interesting to note that as $|\tau^2 - \sigma^2| \to \infty$, $\theta \to n\pi/2$, where $n$ is an integer. This can be seen as follows: When $\theta$ is a function of $\lambda$, from (2.11) we see that as $|\tau^2 - \sigma^2| \to \infty$, $\gamma \to \infty$, and from (2.12), as $\gamma \to \infty$, $\theta \to n\pi/2$. Similarly, when $\theta$ is a function of $\chi$, we see from (3.5) that as $|\tau^2 - \sigma^2| \to \infty$, $\delta \to \infty$, and as $\delta \to \infty$, $\theta \to n\pi/2$.

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[1]F. Lund and T. Regge, Phys. Rev. D 14, 1524 (1976).

[2]It may be noted from (1.5) that saying $\theta$ is a function of $\lambda$ is equivalent to saying that $\chi$ satisfies the wave equation $\chi_{\tau\tau} - \chi_{\sigma\sigma} = 0$, and saying $\theta$ is a function of $\chi$ is equivalent to saying that $\lambda$ satisfies the wave equation $\lambda_{\tau\tau} - \lambda_{\sigma\sigma} = 0$.