PHYSICAL REVIEW D VOLUME 18, NUMBER 10 15 NOVEMBER 1978

Discrete sine-Gordon equations

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(Received 5 July 1978)

Various discretizations of the sine-Gordon equation are studied. Hirota's discretization scheme is extended and two alternative discretization schemes are constructed. The associated soliton solutions, Bäcklund transformations, conservation laws, and the inverse scattering equations are obtained.

I. INTRODUCTION

The sine-Gordon equation has served repeatedly as a prototype for a two-dimensional nonlinear field theory. It can be solved exactly by the inverse-scattering-transform method and possesses all the remarkable properties[1] that follow from this method,[2] namely particlelike solutions, the solitons, and the breather; Bäcklund transformations; infinite number of conservation laws; complete integrability as a Hamiltonian system.[3]

Recently, there has been some interest in exactly soluble discretized nonlinear problems, and the inverse-scattering-transform method has been extended to a wide class of such problems.[4] Hirota[5] has discretized the sine-Gordon equation in both the space and time variables and has obtained discrete soliton solutions, the associated Bäcklund transformations, and the inverse scattering equations.

In this paper we consider a generalization of Hirota's discretization scheme for the sine-Gordon equation and compare it with two alternative schemes. Specifically our results are as follows: (a) We extend Hirota's formalism to incorporate different lattice spacings for the space and time variables. (b) This allows us to take the continuous limit separately in the time or space variables, thus obtaining semidiscrete sine-Gordon equations. (c) We construct two alternative discretization schemes for the sine-Gordon equation, and find that they agree with Hirota's in the semidiscrete limit. (d) We derive Bäcklund transformations for all three schemes, and (e) solve them for the one-soliton and two-soliton solutions. (f) We construct the discrete analogs of the energy conservation laws and show how, in conjunction with the discrete Bäcklund transformation, they can be used to generate an infinite number of conserved quantities. (g) We consider the inverse scattering equations, solve them for the one- and two-soliton solutions, and generalizing the results of a previous paper[6] we derive the discrete analogs of the Coleman correspondences between the sine-Gordon equation and the massive Thirring model.[7]

The sine-Gordon equation in light-cone coordinates $x=(x^1-x^0)/2$ and $t=(x^1+x^0)/2$ is

$$\partial_x\partial_t\phi=\sin\phi\,. \tag{1}$$

Introducing the auxiliary field $\rho(x,t)$ through the equation

$$\partial_x\partial_t\rho=1-\cos\phi\,, \tag{2}$$

we can rewrite (1) and (2) in terms of a single complex variable as

$$w=e^{(\rho+i\phi)/4}\,, \tag{3}$$

$$w\partial_x\partial_t w-\partial_x w\partial_t w=\tfrac{1}{4}(w^2-w^{*2})\,. \tag{4}$$

Hirota's g/f notation is also used, where $g=e^{\rho/4}\sin(\phi/4)$, $f=e^{\rho/4}\cos(\phi/4)$ so that $w=f+ig$ and $\tan(\phi/4)=g/f$. The Bäcklund transformation

$$\begin{aligned}\partial_x\phi-\partial_x\hat\phi&=2a\sin\left(\frac{\phi+\hat\phi}{2}\right),\\ \partial_t\phi+\partial_t\hat\phi&=2a^{-1}\sin\left(\frac{\phi-\hat\phi}{2}\right),\end{aligned} \tag{5}$$

can be expressed[8] in terms of the corresponding fields w and $\hat w=e^{(\hat\rho+i\hat\phi)/4}=\hat f+i\hat g$ as

$$\begin{aligned}\hat w\partial_x w-w\partial_x\hat w&=-\tfrac{1}{2}a(w^*\hat w^*)\,,\\ \hat w^*\partial_t w-w\partial_t\hat w^*&=-\frac{1}{2a}(w^*\hat w)\,.\end{aligned} \tag{6}$$

The associated inverse scattering representation is obtained from (6) by defining[8]

$$\chi_1=\frac{\hat w^*}{w}\,,\quad \chi_2=\frac{\hat w}{w^*}\,,\quad \chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}. \tag{7}$$

Then, Eqs. (6) become the usual[2,8] inverse scattering equations

$$\begin{aligned}\partial_x\chi&=\frac{1}{2}\begin{pmatrix}-i\partial_x\phi & a\\ a & i\partial_x\phi\end{pmatrix}\chi\,,\\ \partial_t\chi&=\frac{1}{2a}\begin{pmatrix}0 & e^{-i\phi}\\ e^{i\phi} & 0\end{pmatrix}\chi\,.\end{aligned} \tag{8}$$

Instead of discretizing the sine-Gordon equation (1) directly, it proves very convenient to discretize first Eq. (4) and then obtain the discrete form of (1) through the identification (3).

II. DISCRETE SINE-GORDON EQUATIONS

A. Hirota's discretization method

In this section we consider Hirota's method of discretizing the sine-Gordon equation. We have slightly changed Hirota's notation and somewhat generalized his results to include different lattice spacings for the space and time variables. Thus, the x and t variables are discretized according to

$$x=hn,\quad n=0,\pm 1,\pm 2,\ldots,\qquad t=\tau m,\quad m=0,\pm 1,\pm 2,\ldots, \tag{9}$$

where h and τ are the lattice spacings. We introduce the notation $\phi_x(x,t)\equiv\phi(x+h,t)=\phi(h(n+1),t)$ for the x-shift operation and similarly for the t shift, so that, for example, $\phi_{xt}(x,t)=\phi(x+h,t+\tau)=\phi(h(n+1),\tau(m+1))$. Hirota's way[5] of discretizing Eq. (4) is to replace it by the partial difference equation[9]

$$ww_{xt}-w_xw_t=\frac{h\tau}{4}\left(\frac{ww_{xt}+w_xw_t}{2}-\frac{w^*w^*_{xt}+w^*_xw^*_t}{2}\right). \tag{10}$$

Defining ρ and ϕ again by $w=e^{(\rho+i\phi)/4}$ the discrete version of the sine-Gordon equation (1) and of the related Eq. (2) can be obtained from (10) as

$$\sin\left(\frac{\phi_{xt}+\phi-\phi_x-\phi_t}{4}\right)=\frac{h\tau}{4}\sin\left(\frac{\phi_{xt}+\phi+\phi_x+\phi_t}{4}\right), \tag{11}$$

$$\exp[(\rho_x+\rho_t-\rho-\rho_{xt})/4]=\frac{\cos[\frac{1}{4}(\phi_{xt}+\phi)]}{\cos[\frac{1}{4}(\phi_x+\phi_t)]}. \tag{12}$$

We observe that in the simultaneous continuous limit ($h=\tau\to 0$) these equations reduce to (1) and (2), while in the continuous-time limit ($\tau\to 0$) they reduce to

$$w\dot{w}_x-w_x\dot{w}=\tfrac{1}{4}h(ww_x-w^*w^*_x),\qquad \dot{\phi}_x-\dot{\phi}=h\sin\left(\frac{\phi+\phi_x}{2}\right),\qquad \dot{\rho}_x-\dot{\rho}=h\left[1-\cos\left(\frac{\phi+\phi_x}{2}\right)\right], \tag{13}$$

where $\dot{\phi}=\partial_t\phi$ etc. In the continuous-space limit ($h\to 0$) Eqs. (10)–(12) reduce to

$$ww'_t-w_tw'=\tfrac{1}{4}\tau(ww_t-w^*w^*_t),\qquad \phi'_t-\phi'=\tau\sin\left(\frac{\phi+\phi_t}{2}\right),\qquad \rho'_t-\rho'=\tau\left[1-\cos\left(\frac{\phi+\phi_t}{2}\right)\right], \tag{14}$$

where $\phi'=\partial_x\phi$ etc.

B. Alternative discretization methods

An alternative discretization method consists in replacing Eq. (4) by the following partial difference equation:

$$ww_{xt}-w_xw_t=\tfrac{1}{4}h\tau(ww_t-w^*w^*_t) \tag{15}$$

with a corresponding discrete sine-Gordon equation

$$\sin\left(\frac{\phi_{xt}+\phi-\phi_x-\phi_t}{4}\right)=\frac{h\tau}{2}\cos\left(\frac{\phi_{xt}+\phi}{4}\right)\sin\left(\frac{\phi_x+\phi_t}{4}\right)\times e^{(\rho-\rho_x)/4}, \tag{16}$$

$$\exp[(\rho_x+\rho_t-\rho-\rho_{xt})/4]=\frac{\cos[\frac{1}{4}(\phi_{xt}+\phi)]}{\cos[\frac{1}{4}(\phi_x+\phi_t)]}. \tag{17}$$

The extra exponential factor $e^{(\rho-\rho_x)/4}$ turns (16) into a *nonlocal* nonlinear partial difference equation. In fact, Eq. (17) can be used to solve for this factor in terms of ϕ,

$$e^{(\rho-\rho_x)/4}=\prod_{-\infty}^{m-1}\frac{\cos[\frac{1}{4}(\phi_{xt}+\phi)]}{\cos[\frac{1}{4}(\phi_x+\phi_t)]}. \tag{18}$$

In the continuous limit $h=\tau\to 0$ Eqs. (15)–(17) reduce to the usual ones, Eqs. (1)–(4), while in the continuous-time limit ($\tau\to 0$) they reduce to

$$w\dot{w}_x-w_x\dot{w}=\tfrac{1}{4}h(w^2-w^{*2}),\qquad \dot{\phi}_x-\dot{\phi}=h\sin\phi\, e^{(\rho-\rho_x)/4},\qquad \partial_t[e^{(\rho-\rho_x)/4}]=\frac{h}{4}\sin\phi\tan\left(\frac{\phi+\phi_x}{4}\right). \tag{19}$$

In the continuous-space limit $h\to 0$ Eqs. (15)–(17) become local, and reduce to the same set of equations as in (14). The nonlocality of (16) can be traced to the asymmetric treatment of the x and t variables. Since the original continuous sine-Gordon is symmetric in x and t, we can obtain a third discretization scheme by interchanging the roles of x and t in Eqs. (15):

$$ww_{xt}-w_xw_t=\tfrac{1}{4}h\tau(ww_x-w^*w^*_x),\qquad \sin\left(\frac{\phi_{xt}+\phi-\phi_x-\phi_t}{4}\right)=\frac{h\tau}{2}\cos\left(\frac{\phi_{xt}+\phi}{4}\right)\sin\left(\frac{\phi_x+\phi_t}{4}\right)\times e^{(\rho-\rho_t)/4}, \tag{20}$$

$$\partial_x[e^{(\rho-\rho_t)/4}]=\frac{\cos[\frac{1}{4}(\phi_{xt}+\phi)]}{\cos[\frac{1}{4}(\phi_x+\phi_t)]}.$$

The continuous-time limit ($\tau \to 0$) or continuous-space limit ($h \to 0$) can be taken in a similar manner. In particular, we note that the ($\tau \to 0$) limit results in the same ($\tau \to 0$) limit of Hirota's scheme, i.e., Eqs. (13). The properties of this semidiscrete sine-Gordon equation are studied elsewhere.[10]

III. BÄCKLUND TRANSFORMATIONS

All of the above discretization schemes admit a Bäcklund transformation which is a discretized version of (5) and (6). In Hirota's case[5,9]

$$\begin{aligned} w_x\hat{w} - w\hat{w}_x &= -p_a(w_x^*\hat{w}^* + w^*\hat{w}_x^*)\,, \\ w_t\hat{w}^* - w\hat{w}_t^* &= -q_a(w_t^*\hat{w} + w^*\hat{w}_t)\,, \end{aligned} \tag{21}$$

where w is the new solution of (10) obtained from an old solution $\hat{w}$ of (10). The Bäcklund parameters p_a and q_a are defined by

$$p_a = \tanh(ha/4), \quad q_a = \tanh(\tau\bar{a}/4)\,, \tag{22}$$

where a and $\bar{a}$ are real parameters constrained to satisfy

$$\tanh(ha/2)\tanh(\tau\bar{a}/2) = h\tau/4\,. \tag{23}$$

Setting $w = e^{(\rho + i\phi)/4}$ and $\hat{w} = e^{(\hat{\rho}+i\hat{\phi})/4}$, the discrete version of (5) becomes

$$\begin{aligned} \sin\left(\frac{(\phi_x - \phi) - (\hat{\phi}_x - \hat{\phi})}{4}\right) &= \tanh\left(\frac{ha}{2}\right)\sin\left(\frac{(\phi_x + \phi) + (\hat{\phi}_x + \hat{\phi})}{4}\right), \\ \sin\left(\frac{(\phi_t - \phi) + (\hat{\phi}_t - \hat{\phi})}{4}\right) &= \tanh\left(\frac{\tau\bar{a}}{2}\right)\sin\left(\frac{(\phi_t + \phi) - (\hat{\phi}_t + \hat{\phi})}{4}\right). \end{aligned} \tag{24}$$

In the continuous limit ($h = \tau \to 0$) Eq. (23) gives $\bar{a} = a^{-1}$, and (21) and (24) reduce to the usual Eqs. (5) and (6).

In the alternative discretization scheme of Eq. (15) the Bäcklund transformation is

$$w_x\hat{w} - w\hat{w}_x = -\nu_a(w^*\hat{w}^*)\,, \tag{25a}$$

$$w_t\hat{w}^* - w\hat{w}_t^* = -\mu_a(w_t^*\hat{w})\,, \tag{25b}$$

where $\nu_a = \tanh(ha/2)$, $\mu_a = \tanh(\tau\bar{a}/2)$ and a and $\bar{a}$ are constrained by the same Eq. (23). By interchanging the roles of x and t we can also obtain the corresponding Bäcklund transformations for the third discretization scheme of Eq. (20) as

$$\begin{aligned} w_x\hat{w} - w\hat{w}_x &= -\nu_a(w_x^*\hat{w}^*)\,, \\ w_t\hat{w}^* - w\hat{w}_t^* &= -\mu_a(w^*\hat{w})\,. \end{aligned} \tag{26}$$

The proof that the above equations indeed define Bäcklund transformations is straightforward. We illustrate this for the case of Eqs. (25). Assuming that $\hat{w}$ is a solution of the Eq. (15), and that w and $\hat{w}$ satisfy (25), we are to prove that w also is a solution of (15). Taking the t shift of (25a) we obtain

$$w_{xt}\hat{w}_t = w_t\hat{w}_{xt} - \nu_a(\hat{w}_t^* w_t^*)\,.$$

Multiplying both sides by $w\hat{w}$ and using Eq. (15) for $\hat{w}$ we obtain

$$\begin{aligned} \hat{w}\hat{w}_t(ww_{xt}) &= ww_t(\hat{w}\hat{w}_{xt}) - \nu_a w\hat{w}w_t^*\hat{w}_t^* \\ &= ww_t[\hat{w}_x\hat{w}_t + \tfrac{1}{4}h\tau(\hat{w}\hat{w}_t - \hat{w}^*\hat{w}_t^*)] \\ &\quad - \nu_a w\hat{w}w_t^*\hat{w}_t^* \end{aligned}$$

from which it follows that

$$\begin{aligned} &\hat{w}\hat{w}_t(ww_{xt} - w_xw_t - \tfrac{1}{4}h\tau ww_t) \\ &\quad = ww_t\hat{w}_x\hat{w}_t - \hat{w}\hat{w}_tw_xw_t - \tfrac{1}{4}h\tau ww_t\hat{w}^*\hat{w}_t^* - \nu_a\hat{w}_t^*w_t^*w\hat{w}\,. \end{aligned}$$

Using now the constraint Eq. (23) to replace $h\tau/4$ by $\mu_a\nu_a$ and using the complex conjugate of (25b) we can rewrite

$$\begin{aligned} \hat{w}\hat{w}_t(ww_{xt} - w_xw_t - \tfrac{1}{4}h\tau ww_t) &= w_t\hat{w}_t(w\hat{w}_x - w_x\hat{w}) - \nu_a w\hat{w}_t^*(w_t^*\hat{w} + \mu_a w_t\hat{w}^*) \\ &= w_t\hat{w}_t(\nu_a w^*\hat{w}^*) - \nu_a w\hat{w}_t^*(w^*\hat{w}_t) \\ &= \nu_a w^*\hat{w}_t(w_t\hat{w}^* - w\hat{w}_t^*) \\ &= \nu_a w^*\hat{w}_t(-\mu_a\hat{w}w_t^*) \\ &= \hat{w}\hat{w}_t(-\tfrac{1}{4}h\tau w_t^*w^*)\,, \end{aligned}$$

where we used (25b) and (23) again. Eliminating the common factor $\hat{w}\hat{w}_t$ we see that w must also be a solution of (15).

As in the continuous case, we may think of the Bäcklund transformation as a nonlinear "superposition" principle[1] which generates a new solution w by adding a soliton to an existing solution $\hat{w}$. Starting with the "vacuum" solution $\hat{w}(x,t) \equiv 1$ (with $\hat{\rho} = \hat{\phi} \equiv 0$) we can successively apply the Bäcklund transformation to generate the one-soliton, two-soliton, etc. solutions.

IV. ONE- AND TWO-SOLITON SOLUTIONS

Starting with the initial solution $\hat{w}(x,t) \equiv 1$ of (10), we solve (21) for w to obtain the one-soliton solution in Hirota's case. We have

$$\begin{aligned} & w_a = f_a + ig_a = e^{(\rho_a + i\phi_a)/4}\,, \\ & \tan(\phi_a/4) = g_a/f_a = e^{\theta_a}\,, \\ & g_a = e^{\theta_a/2} = \left(\frac{1+p_a}{1-p_a}\right)^n\left(\frac{1+q_a}{1-q_a}\right)^m, \\ & f_a = e^{-\theta_a/2}\,, \\ & \theta_a(x,t) \equiv ax + \bar{a}t = ahn + \bar{a}\tau m\,, \\ & \tanh(ha/2)\tanh(\tau\bar{a}/2) = h\tau/4\,, \end{aligned} \tag{27}$$

while in the alternative scheme of Eq. (15) the corresponding Bäcklund transformations (25) give for the one-soliton solution

$$\begin{aligned} &\tan(\phi_a/4) = g_a/f_a = e^{\theta_a}\,, \\ &g_a = (1+\nu_a)^n(1+\mu_a)^m\,, \\ &f_a = (1-\nu_a)^n(1-\mu_a)^m\,, \\ &\nu_a = \tanh(ha/2), \quad \mu_a = \tanh(\tau\bar{a}/2)\,, \\ &\mu_a\nu_a = h\tau/4\,. \end{aligned} \tag{28}$$

In the continuous limit ($h=\tau\to 0$) ϕ_a reduces to the usual soliton. Even though the discretization of x and t badly breaks the Lorentz invariance of the theory, the solution remains basically the same. In terms of the laboratory coordinates x^1 and x^0 we have

$$\theta_a(x,t) = ax + \bar{a}t = \gamma_a(x^1 - v_a x^0)\,,$$

where $v_a = (a-\bar{a})/(a+\bar{a})$, $\gamma_a = (a+\bar{a})/2$ and can be thought of as the "velocity" and the "Lorentz contraction factor" of the soliton. The soliton "at rest" would be characterized then by $v_a = 0$ or by $a=\bar{a} = a_0$, and $\theta_a(x,t) = a_0 x^1$, where a_0 is the solution of

$$\tanh(ha_0/2)\tanh(\tau a_0/2) = h\tau/4\,.$$

In the continuous limit $a_0 = \pm 1$ corresponding to the soliton or antisoliton.

Next, we consider the two-soliton solution which can be obtained by applying the Bäcklund transformation a second time. Starting with the one-soliton solution that we just found $w_a = f_a + ig_a$, and using new Bäcklund parameters for the second soliton, Eqs. (21) for w become

$$\begin{aligned} &w_x w_a - w w_{ax} = -p_b(w_x^* w_a^* + w^* w_{ax}^*)\,, \\ &w_t w_a^* - w w_{at}^* = -q_b(w_t^* w_a + w^* w_{at})\,, \end{aligned} \tag{29}$$

where $p_b = \tanh(hb/4)$, $q_b = \tanh(\tau\bar{b}/4)$, and b and $\bar{b}$ are constrained in a similar manner as in (23),

$$\tanh(hb/2)\tanh(\tau\bar{b}/2) = h\tau/4\,.$$

Solving (29) for $w = f + ig$ we find

$$\begin{aligned} &\tan\frac{\phi}{4} = \frac{g}{f} = \kappa\frac{\sinh[\frac{1}{2}(\theta_a-\theta_b)]}{\cosh[\frac{1}{2}(\theta_a+\theta_b)]} = \kappa\tan\left(\frac{\phi_a-\phi_b}{4}\right), \\ &g = \kappa^{1/2}\sinh[\tfrac{1}{2}(\theta_a-\theta_b)]\,, \\ &f = \kappa^{-1/2}\cosh[\tfrac{1}{2}(\theta_a+\theta_b)]\,, \\ &\kappa = \frac{e^{h(a+b)}-1}{e^{ha}-e^{hb}} = \frac{e^{\tau(\bar{b}+\bar{a})}-1}{e^{\tau\bar{b}}-e^{\tau\bar{a}}}\,, \\ &\theta_a(x,t) = ax + \bar{a}t, \quad \theta_b(x,t) = bx + \bar{b}t\,, \end{aligned} \tag{30}$$

where ϕ_a and ϕ_b are the one-soliton solutions given by (27) with parameters a and b, respectively. Because of the constraint Eqs. (23) we have also expressed κ in terms of the $\bar{a}$ and $\bar{b}$ parameters. We note the following: (i) Equations (30) are straightforward generalizations of the continuous case.[6] (ii) Because of the symmetry of the resulting equations with respect to a and b, it follows that the Bäcklund transformation is a commutative transformation just like in the continuous case. (iii) The inverse of the Bäcklund transformation can be obtained by reversing the sign of the a parameters. (iv) Solution (30) corresponds to two solitons (or two antisolitons) when a and b have opposite sign, and to a soliton and an antisoliton when a and b have the same sign. (v) From the last two remarks there follows the same exclusionlike principle as in the continuous case, namely, that two solitons cannot be put together if they have the same "velocity" parameters, i.e., $\phi\equiv 0$ if $a=-b$. (vi) So far the parameters a and b were assumed to be real; by choosing them to be complex conjugates of each other, i.e., $a = b^*$ we obtain in (30) a discrete analog of the breather solution of the sine-Gordon equation.

V. CONSERVATION LAWS

The procedure by which the Bäcklund transformation serves as the generator for an infinite number of conservation laws is well known.[1] It makes use of the energy conservation laws

$$\begin{aligned} &\partial_t[\tfrac{1}{2}(\partial_x\phi)^2] + \partial_x[\cos\phi] = 0\,, \\ &\partial_x[\tfrac{1}{2}(\partial_t\phi)^2] + \partial_t[\cos\phi] = 0\,. \end{aligned} \tag{31}$$

In this section we derive the discrete analogs of these conservation laws and a few others. Using the field equation (11) we can easily verify

$$\begin{aligned} &\Delta_t\left[1-\cos\left(\frac{\phi_x-\phi}{2}\right)\right] + \Delta_x\left[\frac{h\tau}{4}\cos\left(\frac{\phi+\phi_t}{2}\right)\right] = 0\,, \\ &\Delta_x\left[1-\cos\left(\frac{\phi_t-\phi}{2}\right)\right] + \Delta_t\left[\frac{h\tau}{4}\cos\left(\frac{\phi+\phi_x}{2}\right)\right] = 0\,, \end{aligned} \tag{32}$$

where Δ_x and Δ_t are the difference operators, i.e., $\Delta_x f = f_x - f$ etc. These equations reduce to (31) in the continuous limit. From (32) it follows that the quantities

$$\begin{aligned} &I_1 = \sum_x\left[1-\cos\left(\frac{\phi_x-\phi}{2}\right)\right], \\ &I_2 = \sum_x\left[1-\cos\left(\frac{\phi+\phi_x}{2}\right)\right] \end{aligned}$$

are t-shift invariant, that is, conserved. Two additional conservation laws can be obtained from the equation of motion (11), namely,

$$\Delta_t\left[\sin\left(\frac{\phi_x-\phi}{2}\right)\right]=\Delta_x\left[\frac{h\tau}{4}\sin\left(\frac{\phi+\phi_t}{2}\right)\right],$$
$$\Delta_x\left[\sin\left(\frac{\phi_t-\phi}{2}\right)\right]=\Delta_t\left[\frac{h\tau}{4}\sin\left(\frac{\phi+\phi_x}{2}\right)\right]. \qquad (33)$$

The conservation laws (32) can now be used in conjunction with the Bäcklund transformations (24), with an infinitesimal Bäcklund parameter a, to generate an infinite number of conservation laws in exactly the same way as in the continuous case.[1]

VI. INVERSE SCATTERING EQUATIONS

Hirota has shown[5] how to turn the Bäcklund transformation equations into the inverse scattering equations. This is done by rewriting (21) in terms of the quantities

$$\chi_1=\frac{\hat{w}^*}{w},\quad \chi_2=\frac{\hat{w}}{w^*},\quad \chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \qquad (34)$$

then (21) becomes

$$\chi_x=\begin{pmatrix}e^{i(\phi-\phi_x)/2} & p_a\\ p_a & e^{i(\phi_x-\phi)/2}\end{pmatrix}\chi+p_a\begin{pmatrix}0 & e^{i(\phi-\phi_x)/2}\\ e^{i(\phi_x-\phi)/2} & 0\end{pmatrix}\chi_x,$$
$$\chi_t-\chi=q_a\begin{pmatrix}0 & e^{-i(\phi+\phi_t)/2}\\ e^{i(\phi+\phi_t)/2} & 0\end{pmatrix}(\chi_t+\chi).$$

Eliminating the x-shifted and t-shifted quantities from the right-hand sides, we obtain an equivalent set of inverse scattering equations

$$\chi_x=\begin{pmatrix}\cosh(ha/2)e^{i(\phi-\phi_x)/2} & \sinh(ha/2)\\ \sinh(ha/2) & \cosh(ha/2)e^{i(\phi_x-\phi)/2}\end{pmatrix}\chi\equiv M\chi,$$
$$\chi_t=\begin{pmatrix}\cosh(\tau\bar{a}/2) & \sinh(\tau\bar{a}/2)e^{-i(\phi+\phi_t)/2}\\ \sinh(\tau\bar{a}/2)e^{i(\phi+\phi_t)/2} & \cosh(\tau\bar{a}/2)\end{pmatrix}\chi\equiv N\chi, \qquad (35)$$

where a and $\bar{a}$ are again constrained by (23). The "integrability" of this system (i.e., $\chi_{xt}=\chi_{tx}$) results in the condition

$$M_tN=N_xM,$$

which can easily be seen to be equivalent to the discrete sine-Gordon equation (11). In the continuous limit the usual inverse scattering equations (8) are recovered.

The advantage of this method of deriving the inverse scattering equations from the Bäcklund transformation lies in the fact that (34) allow one to determine χ_1 and χ_2 from the knowledge of w by undoing the Bäcklund transformation to obtain $\hat{w}$.

As an illustration, we work out the χ's for the one- and two-soliton solutions that we found in Sec. IV. Since the one-soliton solution $w=w_a=f_a+ig_a$ is obtained from the "vacuum" solution $\hat{w}=1$, we immediately find the corresponding χ from (34)

$$\chi_{a1}=\frac{1}{f_a+ig_a},\quad \chi_{a2}=\frac{1}{f_a-ig_a}$$

with f_a and g_a given in (27).

It can be verified that these solutions satisfy the quadratic relationships

$$-i\cosh(\tau\bar{a}/2)\chi^*_{a1t}\chi_{a2}=\tfrac{1}{4}(1-e^{i(\phi_a+\phi_{at})/2}),$$
$$\sinh(ha/2)\chi^*_{a2x}\chi_{a2}=(1/4i)(1-e^{i(\phi_a-\phi_{ax})/2}), \qquad (36)$$
$$\sinh(\tau\bar{a}/2)\chi^*_{a1t}\chi_{a1}=(1/4i)(1-e^{i(\phi_a-\phi_{at})/2}),$$

which are the discrete analogs of similar ones in the continuous case[6], namely,

$$-i\chi^*_{a1}\chi_{a2}=\tfrac{1}{4}(1-e^{i\phi_a}),$$
$$a\chi^*_{a2}\chi_{a2}=\tfrac{1}{4}\partial_x\phi_a, \qquad (37)$$
$$a^{-1}\chi^*_{a1}\chi_{a1}=\tfrac{1}{4}\partial_t\phi_a.$$

As shown in Ref. 6 Eqs. (37) form the basis for the classical Coleman correspondences between the sine-Gordon theory and the massive Thirring model. Therefore, (36) can be thought of as a discretized version of the Coleman correspondences.

For the two-soliton solution ϕ given by (30) we have two eigenvectors χ_a and χ_b corresponding to the two Bäcklund parameters a and b. To find χ_a we must undo the Bäcklund transformation with parameter a and we obtain $\hat{w}=w_b$, i.e., the one-soliton solution with parameter b. Similarly, for χ_b we must undo the Bäcklund transformation with parameter b and we obtain $\hat{w}=w_a$. We have then

$$\chi_{a1}=\frac{f_b-ig_b}{f+ig}\,,\quad \chi_{a2}=\frac{f_b+ig_b}{f-ig}\,,$$

$$\chi_{b1}=\frac{f_a-ig_a}{f+ig}\,,\quad \chi_{b2}=\frac{f_a+ig_a}{f-ig}\,,$$

where the f's and g's are given in (27) and (30).

The analogous equations to (36) are, in the two-soliton case,

$$-i\cosh(\tau\bar{a}/2)\chi^*_{a1t}\chi_{a2}+i\cosh(\tau\bar{b}/2)\chi^*_{b1t}\chi_{b2}=\tfrac{1}{4}(1-e^{i(\phi+\phi_t)/2})\,,$$

$$\sinh(ha/2)\chi^*_{a2x}\chi_{a2}-\sinh(hb/2)\chi^*_{b2x}\chi_{b2}=(1/4i)(1-e^{i(\phi-\phi_x)/2})\,,$$

$$\sinh(\tau\bar{a}/2)\chi^*_{a1t}\chi_{a1}-\sinh(\tau\bar{b}/2)\chi^*_{b1t}\chi_{b1}=(1/4i)(1-e^{i(\phi-\phi_t)/2})\,,$$

which are again the discrete analogs of the continuous case[6]

$$-i\chi^*_{a1}\chi_{a2}+i\chi^*_{b1}\chi_{b2}=\tfrac{1}{4}(1-e^{i\phi})\,,$$

$$a\chi^*_{a2}\chi_{a2}-b\chi^*_{b2}\chi_{b2}=\tfrac{1}{4}\partial_x\phi\,,$$

$$a^{-1}\chi^*_{a1}\chi_{a1}-b^{-1}\chi^*_{b1}\chi_{b1}=\tfrac{1}{4}\partial_t\phi\,.$$

VII. CONCLUSION

In discretizing nonlinear equations one is inevitably faced with a large number of choices (all with the same continuous limit) for discretizing the nonlinear terms. The basic difference between the discretization schemes that we have considered lies indeed in the way the nonlinear term w^2 in (4) is discretized. The particular choice one makes can alter the character of the resulting discrete sine-Gordon equation; it can even turn it, as we have seen, into a nonlocal equation. It is nevertheless rather remarkable that there exist discretization schemes that preserve most of the interesting properties of the continuous case.

[1]A thorough review and references on the sine-Gordon equation and its applications can be found in A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).

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[4]A recent review and references on the discrete-inverse-scattering method can be found in M. J. Ablowitz, Studies Appl. Math. 58, 17 (1978).

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[7]S. Coleman, Phys. Rev. D 11, 2088 (1975); S. Mandelstam, *ibid.* 11, 3026 (1975).

[8]R. Hirota, Prog. Theor. Phys. 52, 1498 (1975).

[9]In Ref. 5 Hirota uses equal lattice spacings $h=\tau=2\delta$, and his equations are recovered by the substitutions $x\to x-\delta$, $t\to t-\delta$ in our equations

[10]S. J. Orfanidis, following paper, Phys. Rev. D 18, 3828 (1978).