Quantum-mechanical formulation of the electron-monopole interaction without Dirac strings

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A relativistic wave equation is used to describe the interaction of an electron with a magnetic monopole at rest. A quasipotential, defined in terms of an integral involving the pseudoscalar potential $\varphi_m = g/r$, is used in this Dirac string-free formalism. This equation, written in nonrelativistic form, is used in conjunction with the appropriate angular momentum operators to obtain the Dirac quantization condition. An argument using gauge invariance shows that this condition is not modified by a relativistic treatment. A detailed comparison is made with conventional string theory, which is also formulated relativistically.

I. INTRODUCTION

We estimate that over 1000 papers have been written about the magnetic monopole following Dirac's original suggestion.[1] This sustained interest is due, in large measure, to the added symmetry of Maxwell's equations, when magnetic as well as electric charge and current densities are included as sources, and to the joint quantization of magnetic and electric charge, which is a fundamental aspect of the various theoretical formulations. The magnetic field due to a stationary magnetic monopole of charge g is radial: $\vec{B} = g\vec{r}/r^3$, like the Coulomb field of an electric charge. For a point monopole, $\nabla \cdot \vec{B} = 4\pi g \delta(\vec{r})$. A well-known problem arises when one attempts to describe the magnetic monopole by means of an appropriate potential $\vec{A}$, whose curl is $\vec{B}$, because the divergence of the curl of any vector is identically zero. This contradicts the foregoing equation. One solution to the problem is to abandon potentials completely,[2] but it has been pointed out[3] that theoretical laboratories simply cannot dispense with potentials. The conventional approach is to construct an $\vec{A}$ whose curl comes as close as possible to yielding the correct $\vec{B}$ field. It is important, as Wentzel[4] emphasizes, to identify and subtract off the fictitious field terms. In Dirac's original treatment, the source of $\vec{A}$ is taken to be a semi-infinite string of magnetic dipoles which terminate with a magnetic monopole at the origin. In his formalism, a single four-potential A_μ is used to describe both magnetic and electric charges and currents. A second point of view, which has been considered by a number of authors,[5] involves the use of two sets of four-potentials: one for electric charges and currents and the other for magnetic charges and currents. In this formalism, which we adopt, $\varphi_m = g/r$ and $\vec{B} = -\nabla\varphi_m = g\vec{r}/r^3$ is free of fictitious terms.

There have been a number[6] of attempts to detect the magnetic monopole. The evidence presented so far has been either negative or unconvincing.

In our approach, we have chosen a formalism which is easy to interpret from a physical point of view and which we believe may be readily extended to a description of electron-monopole binding and scattering. If it exists, we believe that the magnetic monopole will be detected by virtue of the unique features associated with the electron-monopole interaction.

In common with the minority point of view, this paper starts with two sets of four-potentials. In order that $\vec{B}$ remain an axial vector, as it is in conventional electromagnetic theory, g and φ_m must be pseudoscalars. This means that the second set of four-potentials must be a pseudovector. In the second-quantized formulation of this theory, one finds discussions of left-handed and right-handed photons.[7] In this paper, we quantize once not twice.

We construct a relativistic Hamiltonian for the electron-monopole interaction which features a quasipotential $\vec{A}^*$, derived from the pseudoscalar φ_m. This quasipotential is a polar vector, whose curl is $\vec{B}$, plus a fictitious term. We show that $\vec{A}^*$ differs conceptually and mathematically from the more conventional $\vec{A}$'s of the Dirac type.

By means of the Foldy-Wouthuysen[8] transformation we rewrite the wave equation in Schrödinger-Pauli form, which we believe is most suitable for exploring electron-monopole binding.[9]

The Dirac quantization condition has been established in magnetic-monopole theory on the basis of semiclassical, first-quantized and second-quantized arguments using gauge,[10] rotational,[11] and Lorentz invariance.[12] Before proceeding to other aspects of the electron-monopole interaction, we establish the Dirac quantization condition for the nonrelativistic equation with $\vec{A}^*$, using the approach Fierz used for the string potential $\vec{A}$. Later we use gauge arguments to show that the same quantization condition holds for the relativistic equation.

The unique features of this paper are associated with the quasipotential A^*, obtained from the pseudoscalar φ_m, which we incorporate into Dirac, Schrödinger-Pauli, and Schrödinger wave equations for the electron-monopole interaction. As a necessary condition for further successful utilization of these equations, the Dirac quantization condition is deduced by means of angular momentum and gauge invariance arguments.

Our purpose, in this paper, is to study the Dirac Hamiltonian for an electron interacting with a stationary magnetic monopole. Our Dirac string-free relativistic equation is obtained from a classical Hamiltonian which we discuss in Sec. II. In Sec. III we compare the properties of our quasipotential with Dirac's string potential. In Sec. IV, we consider the relativistic behavior of an electron in four different physical environments: the free electron, the electron interacting with a proton at rest, and the electron interacting with a stationary magnetic monopole with and without a semi-infinite string of magnetic dipoles. We follow the approach developed by Fierz[13] to obtain the Dirac quantization condition from nonrelativistic theory. This theory focuses on the structure of the angular momentum operators, which depend of course on the nature of the electron's environment. With the successful derivation of the quantization condition, we suggest in the concluding Sec. V additional problems which could be studied with this formalism.

Finally, in the Appendix, we write the classical Hamiltonian generalized to include magnetic charges in motion. And we discuss gauge functions which may be used to establish Dirac's quantization condition on relativistic grounds with or without Dirac strings.

II. RELATIVISTIC HAMILTONIAN WITHOUT DIRAC STRINGS

In this theory $A_\mu^e=(\vec{A}_e, i\varphi_e)$ and $A_\mu^m=(\vec{A}_m, i\varphi_m)$ are the four-vector potentials whose sources are electric and magnetic charges and current densities, respectively. The electric and magnetic field quantities are

$$\vec{E}=-\nabla\varphi_e-\nabla\times\vec{A}_m-c^{-1}\partial\vec{A}_e/\partial t \tag{1}$$

and

$$\vec{B}=\nabla\times\vec{A}_e-\nabla\varphi_m-c^{-1}\partial A_m/\partial t\,. \tag{2}$$

As we consider here the electron to be moving in the field of a monopole at rest, the only nonzero potential in (1) and (2) is $\varphi_m=g/r$. (We develop a more general Hamiltonian in the Appendix.)

We use the conventional Lagrangian-Hamiltonian formalism[14] to obtain a classical Hamiltonian for a charge q of mass m and velocity $\vec{v}$ in the magnetic field $\vec{B}=g\vec{r}/r^3$.

$$L=\tfrac{1}{2}mv^2+qc^{-1}\vec{v}\cdot\vec{A}^*, \tag{3}$$

where the quasipotential $\vec{A}^*$ is defined in terms of φ_m,

$$\vec{A}^*\equiv\int d\vec{l}\times\nabla\varphi_m \tag{4}$$

with $d\vec{l}=dz\vec{k}$. Note that $\vec{A}^*$ is a polar vector, as it is the cross product of a pseudovector with a polar vector. The generalized coordinates are the rectangular Cartesian coordinators x_1, x_2, and x_3. The canonical momentum is

$$\vec{p}=m\vec{v}+qc^{-1}\vec{A}^*. \tag{5}$$

The corresponding Hamiltonian is

$$H=(2m)^{-1}\sum_{k=1}^{3}(p_k-qc^{-1}A_k^*)^2$$
$$=\vec{v}\cdot(\vec{p}-q\vec{A}^*c^{-1})/2\,. \tag{6}$$

The canonical equations of motion are

$$\frac{\partial H}{\partial p_k}=\dot{x}_k=\frac{v_k}{m},\quad -\frac{\partial H}{\partial x_k}=\dot{p}_k\,. \tag{7}$$

The second of Eqs. (7) leads to the Lorentz force equation

$$m\ddot{x}_k=qc^{-1}(\nabla\varphi_m\times\vec{v})_k \tag{8}$$

apart from a fictitious term described below.

For the Dirac Hamiltonian, we replace the velocity $\vec{v}$ in the classical Hamiltonian, Eq. (6), by $c\vec{\alpha}$, where $\vec{\alpha}$ is the familiar Dirac matrix vector. We add the rest energy operator βmc^2 and let $q=e$, the electronic charge. Then

$$H_{\rm DIRAC}=\beta mc^2+c\vec{\alpha}\cdot(\vec{p}-ec^{-1}\vec{A}^*)\,. \tag{9}$$

Define

$$\vec{\pi}^*\equiv\vec{p}-ec^{-1}\vec{A}^*. \tag{10}$$

We note that

$$\dot{\vec{r}}=i\hbar^{-1}[H,\vec{r}]=c\vec{\alpha} \tag{11}$$

and

$$\dot{\vec{\pi}}^*=i\hbar^{-1}[H,\vec{r}]$$
$$=e\vec{\alpha}\times(\nabla\times\vec{A}^*)\,. \tag{12}$$

Equation (11) is the operator analog of the velocity. But (12) is not the Dirac operator analog of the Lorentz force equation because the curl of the quasipotential $\vec{A}^*$ does not equal $g\vec{r}/r^3$. We show in the next section that $\nabla\times\vec{A}^*=g\vec{r}/r^3$ plus a δ-function term.

III. PROPERTIES OF THE QUASIPOTENTIAL AND DIRAC'S STRING POTENTIAL

We are using the quasipotential

$$\vec{A}^* = g\int r^{-3}\vec{r}\times d\vec{l}$$

$$= g\int r^{-3}(\vec{r}\times\vec{k})dz$$

$$= gzr^{-1}(r^2-z^2)^{-1}(y\vec{i}-x\vec{j})$$

$$= -gr^{-1}\cot\theta\hat{\varphi}\,, \tag{13}$$

where we take the constant of integration to be zero. It is obtained from the pseudoscalar potential $\varphi_m = gr^{-1}$, due to a point monopole at the origin.

A familiar representation of the Dirac string potential is used by Fierz[15]:

$$\vec{A} = r^{-1}g(x\vec{j}-y\vec{i})(r+z)^{-1}$$

$$= g\sin\theta(1+\cos\theta)^{-1}r^{-1}\hat{\varphi}\,. \tag{14}$$

This expression comes from the line integral

$$\vec{A}(P,L) = g\int_L^P r^{-3}d\vec{l}\times\vec{r}\,, \tag{15}$$

where the negative z axis is taken as the integration path extending from $-\infty$ to the origin. Physically one may envisage this calculation as follows. Consider a string of magnetic dipoles, each consisting of positive and negative magnetic charges, extending from $z=-\infty$ to $z=+\infty$. Now imagine the string is cut in two just between a positive and a negative monopole at the origin and the upper portion is discarded. The remaining semi-infinite string consists of an integral number of dipoles and one positive monopole at the origin. Equation (14) describes this Dirac string.

Apart from algebraic sign, the integrands in (13) and (15) have the same mathematical form. Physically, they have an essentially different character, the explicit integrations reflect the difference, and the resulting expression for $\vec{A}^*$ and $\vec{A}$ are different.

In the case of the quasipotential,

$$\vec{A}^*(x,y,z) = \int d\vec{l}\times\nabla\phi_m\,, \tag{16}$$

where $\varphi_m = gr^{-1}$. Note that the source is a monopole of charge g, at the origin. We choose $d\vec{l} = dz\vec{k}$ as a matter of convenience.

By contrast, the Dirac potential, is based on the vector potential expression of a dipole of magnetic moment $\vec{m}$: $\vec{A} = (\vec{m}\times\vec{r})r^{-3}$. On replacing $\vec{m}$ by $g\,d\vec{l}$ and integrating from infinity to P, taken as the origin, along a string $L\to P$

$$\vec{A}(P,L) = \vec{A}(x,y,z) = g\int_L^P \frac{d\vec{l}\times\vec{r}'}{r'^3}\,. \tag{17}$$

To evaluate this integral, choose, for example, the negative z axis as the integration line. The element $d\vec{l} = dz'\vec{k}$ and the distance from field point to source point is

$$r' = [x^2+y^2+(z-z')^2]^{1/2}\,.$$

Then

$$\vec{A} = g\int_{z'=-\infty}^{0} dz'(x\vec{j}-y\vec{i})r'^{-3}\,.$$

Let

$$\zeta' = z - z'\,;$$

then

$$\vec{A} = g\int_z^\infty (x\vec{j}-y\vec{i})r'^{-3}d\zeta'$$

$$= gr^{-1}(r+z)^{-1}(x\vec{j}-y\vec{i})\,. \tag{18}$$

One might argue that any relativistic Hamiltonian with the particular $\vec{A}^*$ we have chosen, or the particular $\vec{A}$ chosen by Fierz, is not well defined because there are other possible choices of $\vec{A}^*$ and $\vec{A}$ which may be constructed. Although it is indeed true that $\vec{A}^*$ and $\vec{A}$ are not unique, other choices will lead to essentially the same results. This is familiar in conventional electromagnetic theory, where one may represent a uniform field $B_0\vec{k}$ by three different vector potentials: $\vec{A} = B_0x\vec{j}$ or $-B_0y\vec{i}$ or $\frac{1}{2}B_0(x\vec{j}-y\vec{i})$. In all three cases $\nabla\times\vec{A} = B_0\vec{k}$.

It is not quite so simple in the case of a radial field representing the field of a point monopole. There is no vector potential whose curl is equal to precisely $g\vec{r}r^{-3}$ and nothing more. Our quasipotential $\vec{A}^*$ and Dirac's string potential $\vec{A}$ both yield, when the curl is taken, $g\vec{r}r^{-3}$, plus a fictitious field which must be subtracted off.

There is a straightforward way to evaluate fictitious terms, due to Wentzel.[16] Consider a magnetic monopole of charge g at the origin. Construct a circle of radius ρ around the z axis centered at z_0. Now we compute the magnetic flux Φ through this circular area:

$$\Phi = \int \vec{B}\cdot d\vec{a}$$

$$= g\int \vec{r}r'^{-3}(2\pi\rho'd\rho')\cdot\vec{k}$$

$$= 2\pi g z_0\int_{\rho'=0}^{\rho} \rho'd\rho'(\rho'^2+z_0^2)^{3/2}$$

$$= \begin{cases} 2\pi g(1-\cos\theta)\,, & 0\le\theta\le\pi/2 \\ 2\pi g(-1-\cos\theta)\,, & \pi/2\le\theta\le\pi\,. \end{cases} \tag{19}$$

Next, we compute

$$\int \nabla\times \vec{A}^*\cdot d\vec{a} = \int \vec{A}^*\cdot d\vec{l} = -2\pi g\cos\theta \qquad (20)$$

and

$$\int \nabla\times \vec{A}\cdot d\vec{a} = \int \vec{A}\cdot d\vec{l} = 2\pi g(1-\cos\theta). \qquad (21)$$

Here we have used the spherical polar forms for $\vec{A}^*$ and $\vec{A}$ in (13) and (14). Both expressions are independent of φ and are in the $\hat{\varphi}$ direction. The integration is simple as $d\vec{l} = \rho d\varphi\,\hat{\varphi} = r\sin\theta d\varphi\,\hat{\varphi}$.

In the case of $\vec{A}^*$,

$$\vec{B}_{\text{fictitious}} = \begin{cases} -2\pi g\,\delta(x)\delta(y), & z>0 \\ +2\pi g\,\delta(x)\delta(y), & z<0 \end{cases} \qquad (22)$$

as clearly

$$\int (\nabla\times\vec{A}^* - \vec{B}_{\text{fictitious}})\cdot d\vec{a} = \int g\vec{r}r^{-3}\cdot d\vec{a}.$$

Similarly in the case of the Dirac $\vec{A}$,

$$\vec{B}_{\text{fictitious}} = \begin{cases} 0, & z>0 \\ 4\pi g\,\delta(x)\delta(y), & z<0. \end{cases} \qquad (23)$$

We have constructed other expressions for $\vec{A}^*$ and $\vec{A}$. For example, take $d\vec{l} = dx\,\vec{i}$ in (13) and choose the negative x axis as the flux line or string in (14). The resulting expressions may be obtained by direct integration or by a cyclic permutation of the rectangular Cartesian coordinates. Then

$$\vec{A}^*_{(x)} = g\,x(z\,\vec{j} - y\vec{k})r^{-1}(r^2-x^2)^{-1}, \qquad (24)$$

$$\vec{A}_{(x)} = g(y\vec{k} - z\,\vec{j})r^{-1}(r+x)^{-1}. \qquad (25)$$

The curl of both these expressions is again $g\vec{r}r^{-3}$ with new fictitious fields

$$\vec{B}_f = \begin{cases} -2\pi g\,\delta(y)\delta(z), & x>0 \\ +2\pi g\,\delta(y)\delta(z), & x<0 \end{cases} \qquad (26)$$

and

$$\vec{B}_f = \begin{cases} 0, & x>0 \\ 4\pi g\,\delta(y)\delta(z), & x<0, \end{cases} \qquad (27)$$

respectively. In polar coordinates

$$\vec{A}^*_{(x)} = g\sin\theta(\cos\varphi\,\sin\varphi\,\hat{\theta} + \cos\theta\,\cos\varphi\hat{\varphi})$$
$$\times r^{-1}(1-\sin^2\theta\cos^2\varphi)^{-1},$$

which is much more complicated than the polar form of $\vec{A}^*$ in (13). Our choice of $\vec{A}^* = -g r^{-1}\cot\theta\hat{\varphi}$, corresponding to $d\vec{l} = dz\vec{k}$, is based on simplicity. Further, it is consistent with our subsequent choice of the z axis as the axis of quantization.

IV. RELATIVISTIC AND NONRELATIVISTIC HAMILTONIAN, ANGULAR MOMENTUM OPERATORS, AND THE DIRAC QUANTIZATION CONDITION

We label the equations associated with the free electron F, the hydrogen atom H, the electron string interaction S, and the electron monopole interaction M.

The appropriate relativistic Hamiltonians are

$$\mathcal{H} = \beta mc^2 + c\vec{\alpha}\cdot\vec{p}, \qquad (28\text{F})$$

$$\mathcal{H} = \beta mc^2 + c\vec{\alpha}\cdot\vec{p} - e^2r^{-1}, \qquad (29\text{H})$$

$$\mathcal{H} = \beta mc^2 + c\vec{\alpha}\cdot\vec{\pi}, \qquad (30\text{S})$$

$$\mathcal{H} = \beta mc^2 + c\vec{\alpha}\cdot\vec{\pi}^*. \qquad (31\text{M})$$

In string theory $\vec{\pi}$ is the kinetic momentum

$$\vec{\pi} \equiv \vec{p} - ec^{-1}\vec{A}, \qquad (32\text{S})$$

where $\vec{A}$ is often represented by (14).

It is useful to write the four different equations in the form of the nonrelativistic expansions. For this purpose we use the Foldy-Wouthuysen transformation[17] and set the Dirac matrix $\beta = 1$ corresponding to the positive-energy states

$$\mathcal{H}_{\text{NR}}\,\psi = \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots\right)\psi = E\psi, \qquad (33\text{F})$$

$$\mathcal{H}_{\text{NR}}\,\psi = \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2} - \frac{e^2}{r} + \frac{\pi e^2\hbar^2}{2m^2c^2}\delta(\vec{r}) + \frac{e^2\hbar}{4m^2c^2r^3}\vec{\sigma}\cdot\vec{l} + \cdots\right)\psi = E\psi, \qquad (34\text{H})$$

$$\mathcal{H}_{\text{NR}}\,\psi = \left(\frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{ieg\hbar}{mcr^2}\frac{1}{1+\cos\theta}\frac{\partial}{\partial\varphi} + \frac{e^2g^2}{2mc^2r^2}\frac{(1-\cos\theta)}{(1+\cos\theta)} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\nabla\times\vec{A} + \cdots\right)\psi = E\psi, \qquad (35\text{S})$$

$$\mathcal{H}_{\rm NR}\,\psi=\left(\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}-\frac{iegh}{mcr^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}+\frac{e^2g^2}{2mc^2r^2}\cot^2\theta-\frac{e\hbar}{2mc}\vec{\sigma}\cdot\nabla\times\vec{A}^*+\cdots\right)\psi=E\psi\,. \tag{36M}$$

The first two terms in the expansion of the relativistic kinetic energy operator

$$T=m_0c^2[(1+p^2/m_0^2c^2)^{1/2}-1]$$

are common to each of the foregoing wave equations. In the H-atom equation the Darwin and spin-orbit terms represent relativistic corrections to the Coulomb interaction. In the string and monopole equations the appropriate fictitious magnetic field must be subtracted from the $\vec{\sigma}\cdot$curl terms. The two middle terms obtained by evaluating $\pi^2/2m$ and $\pi^{*2}/2m$ in spherical coordinates differ because the string potential $\vec{A}$ differs from the monopole quasipotential $\vec{A}^*$.

The basic equation considered by Fierz[18] [(2.2) in his discussion of the theory of magnetic charged particles] follows immediately from (35S) if we neglect the relativistic correction to the kinetic energy and the Zeeman term. For purposes of comparison, then, we are examining essentially the two Schrödinger wave equations

$$(\pi^2/2m)\psi=E\psi \text{ and } (\pi^{*2}/2m)\psi=E\psi\,.$$

In the discussion of the angular momentum operators, we will find it convenient to refer to the relativistic Hamiltonians. Clearly, the operators which commute with a given relativistic Hamiltonian also commute with the corresponding nonrelativistic Hamiltonian as the latter is a converging representation of the former.

It is well known that the z component of the orbital angular momentum operator

$$\vec{l}=\vec{r}\times\vec{p} \tag{37}$$

introduced in (34H), does not commute with (28F). Now, a time-changing component of angular momentum could be due to a torque, but this is precluded for a free-particle Hamiltonian. Hence it was necessary to expand the meaning of angular momentum to include spin. Then

$$j_z=l_z+\hbar\sigma_z/2 \tag{38F, H}$$

does indeed commute with (28F) [note that the σ_z here is the four-component Dirac spin matrix, whereas the $\vec{\sigma}$ in (35S) and (36H) is the two-component Pauli matrix]. The fact that j_z also commutes with the H-atom Hamiltonian but does not commute with either the string or monopole Hamiltonians is understandable on classical grounds. The force corresponding to the Coulomb potential is central. Hence there is no torque to induce a change in the angular momentum. On the other hand the force between an electron and a monopole is noncentral leading to a change in the electron's angular momentum. These considerations enable us readily to find enhanced angular momentum operators which commute with the string and monopole Hamiltonians.

From the Lorentz force equation (8)

$$m\ddot{\vec{r}}=-c^{-1}eg\vec{v}\times\vec{r}\,r^{-3} \tag{39S, M}$$

it is easy to show that

$$d\vec{d}_c/dt=0\,, \tag{40S, M}$$

where

$$\vec{d}_c=m\vec{r}\times\vec{v}-c^{-1}eg\vec{r}\,r^{-1}$$

plays the role of a more general angular momentum. Obviously the first term in $\vec{d}_c$ is all that is required if the force equation is either for a free particle

$$m\ddot{\vec{r}}=0 \tag{41F}$$

or for the hydrogen atom

$$m\ddot{\vec{r}}=-e^2\vec{r}\,r^{-3}\,. \tag{42H}$$

Finally, we must add one more term to the angular momentum operators for the string and monopole cases. This arises from $\vec{A}$ and $\vec{A}^*$, respectively. Just as $-e\vec{A}/c$ and $-e\vec{A}^*/c$ are added to $\vec{p}$ to give the kinetic momentum, so must $-ec^{-1}\vec{r}\times\vec{A}$ and $-ec^{-1}\vec{r}\times\vec{A}^*$ be added to $\vec{r}\times\vec{p}$ in the quantum mechanical formulation of angular momentum.

We now can write down the angular momentum operators associated with our four relativistic Hamiltonians:

$$\vec{j}=\vec{l}+\hbar\sigma/2\,, \tag{43F, H}$$

$$\vec{d}=\vec{j}-eg\vec{r}\,r^{-1}-ec^{-1}\vec{r}\times\vec{A}\,, \tag{44S}$$

$$\vec{D}=\vec{j}-eg\vec{r}\,r^{-1}-ec^{-1}\vec{r}\times\vec{A}^*\,. \tag{45M}$$

It is straightforward but tedious to establish the following commutation relations

$$[\mathcal{H}_F, j_z]=[\mathcal{H}_F, j^2]=[\mathcal{H}_H, j_z]=[\mathcal{H}_H, j^2]=0\,, \tag{46F, H}$$

$$[\mathcal{H}_s, d_z]=[\mathcal{H}_s, d^2]=0\,, \tag{47S}$$

$$[\mathcal{H}_m, D_z]=[\mathcal{H}_m, D^2]=0\,, \tag{48M}$$

$$[j_x, j_y]=i\hbar j_z\,, \tag{49F, H}$$

$$[d_x, d_y]=i\hbar d_z\,, \tag{50S}$$

and

$$[D_x, D_y]=i\hbar D_z\,. \tag{51M}$$

Now in order to establish the Dirac quantization condition, we focus on the Schrödinger wave equations, which we can easily write down from (33)–(36) and the corresponding z component and raising and lowering angular momentum operators, written in spherical polar coordinates, from (43)–(45)

$$\frac{p}{2m}\psi=E\psi\,, \tag{52F}$$

$$\left(\frac{p^2}{2m}-\frac{e^2}{r}\right)\psi=E\psi\,, \tag{53H}$$

$$\left(\frac{p^2}{2m}-\frac{eg\hbar}{imcr^2}\,\frac{1}{1+\cos\theta}\,\frac{\partial}{\partial\varphi}+\frac{e^2g^2}{2mc^2r^2}\,\frac{(1-\cos\theta)}{(1+\cos\theta)}\right)\psi=E\psi\,, \tag{54S}$$

$$\left(\frac{p^2}{2m}-\frac{eg\hbar}{imcr^2}\,\frac{\cos\theta}{\sin^2\theta}\,\frac{\partial}{\partial\varphi}+\frac{e^2g^2}{2mc^2r^2}\cot^2\theta\right)\psi=E\psi\,, \tag{55M}$$

$$l_z=\frac{1}{i}\,\frac{\partial}{\partial\varphi}\,,\qquad l_\pm=l_x\pm il_y=e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\,\frac{\partial}{\partial\varphi}\right), \tag{56F,H}$$

$$d_\pm=d_x\pm id_y=l_\pm-e^{\pm i\varphi}\left(\frac{\mu\sin\theta}{1+\cos\theta}\right),\qquad d_z=l_z-\mu\,, \tag{57S}$$

$$D_z=l_z\,,\qquad D_\pm=l_\pm-e^{\pm i\varphi}\mu/\sin\theta\,. \tag{58M}$$

In (56)–(58), we have written the angular momentum operators in units of $\hbar$, truncated to eliminate the spin operator $\vec{\sigma}$, and we use the abbreviation

$$\mu\equiv eg/\hbar c\,. \tag{59S,M}$$

Clearly, if $g=0$, all the angular momentum operators and the string and monopole wave equations reduce to the operators and equations for a free particle.

From (56)–(58), we can construct, in spherical polar coordinates, the operator equations for the total angular momentum squared:

$$\sum_k l_k^2=l_z^2+\tfrac{1}{2}[l_+,l_-]_+=-\sin\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\,,\quad k=x,y,z\,, \tag{60F,H}$$

$$\sum_k d_k^2=l_z^2-2\mu l_z+\mu^2+\tfrac{1}{2}[d_+,d_-]_+=\sum_k l_k^2-\frac{2\mu}{1+\cos\theta}\,\frac{1}{i}\,\frac{\partial}{\partial\varphi}+\frac{\mu^2(1-\cos\theta)}{1+\cos\theta}+\mu^2: \tag{61S}$$

$$\sum_k D_k^2=l_z^2+\tfrac{1}{2}[D_+,D_-]_+=\sum_k l_k^2+\frac{2\mu}{i}\,\frac{\cos\theta}{\sin^2\theta}\,\frac{\partial}{\partial\varphi}+\frac{\mu^2}{\sin^2\theta}\,. \tag{62M}$$

We take as the general form for the solutions of each of the Schrödinger wave equations

$$\psi=\psi(r,\theta,\varphi)=R(r)Y_m(\cos\theta)e^{(m+\mu)\varphi}\,. \tag{63F,H,S,M}$$

This is a representation in which l_z, d_z, D_z, and $\sum_k d_k^2$, and D_k^2 are diagonal, but the x and y components of the three angular momentum operators are not. We make the substitution $x=\cos\theta$ and find the eigenvalue equations:

$$l_z\psi=D_z\psi=(m+\mu)\psi\,,\qquad d_z\psi=m\psi\,, \tag{64F,H,M}$$

$$(1-x^2)Y_m''-2xY_m'-(1-x^2)^{-1}(m+\mu)^2Y_m+\lambda Y_m=0\,, \tag{65F,H}$$

$$(1-x^2)Y_m''-2xY_m'-(1-x^2)^{-1}(m^2+\mu^2+2\mu mx)Y_m+\lambda Y_m=0\,, \tag{66S}$$

$$(1-x)^2Y_m''-2xY_m'-(1-x^2)^{-1}[(m+\mu)+\mu(\mu+2xm+2x\mu)]Y_m+\lambda Y_m=0\,, \tag{67M}$$

Clearly, for the free electron and the electron interacting with a proton we must set $\mu=0$ in (63), (64), and (65). Then we find that (65) is the associated Legendre equation and that (66) and (67) reduce to this equation as they should for $\mu=0$.

We note that the string equation (66S) becomes the monopole equation under the substitution $m\to m+\mu$, a fact which greatly simplifies the remaining discussion and can be understood in terms of the gauge transformation described in the Appendix.

We now use the raising and lowering operators to construct recursion relations of the Rodrigues type. Operating with $d_\pm$ on ψ yields

$$(1-x^2)^{1/2}\left(\frac{d}{dx} \pm \frac{mx+\mu}{1-x^2}\right)Y_m = Y_{m\pm1}\,. \qquad (68S)$$

The monopole equation

$$(1-x^2)^{1/2}\left(\frac{d}{dx} \pm \frac{(m+\mu)x+\mu}{1-x^2}\right)Y_m = Y_{m\pm1} \qquad (69M)$$

may be found by direct calculation with $D_\pm$ acting on ψ or replacing m with $m+\mu$ in (66S). The corresponding free particle and hydrogen atom equation

$$(1-x^2)^{1/2}\left(\frac{d}{dx} \pm \frac{mx}{1-x_2}\right)Y_m = Y_{m\pm1} \qquad (70F,H)$$

is readily obtained from $l_\pm$ acting on ψ with $\mu=0$ or by taking $\mu=0$ in (56) and (57).

The general relationship

$$y'+fy=\exp\left(-\int f dx\right)\frac{d}{dx}\left[\exp\left(\int f dx\right)y\right] \qquad (71F,H,S,M)$$

with

$$f=\pm(mx+\mu)/(1-x^2)\,, \qquad (72S)$$

$$f=\pm[(m+\mu)x+\mu]/(1-x^2)\,, \qquad (72M)$$

$$f=\pm mx/(1-x^2) \qquad (72F,H)$$

enables us to write

$$Y_{m+n}=(1-x^2)^{(m+n)/2}\left(\frac{1-x}{1+x}\right)^{\mu/2}\times\frac{d^n}{dx^n}\left[\left(\frac{1+x}{1-x}\right)^{\mu/2}(1-x^2)^{-m/2}Y_m\right],$$

$$Y_{m-n}=(1-x^2)^{(n-m)/2}\left(\frac{1+x}{1-x}\right)^{\mu/2}\times\frac{d^n}{dx^n}\left[\left(\frac{1-x}{1+x}\right)^{\mu/2}(1-x^2)^{m/2}Y_m\right]. \qquad (73S)$$

These solutions are analytic, only if both of the quantities in square brackets on the right-hand side are polynomials. It is only in this case that the number of functions $Y_{m\pm n}$, for a given eigenvalue λ, is finite.

If we set the first of (73S) equal to a polynomial P, then the second relationship is equal to

$$(1+x)^{m-\mu}(1-x)^{m+\mu}P\,. \qquad (74S)$$

This is only a polynomial if m and μ are both simultaneously equal to an integer or half integer. But

$$\mu=eg/\hbar c=n/2 \qquad (75S)$$

is the Dirac quantization condition.

Parallel calculations for the monopole yield

$$(1+x)^m(1-x)^{m+2\mu}P \qquad (76M)$$

which is a polynomial if m is an integer and μ is a half integer. This establishes the Dirac quantization condition for the monopole.

Fierz points out that (66S) is identical with the equation for a symmetric top.[19] This is formally true also for the monopole equation (67M). For each one of the $Y_{\hat{m}}$'s the integral $\int_{-1}^{1}|Y_m(x)|^2dx$ exists. Indeed from the recursion relations one can construct explicit solutions of the Y_m's and find the normalization factors for both positive and negative values of m and μ.

V. CONCLUDING REMARKS

In this paper we have constructed a relativistic Hamiltonian in which the interaction of an electron with a monopole at rest is described by means of the pseudoscalar potential $\varphi_m=g/r$. The Hamiltonian is cast into the same form as the Hamiltonian for a string by the introduction of a quasipotential $\vec{A}^*=\int dz\vec{k}\times\nabla\varphi_m$. Two approximations are made to write the corresponding Schrödinger operators and the associated angular momentum operators. The method of Fierz is then followed to obtain Dirac's quantization condition, a result without which this formalism would have no merit.

As a result of this successful outcome, it seems that the formalism presented here could be used to consider other problems of interest. What is the structure and physical significance of the relativistic wave equation describing the interaction between an electron and a magnetic current density? What are the conservation laws which can be obtained by applying Noether's theorm[20] to a covariant Lagrangian density formalism? What are the essential physical differences between a monopole of postulated spin $\hbar/2$ interacting with a charge at rest and a monopole of postulated spin 0 in the same environment? For the former a Dirac equation such as (9) would be used with $e\to g$ and $\vec{A}^*\to\int d\vec{r}\times\nabla\varphi_e$. For the latter the Klein-Gordon equation $(\pi^{*2}/2m+mc^2)\psi=E\psi$ would be used, with π^* redefined in terms of the above $\vec{A}^*$. In the electron-monopole interaction, do bound states exist for the Schrödinger-Pauli equation, with the magnetic moment of the electron $\mu_e=e\hbar(1+k)/2mc$?

The theory of the magnetic monopole, with two sets of four-potentials, has been investigated already,[21] but attention has been focused on the second-quantized form of the theory. The suggestions we make here for further study relate primarily to the first-quantized form of the theory. The objective is to expose as many features of the hypothetical magnetic monopole as possible, with the hope that additional detection schemes may emerge.

APPENDIX

1. General classical Hamiltonian

The Hamiltonian used to describe the interaction of a charge q moving with a velocity $\vec{v}$ in the field of a stationary monopole characterized by a potential $\varphi_m = g/r$ may be extended to include fields generated by magnetic monopoles in motion and, of course, by conventional electrical charges at rest or in motion. This brings into play $\vec{A}_m$ and $\vec{A}_e$ of (1) and (2).

We find that the appropriate Langrangian

$$L = mv^2/2 + c^{-1}q(\vec{A}_e + \vec{A}^*)\cdot\vec{v} - q(\varphi_e + \varphi^*), \quad (A1)$$

where the quasiscalar potential

$$\Phi^* \equiv \int \nabla\times\vec{A}_m \cdot d\vec{r} \quad (A2)$$

and the quasivector potential

$$\vec{A}^* \equiv \int d\vec{l}\times\left(-\nabla\phi_m - \frac{1}{c}\frac{\partial\vec{A}_m}{\partial t}\right). \quad (A3)$$

The magnetic monopole is the source of the pseudoscalar potential ϕ_m and the pseudovector potential $\vec{A}_m$. The physical field $\vec{B}_m$ due to magnetic monopoles is

$$\vec{B}_m = -\nabla\phi_m - \frac{1}{c}\frac{\partial\vec{A}_m}{\partial E} = \nabla\times\vec{A}^* - \vec{B}_{\text{fictitious}}. \quad (A4)$$

The corresponding electric field

$$\vec{E}_m = -\nabla\times\vec{A}_m = -\nabla\phi^* - \frac{1}{c}\frac{\partial A^*}{\partial t}. \quad (A5)$$

We use the subscript e to denote potentials and fields arising from electric charges. These have their familiar form

$$\vec{E}_e = -\nabla\phi_e - \frac{1}{c}\frac{\partial\vec{A}_e}{\partial t}, \qquad \vec{B}_e = \nabla\times\vec{A}_e. \quad (A6)$$

If we use symbols without subscripts or asterisks for the total potential and total field

$$\varphi = \varphi_e + \varphi^*, \quad \vec{A} = \vec{A}_e + \vec{A}^*, \quad (A7)$$

$$\vec{E} = \vec{E}_e + \vec{E}_m = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad (A8)$$

and

$$\vec{B} = \vec{B}_e + \vec{B}_m. \quad (A9)$$

The Lagrangian is in standard form

$$L = m\,v^2/2 + c^{-1}q\vec{A}\cdot\vec{v} - q\varphi \quad (A10)$$

but the canonical momentum

$$\vec{p} = m\vec{v} + qc^{-1}\vec{A} \quad (A11)$$

includes the quasipotential $\vec{A}^*$ as well as the conventional vector potential $\vec{A}_e$. Similarly, the Lorentz force equations, derived from the Lagrangian equation of motion,

$$m\vec{a} = q\vec{E} + qc^{-1}(\vec{v}\times\vec{B}), \quad (A12)$$

includes the fields arising from both electric and magnetic charges.

The generalized Hamiltonian

$$H = \sum_k \frac{(p_k - qc^{-1}A_k)^2}{2m} + q\varphi \quad (A13)$$

has the usual form, but p_k, A_k have generalized meanings.

2. The gauge function relating the various $\vec{A}$'s and $\vec{A}^*$'s

We found that the differential equation describing a symmetric top, as used by Fierz, with the potential (14), and the corresponding monopole equation have operators which are related by the transcription $m\to m+\mu$. The curl of the string potential (14) and of the monopole quasipotential (13) are both $g\vec{r}/r^3$ apart from singular terms along the z axis. The gauge function connecting these two quantities can easily be found, providing we exclude $z = \pm r$:

$$\vec{A}^* + \nabla\chi = \vec{A}^{(-z)}, \quad (A14)$$

we find

$$\chi = g\phi. \quad (A15)$$

Hence if we use the operator in the string equation we obtain the same eigenvalues in the monopole equation providing we use the gauge transformation of the first kind

$$\psi' = R(r)Y_m(\cos\theta)e^{i(m+\mu)\varphi}e^{ie\chi/\hbar c} = e^{ie\chi/\hbar c}\psi = e^{i\mu\varphi}\,\psi. \quad (A16)$$

It is the gauge invariance of the two Schrödinger equations which leads to the simple transcription $m\to m+\mu$ relating them.

We note that the monopole potential may be constructed from two string potentials:

$$\vec{A}^* = [A^{-(z)} + A^{(z)}]/2, \quad (A17)$$

where

$$\vec{A}^{(z)} = -(g\sin\theta)/r(1-\cos\theta) \quad (A18)$$

represents a semi-infinite string of dipoles extending along the positive z axis and terminating with a monopole.

We find that the two string potentials $\vec{A}^{(-z)}$ and $\vec{A}^{(z)}$ are related by the gauge function $\chi = 2g\varphi$. If we consider the change in χ as φ goes from $\varphi \to \varphi + 2\pi$: $\Delta\chi = 4\pi g$. If we require that the wave function be single valued, we obtain the Dirac quantization condition $\mu = \frac{1}{2}n$. This is an example of Dirac's original argument.[17]

Finally, we find by comparing any two monopole potentials taken from (4), that the gauge connecting them undergoes a change $\Delta\chi = 4\pi g$ as φ and θ are separately allowed to range from 0 to 2π. This again leads to the Dirac quantization condition.

It is easy to demonstrate that the relativistic wave equations $(\beta m c^2 + c\vec{\alpha}\cdot\vec{\pi}^*)\psi = E\psi$ and $(\beta m c^2 + c\vec{\alpha}\cdot\vec{\pi})\psi = E\psi$ are gauge invariant, as are the corresponding Schrödinger equations. Dirac's argument leading to the quantization condition is equally applicable to the relativistic and to the nonrelativistic wave equations. We conclude, therefore, that the relationship $eg/\hbar c = \frac{1}{2}n$ is not modified by a relativistic treatment.

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