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Color–de Sitter and color-conformal superalgebras

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A generalization of superalgebras leading to a new Lie structure of color algebras is proposed. The three-color extensions of osp(4;1) (super–de Sitter) and su(2,2;1) (Wess-Zumino) superalgebras are presented.

I. INTRODUCTION

There are two known ways of introducing a genuine unification of space-time and internal symmetries via the supersymmetry scheme:

(i) graded de Sitter algebras osp(4; N), admitting O(N) as an internal symmetry,[1,2]

(ii) graded Haag-Lopuszanski-Sohnius conformal algebras su(2, 2; N), with U(N) as integral symmetry group.[1,3]

In these unification schemes flavor and color groups are treated *in the same way*—as the subgroups of O(N) or U(N), respectively. It seems desirable to distinguish flavor and color degrees of freedom by introducing *different structures of flavor and color algebras*. An interesting effort in this direction was made by Günaydin and Gürsey[4] by introducing nonassociative octonionic color algebras.[5,6] In this paper we propose for color a new algebraic structure, but we stay in the framework of associative algebras.

The conventional Z_2 grading, leading to superalgebras containing only fermionic and bosonic generators, can be generalized in several ways.[7] In this paper we shall consider one of the simplest generalizations based on a $Z_2\oplus\cdots\oplus Z_2\oplus Z_2$ grading.[8,9] We call these new algebraic structure *color superalgebras*.[10]

In particular we define the color–de Sitter and color-conformal superalgebras in such a way that the charges generating the color group are *parabosons* and the spinorial supercharges are *parafermions*. We see, therefore, that in our scheme *the color algebra is not a Lie algebra*, because its different generators have anticommutation relations.

The mathematical aspects and some results on classification of color superalgebras are considered in Ref. 11. In the present paper, after introducing the color space and the corresponding generalization of the grading structure, we review in some detail the three-color case (Sec. II). Further we present the explicit realizations of the color–de Sitter (Sec. III) and the color-conformal superalgebra with three colors (Sec. IV). Finally in Sec. V we discuss ways in which the $Z_2\oplus Z_2\oplus Z_2$ color algebras have different implications from conventional superalgebras.

II. COLOR SPACE AND COLOR STRUCTURES

In order to introduce the color degrees of freedom we consider a set of generators $X_{i,\vec{\alpha}}$, where $\vec{\alpha}=(\alpha_1,\ldots,\alpha_n)$ is an n-component vector whose components are integer numbers modulo two ($\alpha_r=0$ or 1; $r=1,\ldots,n$). The 2^n vectors $\vec{\alpha}$ form a vector space, which we call a *color space with n colors*. We denote by $X_{\vec{\alpha}}$ ($X_{i,\vec{\alpha}}\in X_{\vec{\alpha}}$) the set of generators corresponding to the same $\vec{\alpha}$. A *color superalgebra X* is given by a bilinear map, denoted by $\langle\,,\,\rangle$ of $X\times X\to X$ with the following three conditions:

(i) closure relations,

$$\langle X_{\vec{\alpha}},X_{\vec{\beta}}\rangle\subset X_{\vec{\alpha}+\vec{\beta}} \tag{2.1}$$

[it follows from (2.1) that the color algebra with n colors is a $Z_2\oplus\cdots\oplus Z_2$ (with n Z_2's) graded algebra];

(ii) symmetry properties,

$$\langle X_{i,\vec{\alpha}},X_{j,\vec{\beta}}\rangle=(-1)^{(\vec{\alpha},\vec{\beta})+1}\langle X_{j,\vec{\beta}},X_{i,\vec{\alpha}}\rangle\,, \tag{2.2}$$

(iii) generalized Jacobi identity,

$$\langle X_{i,\vec{\alpha}},\langle X_{j,\vec{\beta}},X_{k,\vec{\gamma}}\rangle\rangle(-1)^{(\vec{\alpha},\vec{\gamma})}+\text{cycl perm}=0\,. \tag{2.3}$$

One can show[7] that there are only two nonequivalent choices for the scalar product $(\vec{\alpha},\vec{\beta})$ which are consistent with the relations (2.1)–(2.3):

(i) symplectic antisymmetric scalar product,

$$(\vec{\alpha},\vec{\beta})_a=\alpha_1\beta_2-\alpha_2\beta_1+\cdots+\alpha_{n-1}\beta_n-\alpha_n\beta_{n-1} \tag{2.4}$$

[the scalar product (2.4) can be defined only if n is even];

(ii) Euclidean symmetric scalar product,

$$(\vec{\alpha},\vec{\beta})_s=\alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_n\beta_n\,. \qquad (2.5)$$

In this case n can be any positive integer.

The structure of the color superalgebra is determined by

(i) the choice (2.4) or (2.5) of scalar product,

(ii) the number of colors ($n=1,2,3,\ldots$).

We denote the color structures with the scalar product (2.4) by $C(n;a)$ and with the scalar product (2.5) by $C(n;s)$. In particular if $n=1$, we obtain the conventional superalgebras.[1]

The basic difference between the $C(n,a)$ algebras and the $C(n,s)$ algebras is that for the former the "diagonal" product

$$\langle X_{\vec{\alpha},i},X_{\vec{\alpha},j}\rangle=[X_{\vec{\alpha},i},X_{\vec{\alpha},j}]$$

is always a commutator, while for the latter it can be a commutator or an anticommutator. The $C(n,a)$ algebras appear also as subalgebras of the $C(n,s)$ algebras as will be seen from our examples.

We shall now discuss the simplest color structures.

A. Two colors

We have

$$X_{\vec{\alpha}}:\ X_{(0,0)},X_{(1,0)},X_{(0,1)},X_{(1,1)}\,.$$

If $n=2$ we can have two color structures: $C(2;a)$ and $C(2;s)$. The generalized commutation relations appear as follows:

$C(2;a)$

$$\begin{aligned}
&[X_{(0,0)},X_{(\alpha_1,\alpha_2)}]\subset X_{(\alpha_1,\alpha_2)}\,,\\
&[X_{(1,0)},X_{(1,0)}]\subset X_{(0,0)}\,,\\
&[X_{(0,1)},X_{(0,1)}]\subset X_{(0,0)}\,,\\
&\{X_{(1,0)},X_{(0,1)}\}\subset X_{(1,1)}\,,\\
&[X_{(1,1)},X_{(1,1)}]\subset X_{(0,0)}\,,\\
&\{X_{(1,0)},X_{(1,1)}\}\subset X_{(0,1)}\,,\\
&\{X_{(0,1)},X_{(1,1)}\}\subset X_{(1,0)};
\end{aligned} \qquad (2.6a)$$

$C(2;s)$

$$\begin{aligned}
&[X_{(0,0)},X_{(\alpha_1,\alpha_2)}]\subset X_{(\alpha_1,\alpha_2)}\,,\\
&\{X_{(1,0)},X_{(1,0)}\}\subset X_{(0,0)}\,,\\
&\{X_{(0,1)},X_{(0,1)}\}\subset X_{(0,0)}\,,\\
&[X_{(1,0)},X_{(0,1)}]\subset X_{(1,1)}\,,\\
&[X_{(1,1)},X_{(1,1)}]\subset X_{(0,0)}\,,\\
&\{X_{(1,0)},X_{(1,1)}\}\subset X_{(0,1)}\,,\\
&\{X_{(0,1)},X_{(1,1)}\}\subset X_{(1,0)}\,.
\end{aligned} \qquad (2.6b)$$

We see that the generators $X_{(1,0)}$, $X_{(0,1)}$, and $X_{(1,1)}$ are parabosons in the $C(2,a)$ case; in the $(2,s)$ case, the generators $X_{(1,0)}$ and $X_{(0,1)}$ are parafermions.

B. Three colors

We have

$$\begin{aligned}
X_{\vec{\alpha}}:\ &X_{(0,0,0)} \quad \text{(colorless)},\\
&X_{(1,0,0)},X_{(0,1,0)},X_{(0,0,1)} \quad \text{(red, white, blue)},\\
&X_{(1,1,0)},X_{(1,0,1)},X_{(0,1,1)} \quad \text{(bicolored)},\\
&X_{(1,1,1)} \quad \text{(tricolored)}.
\end{aligned}$$

The only possible color structure is $C(3;s)$. The generalized commutation relations appear as follows:

$$[X_{(0,0,0)},X_{(\alpha,\beta,\gamma)}]\subset X_{(\alpha,\beta,\gamma)}\,; \qquad (2.7a)$$

$$\begin{aligned}
&\{X_{(1,0,0)},X_{(1,0,0)}\}\subset X_{(0,0,0)}\,,\\
&\{X_{(0,1,0)},X_{(0,1,0)}\}\subset X_{(0,0,0)}\,,\\
&\{X_{(0,0,1)},X_{(0,0,1)}\}\subset X_{(0,0,0)}\,,\\
&[X_{(1,0,0)},X_{(0,1,0)}]\subset X_{(1,1,0)}\,,\\
&[X_{(1,0,0)},X_{(0,0,1)}]\subset X_{(1,0,1)}\,,\\
&[X_{(0,1,0)},X_{(0,0,1)}]\subset X_{(0,1,1)}\,;
\end{aligned} \qquad (2.7b)$$

$$\begin{aligned}
&[X_{(1,1,0)},X_{(1,1,0)}]\subset X_{(0,0,0)}\,,\\
&[X_{(1,0,1)},X_{(1,0,1)}]\subset X_{(0,0,0)}\,,\\
&[X_{(0,1,1)},X_{(0,1,1)}]\subset X_{(0,0,0)}\,,\\
&\{X_{(1,1,0)},X_{(1,0,1)}\}\subset X_{(0,1,1)}\,,\\
&\{X_{(1,1,0)},X_{(0,1,1)}\}\subset X_{(1,0,1)}\,,\\
&\{X_{(0,1,1)},X_{(1,0,1)}\}\subset X_{(1,1,0)}\,;
\end{aligned} \qquad (2.7c)$$

$$\begin{aligned}
&\{X_{(1,1,1)},X_{(1,1,1)}\}\subset X_{(0,0,0)}\,,\\
&\{X_{(1,1,1)},X_{(1,0,0)}\}\subset X_{(0,1,1)}\,,\\
&\{X_{(1,1,1)},X_{(0,1,0)}\}\subset X_{(1,0,1)}\,,\\
&[X_{(1,1,1)},X_{(1,1,0)}]\subset X_{(0,0,1)}\,,\\
&[X_{(1,1,1)},X_{(1,0,1)}]\subset X_{(0,1,0)}\,,\\
&[X_{(1,1,1)},X_{(0,1,1)}]\subset X_{(1,0,0)}\,;
\end{aligned} \qquad (2.7d)$$

$$\begin{aligned}
&\{X_{(1,1,0)}\,,\,X_{(1,0,0)}\}\subset X_{(0,1,0)}\,,\\
&\{X_{(1,1,0)},X_{(0,1,0)}\}\subset X_{(1,0,0)}\,,\\
&[X_{(1,1,0)},X_{(0,0,1)}]\subset X_{(1,1,1)}\,;
\end{aligned} \qquad (2.7e)$$

and analogous two sets of three relations for $X_{(1,0,1)}$ and $X_{(0,1,1)}$, respectively.

From the set of relations (2.7) we see that

(a) The set of generators $X_{(0,0,0)}$ forms a Lie algebra. In this sector one should put bosonic generators describing space-time as well as flavor symmetries [see Eq. (2.7a)].

(b) The generators $X_{(1,0,0)}$, $X_{(0,1,0)}$, $X_{(0,0,1)}$ are parafermionic [see Eq. (2.7b)].

(c) The generators $X_{(1,1,0)}$, $X_{(1,0,1)}$, $X_{(0,1,1)}$ are parabosonic, and this sector contains the color charges. It is interesting to observe that the set of generators [see Eq. (2.7c)]

$$X_{(0,0,0)}, X_{(1,1,0)}, X_{(1,0,1)}, X_{(0,1,1)}$$

forms a $C(2;a)$ subalgebra.

(d) The charges $X_{(1,1,1)}$ are fermionic [see Eq. (2.7d)]. We shall consider examples of color superalgebras with $X_{(1,1,1)}\equiv 0$. If $X_{(1,1,1)}\neq 0$ it seems plausible to relate these generators with leptonic degrees of freedom.

III. THE COLOR-DE SITTER ALGEBRA

Our first example is the three-color extension of the graded de Sitter algebra osp(4;1), which are denote by osp(4‖1,1,1). The generators of osp(4‖1,1,1) can be represented by the following real 7×7 matrices:

$$\begin{bmatrix} A\ (4\times 4) & -CF\ (4\times 3) \\ F\ (3\times 4) & B\ (3\times 3) \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{3.1}$$

where $A^T C + CA = 0$, F arbitrary, and

$$B^T = B, \quad B_{11}=B_{22}=B_{33}=0\,. \tag{3.2}$$

We introduce the following basis for the matrices (3.2):

$$L_1=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L_3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.3}$$

which are the generators of a $C(2,a)$ algebra [see Eq. (2.6a)],

$$\{L_1,L_2\}=L_3, \quad \{L_2,L_3\}=L_1, \quad \{L_3,L_1\}=L_2\ . \tag{3.4}$$

Choosing 12 fundamental real parafermionic generators $X_{(1,0,0)}=Q_{\alpha;1}$, $X_{(0,1,0)}=Q_{\alpha;2}$, $X_{(0,0,1)}=Q_{\alpha;3}$ ($\alpha=1,\ldots,4$) as real matrices $(Q_{\alpha;i})_{ab}$ ($i=1,2,3$; $a,b=1,\ldots,7$), which have in the $(4+i)$th column the elments $(C_{\alpha 1},\ldots,C_{\alpha 4},0,0,0)$ and in the $(4+i)$th row the elements $(\delta_{\alpha 1},\ldots,\delta_{\alpha 4},0,0,0)$ and using the generalized commutation relations (2.7), one obtains the three-color superalgebra OSp(4‖1,1,1) which reads as follows:

(a) Three copies of the fundamental superalgebra osp(4;1),

$$\{Q_{\alpha;i},Q_{\beta;i}\}=(\sigma^{AB}C)_{\alpha\beta}M_{AB} \tag{3.5}$$
$$i=1,2,3, \quad A,B=1,2,3,4,5\,,$$

where the ten real 4×4 matrices σ_{AB} ($\sigma_{\mu\nu}=\frac{1}{4}[\gamma_\mu,\gamma_\nu]$; $\sigma_{\mu 5}=-\sigma_{5\mu}=\gamma_\mu$; γ_μ in the Majorana representation) form a basis for sp(4;R). The 7×7 matrices M_{AB} span the A sector of the matrix (3.1) and satisfy the commutation relations of de Sitter o(3,2) algebra.

(b) Definition of color charges

$$\begin{aligned} &[Q_{\alpha;1},Q_{\beta;2}]=C_{\alpha\beta}L_3\,, \quad L_3=X_{(1,1,0)}\,, \\ &[Q_{\alpha;1},Q_{\beta;3}]=C_{\alpha\beta}L_2, \quad L_2=X_{(1,0,1)}\,, \\ &[Q_{\alpha;2},Q_{\beta;3}]=C_{\alpha\beta}L_1, \quad L_1=X_{(0,1,1)}\,. \end{aligned} \tag{3.6}$$

The 7×7 matrices L_i ($i=1,2,3$) have only the entries in the B sector given by the formulas (3.3), and they satisfy the algebra (3.4).

(c) Covariance properties of supercharges.

The fundamental supercharges $Q_{\alpha,i}$ transform under the rotations of the de Sitter group as O(3,2) spinors. The color indices tranform as follows:

$$\begin{aligned} &[L_1,Q_{\alpha;1}]=0\,, \quad \{L_2,Q_{\alpha;1}\}=Q_{\alpha;3}\,, \\ &\{L_1,Q_{\alpha;2}\}=Q_{\alpha;3}, \quad [L_2,Q_{\alpha;2}]=0\,, \\ &\{L_1,Q_{\alpha;3}\}=Q_{\alpha;2}\,, \quad \{L_2,Q_{\alpha;3}\}=Q_{\alpha;1}\,, \quad \text{etc}\,. \end{aligned} \tag{3.7}$$

IV. THE COLOR-CONFORMAL ALGEBRA

The second example is the three-color extension of Wess-Zumino superconformal algebra su(2,2;1), which we denote consequently by su(2,2‖1,1,1). The generators of su(2,2‖1,1,1) can be described by the following complex 7×7 matrices [for the definition of C, see Eq. (3.1)] with vanishing supertrace (tr$M$ = tr$H$):

$$\begin{bmatrix} M\ (4\times 4) & -CF\ (4\times 3) \\ F\ (3\times 4) & H\ (3\times 3) \end{bmatrix} \tag{4.1}$$

where

$$M^\dagger\eta=-\eta M\,,$$
$$\eta_{AB}:\ \text{diagonal}, \ \eta_{11}=\eta_{22}=-\eta_{33}=-\eta_{44}=1\,,$$
$$F\ \text{arbitrary}\,,$$
$$H=H^\dagger\,.$$

The eight Hermitian 3×3 matrices H are obtained here as the realization of the following extension of the $C(2,a)$ algebra (3.4) by new six generators I_i, H_i ($i=1,2,3; H_1+H_2+H_3=0$) and we obtain a new $C(2,a)$ algebra:

$$\begin{aligned} &\{I_1,I_2\}=L_3, \quad \{I_2,I_3\}=L_1, \quad \{I_3,I_1\}=L_2, \\ &[I_1,L_1]=2iH_1, \quad \{I_1,L_2\}=I_3, \quad \{I_1,L_3\}=I_2, \\ &\{I_2,L_1\}=I_3, \quad [I_2,L_2]=2iH_2, \quad \{I_2,L_3\}=I_1, \\ &\{I_3,L_1\}=I_3, \quad \{I_3,L_2\}=I_1, \quad [I_3,L_3]=2iH_3, \\ &[H_1,I_1]=2L_1, \quad [H_1,L_1]=2I_1, \\ &[H_1,I_2]=[H_1,I_3]=0, \quad [H_1,L_2]=[H_1,L_3]=0, \end{aligned} \tag{4.2}$$

etc.

It is easy to check that if we choose L_i given by the Eq. (3.3), we get

$$I_1=\begin{bmatrix}0 & i & 0\\ -i & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad I_2=\begin{bmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{bmatrix}, \quad I_3=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & i\\ 0 & -i & 0\end{bmatrix} \tag{4.3}$$

and

$$H_1=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}, \quad H_2=\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad H_3=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{4.4}$$

We see that we have obtained the three-dimensional representation of the su(3) algebra as a representation of a $C(2,a)$ algebra. We represent the 12 fundamental complex parafermionic generators as complex 7×7 matrices $(Q_{\alpha;i})_{ab}$ with the elements $(C^+_{1\alpha},\dots,C^+_{4\alpha},0,0,0)$ in $(4+i)$th column and the elements $(\delta^-_{\alpha 1},\dots,\delta^-_{\alpha 4},0,0,0)$ in $(4+i)$th row, where $C^\pm=\frac{1}{2}(1\pm\gamma_5)C$, $\delta^\pm=\frac{1}{2}(1\pm\gamma_5)$, and in the Majorana representation $(C^+)^*=C^-$, $(\delta^+)^*=\delta^-$. The color superalgebra su(2, 2‖1, 1, 1) has the following form:

(a) Three copies of the fundamental superalgebra su(2, 2; 1) ($i=1,2,3$),

$$\begin{aligned} &\{Q_{\alpha;i},Q_{\beta;i}\}=\tfrac{1}{2}(\gamma^\mu C)_{\alpha\beta}P_\mu, \\ &\{Q^*_{\alpha;i},Q^*_{\beta;i}\}=-\tfrac{1}{2}(\gamma^\mu C)_{\alpha\beta}K_\mu, \\ &\{Q_{\alpha;i},Q^*_{\beta;i}\}=(\sigma_{\mu\nu}C)_{\alpha\beta}M^{\mu\nu}+\tfrac{1}{2}C_{\alpha\beta}D+4(\gamma_5 C)_{\alpha\beta}\pi_i, \end{aligned} \tag{4.5}$$

where the set $J_{KL}=-J_{KL}$ ($K,L=1,2,\dots,6$) of the conformal generators

$$\begin{aligned} &J_{\mu\nu}=M_{\mu\nu}, \quad J_\mu=P_\mu-K_\mu, \\ &J_{56}=D, \quad J_{\mu 6}=P_\mu+K_\mu \end{aligned} \tag{4.6}$$

satisfies the o(4, 2) algebra, and spans the M sector of the matrices (4.1). Three axial charges π_i commute with J_{KL} as well as among themselves; they define the unique flavor charge

$$\pi=\tfrac{1}{3}(\pi_1+\pi_2+\pi_3) \tag{4.7}$$

and the generators of the Cartan subalgebra of the $C(2;a)$ algebra (4.2),

$$\begin{aligned} &H_1=\tfrac{1}{4}(\pi_3-\pi_2), \quad H_2=\tfrac{1}{4}(\pi_1-\pi_3), \\ &H_3=\tfrac{1}{4}(\pi_2-\pi_1). \end{aligned} \tag{4.8}$$

(b) Definition of color charges

$$\begin{aligned} &[Q_{\alpha;1},Q^*_{\beta;2}]=\tfrac{1}{2}[C_{\alpha\beta}L_3+i(\gamma_5 C)_{\alpha\beta}I_3], \\ &[Q_{\alpha;1},Q^*_{\beta;3}]=\tfrac{1}{2}[C_{\alpha\beta}L_2+i(\gamma_5 C)_{\alpha\beta}I_2], \\ &[Q_{\alpha;2},Q^*_{\beta;3}]=\tfrac{1}{2}[C_{\alpha\beta}L_1+i(\gamma_5 C)_{\alpha\beta}I_1], \end{aligned} \tag{4.9a}$$

or

$$[Q^*_{\alpha;1},Q_{\beta;2}]=\tfrac{1}{2}[C_{\alpha\beta}L_3-i(\gamma_5 C)_{\alpha\beta}I_3], \tag{4.9b}$$

etc. We see that for SU(2, 2‖1, 1, 1) the two-colors sector has the form

$$\begin{aligned} &X_{(1,1,0)}=(L_3,I_3), \quad X_{(1,0,1)}=(L_2,I_2), \\ &X_{(0,1,1)}=(L_1,I_1). \end{aligned} \tag{4.10}$$

(c) Convariance properties of supercharges.

The generators $Q_{\alpha;i}$ transform under the conformal group as SU(2, 2) spinors, i.e., twistors. The color indices transform under L_i according to the set of relations (3.7). Besides this we get new relations

$$\begin{aligned} &[I_1,Q_{\alpha;1}]=0, \\ &\{I_1,Q_{\alpha;2}\}=i(\gamma_5)_{\alpha\beta}Q_{\beta;3}, \\ &\{I_1,Q_{\alpha;3}\}=i(\gamma_5)_{\alpha\beta}Q_{\beta;2}, \quad \text{etc.}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &[Q_{\alpha;1},H_1]=0, \\ &[Q_{\alpha;1},H_2]=-Q_{\alpha;1}, \\ &[Q_{\alpha;1},H_3]=Q_{\alpha;1}. \end{aligned} \tag{4.12}$$

The transformation under the single flavor charge (4.7) has the form

$$[\pi,Q_{\alpha;i}]=-\tfrac{1}{3}(\gamma_5)_{\alpha\beta}Q_{\beta;i}. \tag{4.13}$$

V. CONCLUDING REMARKS

We have considered $Z_2\oplus Z_2\oplus Z_2$ graded superalgebras and have shown through two examples [the osp (4‖1, 1, 1) and the su(2, 2‖1, 1, 1) color algebras] how one can obtain a unification of space-time and "color" symmetry. Are these algebraic structures "new"? A similar question appears also when one considers the parafield realization of color.[12] Actually the algebras of the creation and annihilation operators that one encounters in parastatistics for the three-colors case are *solvable* algebras of the type $C(2,a)$ for parabosons and $C(3,s)$ for parafermions; for these algebras there exists a transformation (the Klein transformation) which turns the $C(2,a)$ algebra into a Lie algebra and the $C(3,s)$ algebra into a superalgebra. Such a transformation generally does not exist if the color algebra is not solvable. A first insight in the problem of "novelty" can be obtained if one compares for example the osp(4‖1, 1, 1) algebra with the superalgebra osp(4, 3), one then observes that the structure constants are identical although the product can be a commutator in one case and an anticommutator in the other. Let us consider now the three Pauli matrices τ_i $(i=1,2,3)$ $(\tau_i\tau_j+\tau_j\tau_i=2\delta_{ij})$ and take any representation of the osp(4‖1, 1, 1) color superalgebra. We can now consider the algebra formed by the matrices

$$\begin{aligned} &X_{(0,0,0)},\quad \tau_1 X_{(1,0,0),i},\quad \tau_2 X_{(0,1,0),i},\\ &\tau_3 X_{(0,0,1),i},\quad \tau_1\tau_2 X_{(1,1,0),i},\\ &\tau_1\tau_3 X_{(1,0,1),i},\quad \tau_2\tau_3 X_{(0,1,1),i},\quad \tau_1\tau_2\tau_3 X_{(1,1,1),i}, \end{aligned} \tag{5.1}$$

where $X_{(\alpha,\beta,\gamma),i}$ are a representation of the osp(4‖1, 1, 1) algebra. It is a trivial exercise to show that the generators (5.1) form a representation of the osp(4, 3) algebra. This representation may be irreducible, fully reducible, or not fully reducible. As shown in Ref. 7 this argument generalizes for any $Z_2\oplus Z_2\oplus\cdots\oplus Z_2$ graded color algebra. Thus the novelty of the osp(4‖1, 1, 1) color superalgebra as compared to the osp(4‖3) superalgebra is that they have *different* representations.[11] For physical applications, however, it is not yet clear if there is a novelty (this point still has to be studied). It is possible that when writing a field theory for the osp(4‖1, 1, 1) algebra one is forced to take certain representations such that for all practical purposes through changes of notation one obtains the same results as if he had started with an osp(4, 3) algebra. We hope to come back to this point in a further publication.

The main purpose of this paper was to show that there are mathematical structures which go beyond the Z_2 graded superalgebras and which may be useful. The $Z_2\oplus Z_2\oplus Z_2$ color superalgebras is only one example. There exist other possibilities such as the one based on a $Z_3\oplus Z_3\oplus Z_2$ or $Z_4\oplus Z_4$ grading as suggested in Ref. 7.

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[1]V. Kac, Commun. Math. Phys. 53, 31 (1977); P. G. O. Freund and I. Kaplansky, J. Math. Phys. 17, 228 (1976); W. Nahm, V. Rittenberg, and M. Scheunert, Phys. Lett. 61B, 383 (1976).

[2]P. K. Townsend and P. van Nieuwenhuizen, Phys. Lett. 67B, 439 (1977); S. Ferrara, Phys. Lett. 69B, 481 (1977).

[3]R. Haag, J. T. Lopuszanski, and M. Sohnius, Nucl. Phys. B88, 257 (1975); S. Ferrara, M.Kaku, P. van Nieuwenhuizen, and P. K. Townsend, Nucl. Phys. B129, 125 (1977).

[4]M. Günaydin and F. Gürsey, Phys. Rev. D 9, 3387 (1974).

[5]It should be mentioned that the notion of colored sypersymmetry has also been tried in the context of the unification of space-time and color degrees of freedom via octonionic Hilbert spaces (see Ref. 6).

[6]R. Casalbuoni, G. Domokos, and S. K. Domokos, Nuovo Cimento 33A, 432 (1976).

[7]V. Rittenberg and D. Wyler, Nucl. Phys. (to be published).

[8]The $Z_2\oplus Z_2\oplus\cdots\oplus Z_2$ superalgebras are known to V. Kac, who briefly mentions them in Ref. 9 under the name of generalized superalgebras.

[9]V. Kac, Adv. Math. 26, 8 (1977).

[10]This name was suggested to us by A. Pais.

[11]V. Rittenberg and D. Wyler, J. Math. Phys. (to be published).

[12]O. W. Greenberg and C. A. Nelson, Phys. Rep. (to be published).