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Classical Yang-Mills theory in the presence of external sources

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We study in detail the classical Yang-Mills field equations in the presence of static external sources. Their formulation as an initial-value problem in the $A_0 = 0$ gauge provides us with a powerful tool for determining the existence of new solutions. In the case of point sources, the only static solutions known so far are the various Coulomb solutions which we classify according to their total energy and isospin. In the case of a localized but extended source, there are, besides the well-known Coulomb solution, two new types of solutions: the "magnetic dipole" solution which has the long-range behavior of a magnetic dipole field and which has lower energy than the Coulomb solution when the total external charge is large enough, and the "total screening" solution which has no long-range field strengths at all and which can have an arbitrarily low energy. We present a detailed study of these new solutions.

I. INTRODUCTION AND SUMMARY

Gauge theories offer the greatest promise to describe the elementary forces in nature. In particular, quantum chromodynamics (QCD), the quantum gauge theory of the unbroken local symmetry group SU(3) of color, is widely believed to be the correct theory of the strong interactions. Although this belief is strong and widespread there is actually very little known about QCD, especially in the infrared regime where the usual tool of field theory, perturbation theory, fails to be applicable.[1] In such a situation the investigation of the classical version of the theory[2] appears as a welcome source of relatively straightforward insights. It has already brought forth the existence of the Wu-Yang monopole[3] and of Coleman's non-Abelian plane waves.[4] The fascinating topological properties of non-Abelian gauge theories have further come to light through the discovery of the 't Hooft-Polyakov monopole[5] as a static solution to the broken-SU(2) gauge theory, and of the instanton[6] and meron[7] solutions to the Euclidean version of Yang-Mills theory. These latter solutions, although purely classical, have turned out to be directly relevant to the problem of defining the vacuum state of QCD and possibly also to the problem of confinement.[8]

In this paper, we will study the solutions to the classical Yang-Mills field equations in the presence of external sources, a problem relatively little investigated so far. We will not take into account the dynamics of the external source which we assume to be static.[9] That the study of classical Yang-Mills theories in the presence of external sources could contain some interesting surprises was first indicated by Mandula[10] in his study of small perturbations around the Coulomb solution. The existence of a Coulomb solution for an arbitrary external source is well known. Mandula showed that if the external source is distributed over a thin spherical shell, the Coulomb solution is unstable under small perturbations as soon as $gQ > \frac{3}{2}$ where g is the gauge coupling constant and Q is the total external charge. He also showed that the instability modes produce an inward flow of charge which tends to screen the external charge. Since the energy of the Yang-Mills fields in the presence of a static source is positive-definite, Mandula's result implies the existence of (at least) one solution with energy lower than that of the Coulomb solution.

In a previous letter,[11] we presented two new classes of solutions to the Yang-Mills field equations in the presence of static, localized but extended external sources. Both classes have totally screened electric fields. One class of solutions ("the magnetic dipole" solutions) has the long-range behavior of a magnetic dipole field and has lower energy than the Coulomb solution when gQ is large enough. The other class ("the total screening" solutions) has no long-range field strengths at all and has arbitrarily low energy for all values of gQ.

In Sec. II, we review the general properties of the Yang-Mills equations in the presence of external sources, with particular emphasis on static sources. In Sec. III, we discuss the Yang-Mills equations as an initial-value problem in the $A_0 = 0$ gauge. This is a very useful tool for showing the existence of new types of solutions.

For a point source, the Coulomb solution is the only static solution known so far.[12] In the presence of several point sources, there are in general several Coulomb solutions differing from one another by their total energy and isospin. This is discussed in detail in Sec. IV.

In Secs. V and VI we give a detailed account of the "total screening" and "magnetic dipole" solutions, respectively, and show how they generalize to gauge groups other than SU(2).

II. INTRODUCTION OF EXTERNAL SOURCES INTO THE YANG-MILLS EQUATIONS

The equations of motion for the gauge fields[13] A_a^μ in the presence of an external current j_a^μ are

$$D_\mu F^{\mu\nu} = j^\nu \ , \tag{2.1}$$

where

$$A^\mu = -iA_a^\mu T^a, \quad j^\mu = -ij_a^\mu T^a$$

and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + g[A^\mu, A^\nu]. \tag{2.2}$$

Here T^a form an arbitrary representation of the Lie algebra of a gauge group G,

$$[T^a, T^b] = i c^{abc} T^c \ , \tag{2.3}$$

where c^{abc} are the structure constants of G. (We use $a, b, c, \ldots$ to denote group indices, $\mu, \nu, \ldots$ for space-time indices and $i, j, k, \ldots$ for the spatial components. We use the metric with $g^{00} = -1$.) D_μ is the usual covariant derivative

$$D_\mu \phi = \partial_\mu \phi + g[A_\mu, \phi], \tag{2.4}$$

with $\phi = -i\phi^a T^a$ for any field $\phi^a(x)$ transforming as the adjoint representation of G. The field strengths $F_{\mu\nu}$ also satisfy

$$D_\mu \tilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} D_\mu F_{\alpha\beta} = 0 \tag{2.5}$$

as a direct consequence of their definition, (2.2).

It is well known that under local gauge transformations

$$A^\mu \to U A^\mu U^{-1} - \frac{1}{g}(\partial^\mu U)U^{-1} \ , \tag{2.6}$$

with

$$U = \exp[-ig\lambda^a(x) T^a],$$

the field strengths and their covariant derivatives transform as

$$F^{\mu\nu} \to U F^{\mu\nu} U^{-1}, \quad D_\alpha F^{\mu\nu} \to U(D_\alpha F^{\mu\nu})U^{-1}. \tag{2.7}$$

Thus Eq. (2.1) is covariant provided j^μ also transforms covariantly, i.e.,

$$j^\mu \to U j^\mu U^{-1} \ . \tag{2.8}$$

The equations of motion (2.1) can be derived from a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \mathrm{Tr}(j^\mu A_\mu). \tag{2.9}$$

The Noether current corresponding to the global gauge symmetry [$U(x)$ x independent] is

$$J^\nu = \partial_\mu F^{\mu\nu} = j^\nu - g[A_\mu, F^{\mu\nu}]. \tag{2.10}$$

J^ν is conserved ($\partial_\nu J^\nu = 0$) due to the antisymmetry of $F^{\mu\nu}$. Thus J^ν is a conserved but not a gauge-covariant current, whereas j^ν is gauge covariant but not conserved. $\partial_\nu J^\nu = 0$ implies conservation of the total isospin, i.e. external isospin plus the isospin carried by the Yang-Mills field. The total non-Abelian "charge" corresponding to J^ν is

$$I = \int d^3x J^0(x) = \int d^3x \partial_i F^{0i} = \int_{\substack{\text{surface}\\ \text{at } \infty}} F^{0i} n_i \, d^2x. \tag{2.11}$$

I is time independent and is covariant under gauge transformations which are constant gauge transformations at spatial infinity.

The gauge-covariant current j^ν, although not conserved, satisfies

$$D_\mu j^\mu = \partial_\mu j^\mu + g[A_\mu, j^\mu] = 0 \ . \tag{2.12}$$

Indeed, in general

$$[D_\mu, D_\nu]\phi = g[F_{\mu\nu}, \phi], \tag{2.13}$$

and therefore

$$D_\mu j^\mu = D_\mu D_\nu F^{\mu\nu} = \tfrac{1}{2}[D_\mu, D_\nu]F^{\mu\nu} = \tfrac{1}{2}[F_{\mu\nu}, F^{\mu\nu}] = 0. \tag{2.14}$$

From the Lagrangian (2.9) we can calculate the energy-momentum tensor

$$\begin{aligned} T^{\mu\nu} &= -\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\alpha^a}\,\partial^\nu A_\alpha^a + g^{\mu\nu}\mathcal{L} \\ &= +F_a^{\mu\alpha}\partial^\nu A_\alpha^a + g^{\mu\nu}\mathcal{L} \\ &= +F_a^{\mu\alpha}F_\alpha^{\nu a} - (D_\alpha F^{\mu\alpha})_a A_a^\nu + g^{\mu\nu}\mathcal{L} + \partial_\alpha(F_a^{\mu\alpha}A_a^\nu). \end{aligned} \tag{2.15}$$

Using Eqs. (2.1) and (2.15), the total energy is given by

$$H = \int d^3x T^{00} = \int d^3x \left[\tfrac{1}{2}(E^2 + B^2) + j_a^k A_k^a\right], \tag{2.16}$$

provided the surface integral

$$\int d^2x n_i F_{0i}^a A_0^a = 0 \ . \tag{2.17}$$

Here $E_i = F_{0i}$ and $B_k = \frac{1}{2}\epsilon_{kij}F^{ij}$ ($i, j, k = 1, 2, 3$).

A. Static sources

In this paper we shall be concerned with static sources. We call a source distribution $j^\mu(x)$ static if $j^i(x) = 0$, $i = 1, 2, 3$. Equation (2.12) then implies $\partial_0 q(x) = -g[A_0(x), q(x)]$ with $q(x) = j^0(x)$. This means that the time development of $q(\vec{x}, t)$ is given by a gauge transformation which depends upon $A_0(\vec{x}, t)$,

$$q(\vec{x}, t) = U(\vec{x}; t, t_0) q(\vec{x}, t_0) U^\dagger(\vec{x}; t, t_0)$$

with

$$U(\vec{x}; t, t_0) = \exp\left(-\int_{t_0}^{t} dt' \, g A_0(\vec{x}, t')\right)_{\text{path-ordered}} . \tag{2.18}$$

Consequently, our assumption that the source is static implies that the Casimir invariants build out of the $q^a(\vec{x}, t)$—i.e., the polynomial expressions in q^a which are invariants under the group transformations—are time independent. For example, for SU(n), in which case $q(x)$ is an anti-Hermitian traceless $n \times n$ matrix and the Casimir invariants are $C_l(x) = \mathrm{Tr}[q(x)]^l$, $l = 1, \ldots, n-1$, [$n-1$ = rank of SU(n)] we have

$$\begin{aligned} \partial_0 C_l &= l \, \mathrm{Tr}[(\partial_0 q) q^{l-1}] \\ &= -gl \, \mathrm{Tr}\{[A_0, q] q^{l-1}\} \\ &= -gl[\mathrm{Tr}(A_0 q^l) - \mathrm{Tr}(A_0 q^l)] \\ &= 0 . \end{aligned} \tag{2.19}$$

Thus a static source distribution only "rotates" in the internal isospin space. For a static source, the energy is simply given by

$$H = \tfrac{1}{2} \int d^3x \, (E^2 + B^2) . \tag{2.20}$$

We now consider the Yang-Mills field equations (2.1) in the presence of a static source $j_a^\mu(x) = \delta^{\mu 0} q_a(x)$. Let us specialize for the moment to the gauge group SU(2). In this case, for an arbitrary $q^a(x)$ there exists[14] a gauge transformation U(x) such that $[U(x) q(x)]^a = \delta^{a3} q(\vec{x})$ when $q(\vec{x}) = [q^a(\vec{x}) q_a(\vec{x})]^{1/2}$ is time independent. With the ansatz $A_a^\mu = \delta_{a3} C^\mu$ all the nonlinear terms disappear from Eqs. (2.1). The resulting equations are the Abelian Maxwell equations.

$$\partial_\mu(\partial^\nu C^\mu - \partial^\mu C^\nu) = g^{\nu 0} q(\vec{x}) . \tag{2.21}$$

Its solutions are the static Coulomb potential plus an arbitrary radiation field. They are thus also solutions to the non-Abelian equations (2.1). We will call the solution with the static Coulomb potential only, the *"Coulomb solution"* to Eqs. (2.1).

Now consider a general gauge group G of order n and rank r. By definition, the rank is the maximum number of commuting generators in the Lie albebra of G. This maximal set of commuting generators is called the Cartan subalgebra. The rank of G is also the maximum number of independent polynomial invariants (i.e., Casimir invariants) which can be constructed out of the generators.

Let T_i ($i = 1, \ldots, r$) be the generators of the Cartan subalbegra of G. For an arbitrary source distribution $q^a(x)$ ($a = 1, \ldots, n$) there exists[14] a gauge transformation $U(x)$ which lines up q^a completely within the Cartan subalgebra

$$U(x) q(x) U^\dagger(x) = \sum_{i=1}^{r} (-i) T_i q_i'(\vec{x}) . \tag{2.22}$$

The $q_i'(\vec{x})$ are time independent because they can always be expressed in terms of the r time-independent Casimir invariants build out of the $q^a(x)$. If we make the ansatz $A_\mu(x) = (-i) \sum_{i=1}^{r} T_i C_\mu^i(x)$, Eqs. (2.1) become linear and identical to the set of r Abelian Maxwell equations

$$\partial_\mu(\partial^\nu C_i^\mu - \partial^\mu C_i^\nu) = g^{\nu 0} q_i'(\vec{x}), \quad i = 1, \ldots, r. \tag{2.23}$$

Thus for any gauge group and for an arbitrary[14] static source distribution, there always exists a static Coulomb solution (plus an arbitrary radiation field) to the Yang-Mills equations which can be obtained by solving the linear Maxwell equations (2.23).

Let us in particular consider the case of the gauge group SU(3) which is of order 8 and rank 2. The commuting generators of SU(3) are those corresponding to the two diagonal Gell-Mann matrices λ_3 and λ_8. A general source distribution is given by $q(x) = \sum_{a=1}^{8} (-i/2) \lambda^a q^a(x)$. The Casimir invariants are

$$C_1(\vec{x}) = -2 \, \mathrm{Tr}[q(x)]^2 = q^a(x) q^a(x)$$

and

$$C_2(\vec{x}) = -4i \, \mathrm{Tr}[q(x)]^3 = d_{abc} q^a(x) q^b(x) q^c(x) . \tag{2.24}$$

For an arbitrary source distribution $q(x)$, there exists[14] a local gauge transformation $U(x)$ that lines up $q(x)$ into the commuting directions of internal space,

$$U(x) q(x) U^\dagger(x) = \frac{-i}{2} [q_3'(\vec{x}) \lambda_3 + q_8'(\vec{x}) \lambda_8] . \tag{2.25}$$

Indeed one can always find a special unitary matrix to diagonalize an arbitrary Hermitian matrix. q_3' and q_8' are time independent bacause they are related to the Casimir invariants by

$$\begin{aligned} C_1(\vec{x}) &= [q_3'(\vec{x})]^2 + [q_8'(\vec{x})]^2 , \\ C_2(\vec{x}) &= \frac{-1}{\sqrt{3}} [q_8'(\vec{x})]^3 + \sqrt{3} \, [q_3'(\vec{x})]^2 q_8'(\vec{x}) . \end{aligned} \tag{2.26}$$

III. THE INITIAL-VALUE PROBLEM IN THE $A_0 = 0$ GAUGE

In the following sections we shall discuss several static solutions to the classical Yang-Mills equations. Sometimes, however, when we are unable to find static solutions, it will be useful to know that certain nonstatic solutions exist with a given energy and total isospin. The initial-value problem for the Yang-Mills equation in the $A_0=0$ gauge provides a very powerful technique for accomplishing this.

It is well known that for any gauge field configuration $A_a^\mu(x)$ one can find a gauge transformation $U(x)$ such that $A^{\mu\prime}=UA^\mu U^\dagger-(1/g)(\partial^\mu U)U^\dagger$ has $A_0'=0$. Having fixed $A_0=0$, there remains the freedom to perform local gauge transformations $U(\vec{x})$ which are time independent, since they preserve $A^0=0$. Let us thus consider Eqs. (2.1) in the $A_0=0$ gauge:

$$(D_i E_i)^a=\partial_i E_i^a+g(A_i\times E_i)^a=q^a(\vec{x})\ , \tag{3.1a}$$

$$\frac{dE_i^a}{dt}=(D_j F_{ji})^a\ , \tag{3.1b}$$

with

$$E_i^a=\frac{dA_i^a}{dt}\ , \tag{3.1c}$$

where $(A_i\times E_i)^a=c^{abc}A_i^b E_i^c$. Equation (2.12) implies that in the $A_0=0$ gauge $\partial_0 q^a=0$. Thus both the magnitude and the direction of the sources in the internal space is fixed.

Equation (3.1b) is a second-order equation for the time evolution of $A_i^a(x)$. Thus, given A_i^a and $E_i^a=\partial_0 A_i^a$ at some initial time $t=t_0$, the Yang-Mills fields are specified for all time through Eqs. (3.1b) and (3.1c). Equation (3.1a) provides a constraint on the values of $A_i^a(\vec{x},t)$ and $E_i^a(\vec{x},t)$. However, since Eq. (3.1b) ($D_\mu F^{j\mu}=0$) combined with Eq. (2.14) ($D_\mu D_\nu F^{\mu\nu}=0$) implies $\partial_0(D_i E_i)=0$ in the $A_0=0$ gauge, it is sufficient that A_i^a and E_i^a satisfy the constraint (3.1a) at $t=t_0$ for them to satisfy the constraint at all times.

The total energy of the system and its total isospin I can be computed at $t=t_0$ and will be conserved. If the energy is finite then the positive-definiteness of the Hamiltonian density $\mathcal{H}=\frac{1}{2}(E^2+B^2)$ ensures that no nonintegrable singularities in $\mathcal{H}$ will develop in time.

IV. POINT SOURCES

In this section we discuss the Yang-Mills fields produced by a system of point sources. Let us first consider the case of the gauge group SU(2) and two point sources separated by a distance $r=|\vec{a}_1-\vec{a}_2|$. We have

$$q^a(x)=Zg[e_1^a\delta(\vec{x}-\vec{a}_1)+e_2^a\delta(\vec{x}-\vec{a}_2)]\ , \tag{4.1}$$

where Z is the charge of both point sources in units of the gauge coupling constant g, and e_1^a and e_2^a are unit vectors giving the orientation of the two sources in isospin space. Equation (4.1) does not give a gauge-invariant characterization of the sources, however. A locally gauge-invariant characterization is the following:

$$Q(\vec{x})=[q^a(\vec{x})q^a(\vec{x})]^{1/2}=Zg[\delta(\vec{x}-\vec{a}_1)+\delta(\vec{x}-\vec{a}_2)]\ . \tag{4.2}$$

The situation corresponds to having two δ-function localized particles represented by vectors ψ_1 and ψ_2 which transform as the n-dimensional representation of SU(2),

$$q^a(x)=\sum_{i=1}^{2} g\psi_i^\dagger T^a\psi_i\delta(\vec{x}-\vec{a}_i)\ . \tag{4.3}$$

The statement that two such particles are located at $\vec{a}_1$ and $\vec{a}_2$ is gauge invariant whereas a specification of their orientation in isospin space is not gauge invariant.

The gauge invariance of the Yang-Mills equations (2.1) means that if we find a solution to those equations with the orientations e_1^a and e_2^a on the right-hand side, this is also, after an appropriate gauge transformation, a solution for any other orientations $e_1^{a\prime}$ and $e_2^{a\prime}$. Therefore, when we write down an expression for a solution A_μ^a, although this expression will always correspond to particular values of e_1^a and e_2^a, these values of e_1^a and e_2^a do not characterize the solution. Instead we will characterize a solution by its energy and its total isospin (isospin of the source plus isospin carried by the Yang-Mills fields) because these quantities are gauge invariant in addition to being conserved.

Following our discussion of the initial value problem in the $A_0=0$ gauge, we solve the $t=t_0$ constraint equation (3.1a) for the source (4.2) by

$$E_i^a(\vec{x})=\frac{Zg}{4\pi}\left(e_1^a\frac{x_i-a_{1i}}{|\vec{x}-\vec{a}_1|^3}+e_2^a\frac{x_i-a_{2i}}{|\vec{x}-\vec{a}_2|^3}\right)\ , \tag{4.4}$$

$$A_i^a(\vec{x})=0\ .$$

The resulting solution to the equations of motion will not be static unless $e_1^a=\pm e_2^a$ in which cases $dE_j^a/dt=(D_i F_{ij})^a=0$ and $A_i^a=(t-t_0)E_i^a$. The energy of the configuration (4.4) is infinite but by subtracting the infinite self-energy of each source we obtain a finite interaction energy

$$H_{\rm int}=\frac{Z^2g^2}{4\pi r}e_1^a e_2^a\ . \tag{4.5}$$

The total isotopic charge is given by Eq. (2.11),

$$I^a=Zg(e_1^a+e_2^a)\ . \tag{4.6}$$

We thus have an infinite set of solutions to the Yang-Mills equations with the source (4.2). e_1^a and e_2^a are just a convenient way of labeling these solutions.

Among these solutions, only two (up to gauge transformations) are static. They correspond to setting $e_1^a = +e_2^a$ and $e_1^a = -e_2^a$. In the Coulomb gauge they have the form

$$A_0^a(x) = \frac{Zg}{4} e_1^a \left(\frac{1}{|\vec{x} - \vec{a}_1|} \pm \frac{1}{|\vec{x} - \vec{a}_2|} \right), \quad A_i^a(x) = 0. \tag{4.7}$$

Their energy is

$$H_{\rm int} = \pm \frac{Z^2 g^2}{4\pi r}, \tag{4.8}$$

and their total isospin

$$I^a = \begin{Bmatrix} 2Zge_1^a \\ 0 \end{Bmatrix}. \tag{4.9}$$

In a loose way, these two static solutions correspond to the $I = 0$ and $I = 1$ representations that figure in the Kronecker product of two doublets,

$$\underline{2} \times \underline{2} = \underline{1} + \underline{3}. \tag{4.10}$$

The $I = 0$ state is attractive, while the $I = 1$ state is repulsive.

A. Point sources in SU(3)

We now turn to point sources for the gauge group SU(3). In the case of SU(2) both a neutral attractive ($I = 0$) and a repulsive ($I = 1$) solution exist for the source distribution corresponding to two point sources of equal magnitude. In fact there always exists U in SU(2) such that $(Ue)^a = -e^a$ for any three vector e^a. This result does not hold in general for other representations and other groups. In particular we shall see that there is no U in SU(3) such that $U_{ab}\delta_{b8} = -\delta_{a8}$. As a consequence the source distribution $Zg\delta^{a8}[\delta(\vec{x} - \vec{a}_1) + \delta(\vec{x} - \vec{a}_2)]$ has a repulsive solution with $I^a = 2Zg\delta^{a8}$ but it has no attractive solution with $I^a = 0$. Let us now discuss this situation in more detail.

Recall that for SU(3) there are two gauge-invariant objects—the Casimir invariants (2.24),

$$\begin{aligned} C_1(\vec{x}) &= -2\,\mathrm{Tr}[q^2(x)] = q^a(x) q_a(x), \\ C_2(\vec{x}) &= -4i\,\mathrm{Tr}(q^3(x)) = d_{abc} q^a(x) q^b(x) q^c(x). \end{aligned} \tag{4.11}$$

Observe that for $q^a = \delta^{a8}$, $C_2 = -1/\sqrt{3}$, whereas for $q^a = -\delta^{a8}$, $C_2 = 1/\sqrt{3}$. Thus no group element can change $+\delta^{a8}$ to $-\delta^{a8}$; they are in different orbits.

Consider a field $\psi^a(x)$, $a = 1, 2, 3$ in the color-triplet representation of SU(3); we shall call its components red, white, and blue, respectively. The color charge distribution carried by ψ will be (apart from a normalization constant)

$$q^a(x) = \psi^\dagger(x) \tfrac{1}{2} \lambda^a \psi(x). \tag{4.12}$$

$\psi^a(x)$ is always related, by a gauge transformation, to $\psi'^a(x) = \delta^{a3}\psi'^3(x)$ in which case

$$\begin{aligned} q'^a(x) &= -\frac{1}{\sqrt{3}} [\psi'^\dagger(x)\psi'(x)]\delta_{a8} \\ &= \frac{-1}{\sqrt{3}} [\psi^\dagger(x)\psi(x)]\delta_{a8}. \end{aligned} \tag{4.13}$$

Thus the source produced by a field ψ in the triplet representation has the characteristic property that it is in the orbit of $-(1/\sqrt{3})[\psi^\dagger(x)\psi(x)]\delta_{a8}$. In other words its source distribution is related to $-(1/\sqrt{3})\psi^\dagger(x)\psi(x)\delta_{a8}$ by a gauge transformation. This can be expressed in a gauge-invariant manner by saying that for a color triplet

$$\frac{C_2(x)}{C_1(x)^{3/2}} = \frac{1}{\sqrt{3}}. \tag{4.14}$$

For a color antitriplet ($\lambda^a \to -\lambda^{aT}$) we have

$$\frac{C_2(x)}{C_1(x)^{3/2}} = -\frac{1}{\sqrt{3}}. \tag{4.15}$$

The color charge distribution carried by a color antitriplet is in the orbit of $(+1/\sqrt{3})\delta_{a8}\psi^\dagger(x)\psi(x)$.

Another interesting property of SU(3) is that although no group transformation can change δ_{a8} to $-\delta_{a8}$ there exist transformations which change δ_{a8} to

$$\delta_{a8} \to -\tfrac{1}{2}\delta_{a8} + \frac{\sqrt{3}}{2} \sum_{i=1}^{3} \alpha_i \delta_{ai}, \tag{4.16}$$

with

$$\sum_{i=1}^{3} |\alpha_i|^2 = 1.$$

In particular the currents corresponding to blue, white, and red are, respectively,

$$\text{blue:}\quad \psi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to q^a = -\frac{1}{\sqrt{3}} \delta^{a8}, \tag{4.17a}$$

$$\text{white:}\quad \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to q^a = \frac{1}{\sqrt{3}} \left(\tfrac{1}{2}\delta_{a8} - \frac{\sqrt{3}}{2}\delta_{a3} \right), \tag{4.17b}$$

$$\text{red:}\quad \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to q^a = \frac{1}{\sqrt{3}} \left(\tfrac{1}{2}\delta_{a8} + \frac{\sqrt{3}}{2}\delta_{a3} \right). \tag{4.17c}$$

These q^a are all related by group transformations. In fact (4.17) correspond to the three different ways in which the matrix $q = -\frac{1}{2} i \lambda^a q^a$ with $q^a = \psi^\dagger(x) \frac{1}{2} \lambda^a \psi(x)$, can be diagonalized.

We now turn to the problem of two point sources both in the triplet representation of SU(3) at $\vec{a}_1$ and $\vec{a}_2$ with $r = |\vec{a}_1 - \vec{a}_2|$. As for SU(2) [Eqs. (4.1) and (4.2)] we can consider the sources at $\vec{a}_1$ and $\vec{a}_2$ to be arbitrary vectors in the orbit of $-(1/\sqrt{3})\delta_{a8}$. We can then write an $A^0 = 0$ gauge initial condition as in Eq. (4.4) which leads to a solution of the equations of motion. Up to global gauge transformation only two of these solutions are static. They can be written in Coulomb gauge as

$$A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\delta^{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}-\frac{1}{|\vec{x}-\vec{a}_2|}\right),\tag{4.18a}$$

$$A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\left[\delta^{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}+\frac{1}{2}\frac{1}{|\vec{x}-\vec{a}_2|}\right),\right.$$
$$\left.-\delta^{a3}\left(\frac{\sqrt{3}}{2}\frac{1}{|\vec{x}-\vec{a}_2|}\right)\right].\tag{4.18b}$$

They correspond to the two inequivalent ways one can diagonalize the sources [blue-blue for (4.18a) and blue-white for (4.18b)].

Thus, as for SU(2), we have an infinity of nonequivalent solutions for the same source distribution. These solutions are characterized by their total energies and by their total color, I^a. Note that in their ground state, the two color-triplet point sources attract but are not neutral [Eq. 4.18b)]. Furthermore, the two static Coulomb solutions (4.18a) and (4.18b) of the system are analogous to the two representations which can be constructed from two color triplets

$$\underline{3}\times\underline{3}=\underline{\bar{3}}+\underline{6}\,.\tag{4.19}$$

Equation (4.18a) corresponds to the sextet whereas (4.18b) corresponds to the antitriplet.

Now consider the situation of a pointlike color triplet at $\vec{a}_1$, and a pointlike antitriplet at $\vec{a}_2$. Again there are an infinity of solutions with various energies and total color corresponding to all the combined orientations in the orbit of $(-1/\sqrt{3})\delta_{a8}$ for the color triplet $(1/\sqrt{3})\delta_{a8}$ for the antitriplet. Among them there are again two static solutions,

$$A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\delta^{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}+\frac{1}{|\vec{x}-\vec{a}_2|}\right),\tag{4.20a}$$

$$A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\left[\delta^{a8}\left(\frac{-1}{|\vec{x}-\vec{a}_1|}-\frac{1}{2}\frac{1}{|\vec{x}-\vec{a}_2|}\right)\right.$$
$$\left.+\delta^{a3}\left(\frac{\sqrt{3}}{2}\frac{1}{|\vec{x}-\vec{a}_2|}\right)\right].\tag{4.20b}$$

Solution (4.20a) is attractive with $I^a=0$ while (4.20b) is repulsive with $I^a\neq 0$. Once more one can show that (4.20a) has the lowest energy in the aforementioned infinite class of solutions. Equations (4.20a) and (4.20b) correspond to the two nonequivalent ways of diagonalizing the sources: either they have opposite color (blue-$\overline{\text{blue}}$) or they do not have opposite color (blue-$\overline{\text{white}}$). We can again draw an analogy between (4.20a) and (4.20b) and the decomposition

$$\underline{3}\times\underline{\bar{3}}=\underline{1}+\underline{8}\,.\tag{4.21}$$

Finally we consider three color triplets at $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. Of the infinite class of solutions with different energies and total color there now exist three static solutions which are not related by a global gauge transformation. These correspond to blue-blue-blue, blue-blue-white, and blue-white-red and are given by

$$\text{b-b-b},\quad A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\delta_{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}-\frac{1}{|\vec{x}-\vec{a}_2|}-\frac{1}{|\vec{x}-\vec{a}_3|}\right),\tag{4.22a}$$

$$\text{b-b-w},\quad A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\left[\delta_{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}-\frac{1}{|\vec{x}-\vec{a}_2|}+\frac{1}{2}\frac{1}{|\vec{x}-\vec{a}_3|}\right)-\delta_{a3}\left(\frac{\sqrt{3}}{2}\frac{1}{|\vec{x}-\vec{a}_3|}\right)\right],\tag{4.22b}$$

$$\text{b-w-r},\quad A_i^a=0,\quad A_0^a=\frac{1}{4\pi}\frac{1}{\sqrt{3}}\left[\delta_{a8}\left(-\frac{1}{|\vec{x}-\vec{a}_1|}+\frac{1}{2}\frac{1}{|\vec{x}-\vec{a}_2|}+\frac{1}{2}\frac{1}{|\vec{x}-\vec{a}_3|}\right)+\delta_{a3}\left(\frac{\sqrt{3}}{2}\frac{1}{|\vec{x}-\vec{a}_3|}-\frac{\sqrt{3}}{2}\frac{1}{|\vec{x}-\vec{a}_2|}\right)\right],\tag{4.22c}$$

Equation (4.22c) has the lowest energy of these solutions. It has $I^a=0$, and it is thus neutral. Equations (4.22a), (4.22b), and (4.22c) correspond, respectively, to the $\underline{10}$, $\underline{8}$, and $\underline{1}$ in the decomposition

$$\underline{3}\times\underline{3}\times\underline{3}=\underline{10}+2\,\underline{8}+\underline{1}\,.\tag{4.23}$$

We summarize the situation for SU(3) in Table I by showing the relative values of the interaction energies for various states (i.e., static solutions).[15]

V. EXTENDED SOURCES AND SCREENING

Consider the Yang-Mills field produced by an extended, static source distribution $q^a(x)$. For SU(2) we characterize this source by the gauge-invariant distribution $q(\vec{x})=[q^a(x)q_a(x)]^{1/2}$. Let us assume that $q(\vec{x})$ has no δ-function singularities and that $q(\vec{x})\leq Ae^{-br}$ as $r=|\vec{x}|\to\infty$ for some A and $b>0$. The standard "Coulomb" solution for this configuration is obtained by setting

$$q^a(x)=\delta^{a3}q(\vec{x})$$

TABLE I. Relative values of the interaction energies for the Coulomb solutions in the presence of two color triplets, of a triplet and antitriplet, and of three triplets in SU(3) Yang-Mills theory.

Decomposition	Interaction energy
$\underline{3}\times\underline{3}=\underline{\bar{3}}+\underline{6}$	$\underline{\bar{3}}$, solution (4.18b): $\frac{-1}{2r_{12}}$
	$\underline{6}$, solution (4.18a): $\frac{+1}{r_{12}}$
$\underline{3}\times\underline{\bar{3}}=\underline{1}+\underline{8}$	$\underline{1}$, solution (4.20a): $\frac{-1}{r_{12}}$
	$\underline{8}$, solution (4.20b): $\frac{+1}{2r_{12}}$
$\underline{3}\times\underline{3}\times\underline{3}=\underline{1}+2\,\underline{8}+\underline{10}$	$\underline{1}$, solution (4.22c): $(-1/2)(1/r_{12}+1/r_{13}+1/r_{23})$
	$\underline{8}$, solution (4.22b): $+1/r_{12}-\frac{1}{2}(1/r_{13}+1/r_{23})$
	$\underline{10}$, solution (4.22a): $\frac{1}{r_{12}}+\frac{1}{r_{13}}+\frac{1}{r_{23}}$

and

$$A_a^\mu(\vec{x})=\delta_{a3}C^\mu(\vec{x})\,. \tag{5.1}$$

Let us first restrict ourselves to spherically symmetric source distributions, $q(\vec{x})\equiv q(r)$. Let us suppose for the moment that $q(r)=0$ for $r>r_0$. The Coulomb solution with $q^a=\delta^{a3}q(r)$ will have the long-range electric field $E_i^a=(Q/4\pi)(\hat{r}_i/r_2)\delta_{a3}$ for $r>r_0$. Now recall that in SU(2) a source $+\delta_{a3}$ can locally be changed to $-\delta_{a3}$. (Such a feat is impossible in electrodynamics where the sign of the charge is gauge invariant.) To see the consequences of this let us divide the region $r\leq r_0$ into an even number of shells each having an equal total isotopic charge. Then let q^a point in the $+\delta_{a3}$ and $-\delta_{a3}$ directions in alternate shells. If we then make the ansatz $A_a^\mu=\delta_{a3}C^\mu$, the source distribution on the right-hand side of the resulting Maxwell equations is spherically symmetric and has zero net charge. The solution which results thus has a vanishing electric and magnetic field for $r>r_0$. By a gauge transformation this is also a solution for the configuration $q^a(x)=\delta^{a3}q(r)$. We shall see that as the number of shells tends to infinity the energy of the solution tends to zero. Thus for a spherically symmetric extended source distribution there exist solutions of arbitrarily small energy with totally screened electric and magnetic fields.

For the spherically symmetric case it is technically simple to avoid the discontinuities associated with a sharp transition from $+\delta_{a3}$ to $-\delta_{a3}$. Let us return to a general $q(r)$ and define

$$Q=\int_0^\infty 4\pi r^2 q(r)dr\,, \tag{5.2a}$$

$$h(r)=\frac{1}{Q}\int_r^\infty 4\pi R^2 q(R)dR\,, \tag{5.2b}$$

so that

$$q(r)=-\frac{Q}{4\pi}\frac{1}{r^2}\frac{dh}{dr}\,. \tag{5.2c}$$

$h(r)$ is the fraction of the total charge Q outside a radius r. Let us represent the charge distribution as follows:

$$q^a(r)=q(r)[\delta^{a2}\sin 2\pi n h(r)+\delta^{a3}\cos 2\pi n h(r)]\,. \tag{5.3}$$

The Yang-Mills equations now have the following solution:

$$\begin{aligned}A_i^a&=E_i^a t\,,\\ E_i^a&=\frac{Q}{4\pi}\frac{\hat{r}_i}{r^2}\frac{1}{2\pi n}\{\delta^{a2}[\cos 2\pi n h(r)-1]\\ &\qquad-\delta^{a3}\sin 2\pi n h(r)\}\,.\end{aligned} \tag{5.4}$$

This can now be rotated back into a gauge where q^a is parallel to δ_{a3} everywhere. The solution then has the form

$$\begin{aligned}q^a(x)&=q(r)\delta^{a3}\,,\\ A_0^a&=0,\ A_i^a=E_i^a t-\delta^{a1}\frac{1}{g}\partial_i[2\pi n h(r)]\,,\\ E_i^a&=\frac{Q}{4\pi}\frac{\hat{r}_i}{r_2}\frac{1}{2\pi n}\{\delta^{a2}[1-\cos 2\pi n h(r)]\\ &\qquad-\delta^{a3}\sin 2\pi n h(r)\}\,.\end{aligned} \tag{5.5}$$

The electric field is completely screened and there is no magnetic field. The energy of this total screening solution is computed from (5.4) or (5.5) and is given by

$$H^{\text{t.s.}}=\frac{1}{a}\frac{Q^2}{2\pi}\frac{1}{(2\pi n)^2}\int_0^\infty \frac{dx}{x^2}\sin^2\pi n h(xa)\ . \tag{5.6}$$

$H^{\text{t.s.}}$ is finite provided $1-h(r)\sim r^{1/2+\Delta}$ as $r\to 0$ with $\Delta>0$. [(5.2.b) implies $h(0)=1$.] This means that $q(r)$ is less singular than $1/r^{5/2}$ as $r\to 0$; this condition is also necessary and sufficient for the Coulomb solution to have finite energy. If $h(r)$ satisfies the above condition then $H^{\text{t.s.}}\to 0$ as $n\to\infty$.[11] We have thus shown that a static, extended, spherically symmetric source distribution admits solutions of arbitrarily small energies for which the electric field is totally screened and the magnetic field is zero.

Before turning to a general (nonspherically symmetric) source let us see what happens in the case of SU(3). The source is now described by the two gauge-invariant functions,

$$\begin{aligned} C_1(\vec{x}) &= q^a(x)q_a(x)\ ,\\ C_2(\vec{x}) &= d_{abc}q^a(x)q^b(x)q^c(x)\ . \end{aligned} \tag{5.7}$$

We have already noted in Sec. II that C_1 and C_2 are time independent if $\vec{j}=0$. Suppose first that $C_2(\vec{x})=0$. Then $q^a(x)$ can be gauge rotated into the form $q^a(x)=q(\vec{x})\delta^{a3}$. It then lies completely within an SU(2) subalgebra of SU(3). As a consequence $q^a(x)$ can be locally changed from $+\delta_{a3}$ to $-\delta_{a3}$ and the total screening solutions exist as for SU(2).

If $C_2(\vec{x})\neq 0$ then the situation becomes more complicated. As discussed in Sec. II $q^a(x)\lambda^a$ can always be diagonalized by a gauge transformation so that the source can be represented as

$$q^a(\vec{x})=\delta^{a3}q^3(\vec{x})+\delta^{a8}q^8(\vec{x})\ . \tag{5.8}$$

For simplicity we shall restrict ourselves to the case where the source is derived from a field in the $\underline{3}$ representation of SU(3) as in (4.13). [Our results apply to the general case (5.8) as well.] Thus consider a spherically symmetric source with

$$q^a(r)=-\delta^{a8}q(r)=\psi^\dagger(r)\frac{\lambda^a}{2}\psi(r)\ ,$$

where

$$\psi^a(r)=\delta_{a3}[\sqrt{3}\,q(r)]^{1/2}\ .$$

By a local gauge transformation $q^a(r)$ can be rotated to the form

$$q'^a(r)=\sqrt{3}\,q(r)\left[\left(\frac{1}{\sqrt{3}},\sqrt{\tfrac{2}{3}}\cos\theta,\ \sqrt{\tfrac{2}{3}}\sin\theta\right)\frac{\lambda_a}{2}\begin{pmatrix}1/\sqrt{3}\\ \sqrt{2/3}\cos\theta\\ \sqrt{2/3}\sin\theta\end{pmatrix}\right], \tag{5.10}$$

with

$$\theta=2\pi n h(r)\ .$$

where $h(r)$ is given by Eq. (5.2). It is now a simple exercise to compute the components of $q'^a(r)$ and to verify that $\int d^3x q^{a\prime}(x)=0$. In analogy with (5.4) one can now write the electric field as the sum of the Coulomb field for each component of $q'^a(x)$ and write $A_i^a=E_i^a t$, $A_0^a=0$. One then has a solution to the Yang-Mills equations. The electric field is totally screened and the energy tends to zero as $n\to\infty$.

One can understand this result better by recalling that although a source $-\delta^{a8}$ (corresponding to a $\underline{3}$) cannot be rotated into $+\delta^{a8}$ (corresponding to a $\overline{\underline{3}}$) we are still able (as in Sec. IV) to combine three $\underline{3}$'s to neutralize the system. In other words one divides $q(r)$ into equally charged shells, as we did for SU(2) and alternately assigns to these shells the charges $-\delta^{a8}$, $+\frac{1}{2}\delta^{a8}+(\sqrt{3}/2)\delta^{a3}$, and $+\frac{1}{2}\delta^{a8}-(\sqrt{3}/2)\delta^{a3}$ thus neutralizing the charge.

Finally we comment on the case of a general nonspherically symmetric extended source distribution $q(\vec{r})$ which vanishes exponentially at spatial infinity. We consider the gauge group SU(2) for simplicity. There exist infinitely many source distributions $q^a(\vec{r})$ with the same $q(\vec{r})=[q^a(\vec{r})q_a(\vec{r})]^{1/2}$ all of which are related by gauge transformations. However, let us, for the moment, consider each such $q^a(\vec{r})$ separately. The initial condition (3.1a) for the $A^0=0$ gauge initial-value problem is solved by

$$\begin{aligned} A_i^a &= 0,\\ E_i^a(\vec{x}) &= \frac{1}{4\pi}\int d^3x'\,q^a(\vec{x}')\frac{x_i-x_i'}{|\vec{x}-\vec{x}'|^3},\ \text{at } t=t_0. \end{aligned} \tag{5.11}$$

Clearly these configurations for different choices of $q^a(\vec{r})$ [but for the same $q(\vec{r})$] are in general not related by a gauge transformation. In fact they have different energies and total isospin. Most of the solutions resulting from the time-evolution (3.1b) and (3.1c) will be non-Abelian in character (for example $[E_i,E_j]$ is usually nonzero) and in general they will not be static. We seen then that in addition to the Coulomb solution discussed in Sec. II, any source distribution $q(\vec{r})$ admits an infinite class of solutions with varying energies and total isospin.

As a particular example let us take $q^a(\vec{r})\sim\delta^{a3}$

but let us alternate the sign of $q^a(\vec{r})$ within the source distribution (by a gauge transformation), so that

$$q^a(\vec{r}) = \delta^{a3} q^3(\vec{r}) , \tag{5.12}$$

with $q^3(\vec{r}) = +q(\vec{r})$ in some regions (R_+) and $q^3(\vec{r}) = -q(\vec{r})$ in others (R_-). It is clearly possible to choose these regions such that the first n multipole moments of $q^3(\vec{r})$ vanish. In other words for each $n > 0$ there exist regions R_+ and R_- such that

$$\int d^3x q^3(\vec{x}) x_{i_1} \cdots x_{i_k} = 0 , \tag{5.13}$$

for $i_1, \ldots, i_k = 1, \ldots, 3$, $0 \leq k \leq n$.

The configuration (5.11) then yields a static solution to the Yang-Mills equations $[A_i^a = E_i^a (t - t_0)]$ with $E_i^a \sim \delta^{a3}\ F(\theta, \phi)/r^{n+1}$ as $\vec{r} \to \infty$. Thus for a general source distribution there exist static solutions to the Yang-Mills equations with arbitrarily low energy and with an arbitrary number of multipoles of the electric field vanishing.

VI. THE MAGNETIC DIPOLE SOLUTION

If small perturbations are applied to the Coulomb solution of an extended charge distribution, one might presume, although it has not been shown, that by emitting radiation the Yang-Mills fields decay into one of the "total screening" solutions exhibited in the preceding section. Moreover, Mandula[9] has discovered an instability mode of the Coulomb solution which is present specifically when gQ is large enough; more precisely, when $gQ > \frac{3}{2}$ in the case he studied of an external charge distributed over a thin spherical shell. In our preceding publication[11] we found that extended charge distributions can indeed admit a type of solution which has the long-range behavior of a magnetic dipole field, which has lower energy than the corresponding Coulomb solution once gQ is large enough and thus appears to correspond well to Mandula's instability mode. We shall now give a more detailed account of these "magnetic dipole" solutions.

Let us first everywhere line up the external source into the commuting directions of isospin space (Cartan subalgebra),

$$[q(x),\ q(y)] = 0 \quad \text{all } x \text{ and } y. \tag{6.1}$$

In this diagonal form, the source $q(x)$ is time independent (see Sec. II). Equation (6.1) partly fixes the gauge. The remaining gauge freedom consists of all local gauge transformations in the Cartan subgroup $U^{(1)}(1) \times U^{(2)}(1) \times \cdots \times U^{(r)}(1)$, where r is the rank of the group.

From Sec. II, we know that any solution A_μ to the field equations in the presence of a static source $q(x)$ necessarily satisfies

$$D_\mu j^\mu(x) = \partial_0 q(x) + g[A_0(x), q(x)] = 0 . \tag{6.2}$$

Therefore, we can conclude that in our chosen gauge, Eq. (6.1), in which $\partial_0 q(x) = 0$, a solution to the Yang-Mills field equations in the presence of an *extended* charge distribution has $A_0(x)$ also lined up in the commuting directions of isospin space.

If, in addition, we *assume*[16] that the Yang-Mills fields $A_\mu(x)$ are all time independent, the field equations get simplified as follows:

$$E_i = F_{0i} = -\partial_i A_0 + g[A_0, A_i], \tag{6.3}$$

$$D_i E_i = -\partial_i \partial_i A_0 + g[A_i, [A_0, A_i]] + g(2[\partial_i A_0, A_i] + [A_0, \partial_i A_i] = q(\vec{x}) , \tag{6.4}$$

$$D_0 E_j = g[A_0, -\partial_j A_0 + g[A_0, A_j]] = g^2[A_0, [A_0, A_j]] = D_i F_{ij} . \tag{6.5}$$

These equations get simplified considerably further by imposing cylindrical symmetry around an axis in *real* space, which we choose to be in the $\hat{3}$ direction. We make the ansatz

$$A_0 = \phi(\rho, x_3) \quad \text{and} \quad A_i = \epsilon_{i3j} \frac{x_j}{\rho} A(\rho, x_3) , \tag{6.6}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$. As a consequence

$$\begin{aligned} &B_{ij} \equiv F_{ij} = \partial_i A_j - \partial_j A_i, \\ &\partial_i A_i = 0 \quad \text{and} \quad A_i \partial_i A_0 = 0. \end{aligned} \tag{6.7}$$

Thus we obtain

$$-\partial_i \partial_i A_0 + g^2[A_i [A_0, A_i]] = q(\vec{x}) , \tag{6.8a}$$

$$\partial_i \partial_i A_j - g^2 [A_0, [A_0, A_j]] = 0 , \tag{6.8b}$$

or in terms of ϕ and A

$$-\nabla^2 \phi - g^2[A, [A, \phi]] = q(\vec{x}) , \tag{6.9a}$$

$$\nabla^2 A - \frac{1}{\rho^2} A - g^2[\phi, [\phi, A]] = 0 . \tag{6.9b}$$

We will first analyze these equations for the case of the gauge group SU(2) and discuss later how the results generalize to other Lie groups. For SU(2), Eqs. (6.9) become

$$-\nabla^2 \phi^3 + g^2 \phi^3 [(A^1)^2 + (A^2)^2] = q^3(\vec{x}) , \tag{6.10a}$$

$$\phi^3 A^3 A^1 = \phi^3 A^3 A^2 = 0 , \tag{6.10b}$$

$$+\nabla^2 A^a - \frac{1}{\rho^2} A^a + g^2 (\phi^3)^2 A^a = 0, \quad \text{for } a = 1, 2 \tag{6.10c}$$

$$\nabla^2 A^3 - \frac{1}{\rho^2} A^3 = 0 . \tag{6.10d}$$

The last equation plus the boundary condition that $A^3 \to 0$ at infinity and the requirement that A^3 be free of singularities implies that $A^3 = 0$. Moreover, since A^1 and A^2 play symmetrical roles in Eq. (6.10) we shall assume, for simplicity, that $A^2=0$. We then obtain the equations of our previous paper[11]; renaming $\phi^3=\phi$ and $A^1=A$,

$$-\nabla^2\phi+g^2A^2\phi=q^3(\vec{x}) \,, \tag{6.11a}$$

$$\nabla^2 A-\frac{1}{\rho^2}A+g^2\phi^2 A=0 \,. \tag{6.11b}$$

One obtains the Coulomb solution by setting $A=0$. If $A\neq 0$, the full nonlinearity of the equations comes into play and there are no analytical methods available. It is nevertheless possible to show that there exists a large class (a continuous infinity) of localized and integrable (i.e., $Q<\infty$) charge distributions which, besides the Coulomb potential, admit a new type of solution with $A\neq 0$ and $\phi\neq 0$ and finite total energy. To this end let us consider any field $A(\rho,x_3)$ which satisfies the following conditions:

(1) $A(\rho,x_3)$ goes to zero as $r\to 0$.

(2) Away from the origin, $A(\rho,x_3)$ approaches exponentially fast a solution $\mathfrak{A}$ of $\nabla^2\mathfrak{A}-(1/\rho^2)\mathfrak{A}=0$.

The idea is as follows. For that given $A(\rho,x_3)$, we try to successively solve Eq. (6.11b) for $\phi(\rho,x_3)$ and calculate $q(\rho,x_3)$ from ϕ, A, and Eq. (6.11a). For the charge distribution $q(\rho,x_3)$ thus found, $\phi(\rho,x_3)$ and $A(\rho,x_3)$ will be an exact solution of the field equations. The first condition on $A(\rho,x_3)$ has been imposed so that the energy density be integrable at the origin. The second condition on $A(\rho,x_3)$ has been imposed so that both ϕ and q would vanish exponentially fast away from the origin. Whether everything works out or not depends very much on which solution $\mathfrak{A}$ of $\nabla^2\mathfrak{A}-(1/\rho^2)\mathfrak{A}=0$ one uses. For a particular solution $\mathfrak{A}$ to be useful it should have at least the following properties. First, it should go to zero at infinity to assure finiteness of the energy there. Second, it should have no singularities anywhere other than at the origin to ensure that the whole charge distribution will be localized there. Third, it must be possible to find an A satisfying conditions (1) and (2) above such that $(1/A)[\nabla^2 A-(1/\rho^2)A]$ is negative. Otherwise, of course, there is no real solution ϕ to Eq. (6.11b). Of the various solutions $\mathfrak{A}$ which we know of to $\nabla^2\mathfrak{A}-(1/\rho^2)\mathfrak{A}=0$[$\mathfrak{A}=(1/\rho)$, $(x_3/r\rho)$, r/ρ, x_3/ρ, $(1/r^{n+1})P_n^1(\cos\theta)$ and $r^nP_n^1(\cos\theta)$ for $n\geq 1$, where the P_n^m are the Legendre polynomials], only the solutions $(1/r^{n+1})P_n^1(\cos\theta)$ have the required properties. Among these we will only use extensively the one for $n=1$: $\rho/r^3=(\sin\theta)/r^2$.

Let us then consider an A field of the following form:

$$A(\rho,x_3)=ca\frac{\rho}{r^3}f\left(\frac{r}{a},\theta\right) \,, \tag{6.12}$$

where $f((r/a),\theta)$ is an arbitrary function that goes to one exponentially fast as $r/a\to\infty$, and goes to zero as $r\to 0$, in a particular fashion to be determined later. a is a parameter with dimension of length which will of course be the spatial extension of the charge distribution q. c is the (dimensionless) norm of $A(\rho,x_3)$ which will be directly related to Q and the gauge coupling constant g. We will call $f((r/a),\theta)$ the shape function because it gives us all the information complementary to the norm (Q or c) and the spatial size (a). Solving Eq. (6.11b), we find

$$g\phi=\left[\frac{-1}{f}\left(\nabla^2 f-\frac{4}{r}\frac{\partial f}{\partial r}+\frac{2}{r^2\tan\theta}\frac{\partial f}{\partial\theta}\right)\right]^{1/2}$$
$$=\frac{1}{a}\mathfrak{F}(x,\theta) \text{ with } x=\frac{r}{a} \,, \tag{6.13}$$

where $\mathfrak{F}$ is again a dimensionless function which depends only on the "shape". $g\phi$ goes to zero exponentially as $r\to\infty$ because f goes to one exponentially in that limit. There certainly exists a very wide set of shape functions f such that $g\phi$ is real everywhere. The external charge distribution is now given by Eqs. (6.11b), (6.12), and (6.13). The total charge is

$$Q=\int d^3x(-\nabla^2\phi+g^2A^2\phi)$$
$$=\int d^3x g^2A^2\phi=gc^2I_1 \,, \tag{6.14}$$

where

$$I_1=2\pi\int_0^\pi \sin^3\theta\, d\theta\int_0^\infty \frac{dx}{x^2}f^2(x,\theta)\mathfrak{F}(x,\theta) \tag{6.15}$$

is a number which depends only on the "shape".

The field strengths are the following:

$$\vec{B}^2=\vec{B}^3=\vec{E}^1=0 \,,$$
$$\vec{E}^3=-\vec{\nabla}\phi=-\frac{1}{ag}\vec{\nabla}\mathfrak{F}\left(\frac{r}{a},\theta\right) \,,$$
$$\vec{E}^2=g\phi\vec{A}=c\mathfrak{F}\left(\frac{r}{a},\theta\right)f\left(\frac{r}{a},\theta\right)\frac{\hat{3}\times\vec{x}}{r^3} \,, \tag{6.16}$$
$$\vec{B}^1=\vec{\nabla}\times\vec{A}^1$$
$$=\frac{3(\vec{m}\cdot\vec{x})\vec{x}-\vec{m}r^3}{r^5}f\left(\frac{r}{a},\theta\right)+\frac{\vec{m}(\vec{x}\cdot\vec{\nabla}f)-\vec{x}(\vec{m}\cdot\vec{\nabla}f)}{r^3}$$

with $\vec{m}=ca\hat{3}$. We thus find that the long-range behavior of this new type of solution is that of a magnetic dipole field. All the field strengths but $\vec{B}^1$ are either zero or short range. The physical situation is as follows. The Yang-Mills field $\vec{A}_1$ and ϕ create a charge distribution $-g^2(\vec{A}^1)^2\phi$ whose total charge exactly cancels Q. The electric fields thus become short range. On the other hand, those Yang-Mills fields create a current loop distribution,

$$\vec{j}^1 = g^2\phi^2\vec{A}^1 = g^2\phi^2 A\,\frac{1}{\rho}\,(\hat{3}\times\vec{x})\,, \tag{6.17}$$

whose total magnetic moment is precisely $\vec{m} = ca\vec{3}$.

Let us now consider the energy of the magnetic dipole solution,

$$\begin{aligned}H^{\text{m.d.}} &= \int d^3x\,\tfrac{1}{2}[(\vec{E}^3)^2 + (\vec{E}^2)^2 + (\vec{B}^1)^2] \\ &= \int d^3x\,\tfrac{1}{2}[(\vec{\nabla}\phi)^2 + g^2\phi^2(\vec{A}^1)^2 + (\vec{\nabla}\times\vec{A}^1)^2] \\ &= \int d^3x\,[\tfrac{1}{2}(\vec{\nabla}\phi)^2 + g^2\phi^2 A^2]\,,\end{aligned} \tag{6.18}$$

where we have integrated the last term by parts and made use of Eq. (6.11b). Let us separate the dependence of $H^{\text{m.d.}}$ on the gauge coupling constant g, the total external charge Q, the size a of the source, and the "shape":

$$\begin{aligned}H^{\text{m.d.}} &= \int d^3x\,\left\{\tfrac{1}{2}\left[\frac{1}{ga}\vec{\nabla}\mathfrak{F}\left(\frac{r}{a},\theta\right)\right]^2 + \left[\frac{1}{a}\mathfrak{F}\left(\frac{r}{a},\theta\right)\right]^2 c^2a^2\,\frac{\sin^2\theta}{r^4}f^2\left(\frac{r}{a},\theta\right)\right\} \\ &= \frac{1}{a}\left(\frac{1}{g^2}I_2 + \frac{Q}{g}\,\frac{I_3}{I_1}\right)\,,\end{aligned} \tag{6.19}$$

where we have used Eq. (6.14) and where

$$I_2 = \pi\int_0^\pi \sin\theta d\theta \int_0^\infty x^2dx\left[\left(\frac{\partial\mathfrak{F}}{\partial x}(x,\theta)\right)^2 + \frac{1}{x^2}\left(\frac{\partial\mathfrak{F}}{\partial\theta}(x,\theta)\right)^2\right], \tag{6.20}$$

$$I_3 = 2\pi\int_0^\pi \sin^3\theta d\theta\int_0^\infty \frac{dx}{x^2}\,\mathfrak{F}^2(x,\theta)f^2(x,\theta)\,,$$

I_2 and I_3, as I_1 depend only on the "shape".

The requirement of convergence for I_2 puts a restriction on the way $f((r/a),\theta)$ approaches zero as $r\to 0$, since otherwise $\mathfrak{F}((r/a),\theta)\sim 1/r$ as $r\to 0$ [see Eq. (6.13)]. Let

$$f\left(\frac{r}{a},\theta\right) = r^n H_1(\theta) + r^{n+1}H_2(\theta) + O(r^{n+2}) \text{ as } r\to 0. \tag{6.21}$$

Then, from Eq. (5.13), one can verify that $\mathfrak{F}((r/a),\theta)\sim$constant as $r/a\to 0$ and therefore that the integral I_2 is convergent, provided

$$\frac{d^2H_1}{d\theta^2} + \frac{3}{\tan\theta}\,\frac{dH_1}{d\theta} + n(n-3)H_1 = 0, \tag{6.22a}$$

$$\frac{d^2H_2}{d\theta^2} + \frac{3}{\tan\theta}\,\frac{dH_2}{d\theta} + (n+1)(n-2)H_2 = 0\,, \tag{6.22b}$$

and the θ integral is itself convergent. For example if f is θ independent it is necessary that

$$f(r)\sim r^3 + O(r^5)\,. \tag{6.22c}$$

The convergence of the I_1 and I_3 integrals requires that $n > \frac{1}{2}$.

The energy of the Coulomb solution corresponding to the external charge distribution $q(\vec{x})$ has the general form

$$H^{\text{Coulomb}} = \frac{Q^2}{a}I_4, \tag{6.23}$$

where I_4 depends only on the "shape". For a given "shape", size, and gauge coupling constant, the energy of the magnetic dipole solution rises linearly with the total external charge Q whereas the energy of the Coulomb solution rises quadratically in Q. Thus there is a critical value of Qg,

$$(Qg)_{\text{crit}} = \frac{1}{2I_4}\left\{\frac{I_3}{I_1} + \left[\left(\frac{I_3}{I_1}\right)^2 + 4I_2I_4\right]^{1/2}\right\}, \tag{6.24}$$

above which the magnetic dipole solution has lower energy than the Coulomb solution.

In summary, to *every* function $f((r/a),\theta)$ which goes to one as $r\to\infty$, which goes to zero as $r\to 0$ in the manner specified by Eqs. (6.21) and (6.22) with $n > \frac{1}{2}$, and which is such that the $\mathfrak{F}(x,\theta)$ in Eq. (6.13) is everywhere real, there corresponds an external charge distribution which admits a static solution of the magnetic dipole type given by Eqs. (6.6), (6.12), and (6.13). The energy of these magnetic dipole type solutions become lower than that of the corresponding Coulomb solutions if the external source strength is sufficiently large.

We shall now explore the generalization of our results to any gauge group. To this end the Cartan-Weyl representation of Lie algebras is useful. This representation generalizes the use of raising and lowering operators SU(2) to other groups. According to the Cartan-Weyl representation, a basis can be chosen for any Lie algebra where every generator is either inside the Cartan subalgebra $\{T_i, i=1,\ldots,r,\ r=\text{rank}\}$ which is the maximal set of commuting generators or is associated with one of the $(n-r)$, r-component "root vectors" of the Lie algebra (n = order). Let us then label the generators which are outside the Cartan subalgebra T_a where $a=(a_1,\ldots,a_r)$ is the associated root vector. Let us also recall that

(1) if a is a root vector, so is $-a=(-a_1,\ldots,-a_r)$;

(2) all root vectors are sums of a subset among them called the simple root vectors.

This having been said, we can write down the commutation relations in the Cartan-Weyl representation,

$$[T_i, T_j]=0 \quad \text{for all } i,\ j=1,\ldots,r,$$
$$[T_i, T_a]=a_i T_a,$$
$$[T_a, T_{-a}]=\sum_{i=1}^{r} a_i T_i \ . \tag{6.25}$$

For $a+b\neq 0$

$$[T_a, T_b]=\begin{cases} 0 \text{ if } a+b \text{ is not a root vector} \\ N_{ab}T_{a+b} \text{ if } a+b \text{ is a root vector.}\end{cases}$$

The root vectors and the N_{ab} thus play the role of structure constants in this representation. Now, we know that ϕ must lie completely within the Cartan subalgebra [see Eq. (6.2)] while there is no *a priori* restriction on where A lies. Thus

$$\phi=(-i)\sum_{k=1}^{r} \phi_k T_k , \tag{6.26a}$$

$$A=(-i)\left(\sum_{k=1}^{r} A_k T_k + \sum_a A_{-a} T_a\right). \tag{6.26b}$$

However, projecting Eq. (6.9b) onto the Cartan subalgebra implies

$$\nabla^2 A_k - \frac{1}{\rho^2}A_k=0, \quad k=1,\ldots,r\ . \tag{6.27}$$

With the boundary condition that $A\to 0$ at infinity and the requirement that it has no singularities, we obtain that $A_k=0$ for $k=1\ldots r$, i.e., A must lie completely outside the Cartan subalgebra. To satisfy Eq. (6.9a) one must then require

$$\phi_i \sum_{\substack{a,b \\ a+b\neq 0}} A_{-a}A_{-b}(a_i-b_i)N_{ab}T_{a+b}=0, \quad i=1,\ldots,r \tag{6.28}$$

and

$$-\nabla^2\phi_k + g^2\sum_a A_a A_{-a} a_k a^i \phi_i = q_k, \quad k=1,\ldots,r, \tag{6.29a}$$

while Eq. (5.9b) becomes

$$\nabla^2 A_a - \frac{1}{\rho^2}A_a + g^2\left(\sum_{k=1}^{r} a_k\phi_k\right)^2 A_a=0. \tag{6.29b}$$

Let us now specialize to the case of the gauge group SU(3). After adding some tedious analysis, we find that within our original ansatz Eq. (6.6) there are three types of solutions to the field equations [SU(3) notation, $\phi=-\frac{1}{2}i\,(\phi_3\lambda_3+\phi_8\lambda_8)$, $A=-\frac{1}{2}i\sum_{k=1,2,4,5,6,7}A_k\lambda_k$]:

(i) All A_k vanish except A_1 and A_2, and

$$-\nabla^2\phi_3 + g^2[(A_1)^2+(A_2)^2]\,\phi_3=q_3,$$
$$-\nabla^2\phi_8=q_8\ , \tag{6.30a}$$
$$\left[\nabla^2-\frac{1}{\rho^2}+g^2(\phi_3)^2\right]A_{1,2}=0\ .$$

(ii) All A_k vanish except A_4 and A_5, and

$$-\nabla^2\left(\frac{\phi_3+\sqrt{3}\,\phi_8}{2}\right)+g^2[(A_4)^2+(A_5)^2]\left(\frac{\phi_3+\sqrt{3}\,\phi_8}{2}\right)=\frac{q_3+\sqrt{3}\,q_8}{2},$$
$$-\nabla^2\left(\frac{-\sqrt{3}\,\phi_3+\phi_8}{2}\right)=\frac{-\sqrt{3}\,q_3+q_8}{2}, \tag{6.30b}$$
$$\left[\nabla^2-\frac{1}{\rho^2}+g^2\left(\frac{\phi_3+\sqrt{3}\,\phi_8}{2}\right)^2\right]A_{4,5}=0\ .$$

(iii) All A_k vanish except A_6 and A_7, and

$$-\nabla^2\left(\frac{\phi_3-\sqrt{3}\,\phi_8}{2}\right)+g^2[(A_6)^2+(A_7)^2]\left(\frac{\phi_3-\sqrt{3}\,\phi_8}{2}\right)=\frac{q_3-\sqrt{3}\,q_8}{2},$$
$$-\nabla^2\left(\frac{\sqrt{3}\,\phi_3+\phi_8}{2}\right)=\frac{\sqrt{3}\,q_3+q_8}{2}, \tag{6.30c}$$
$$\left[\nabla^2-\frac{1}{\rho^2}+g^2\left(\frac{\phi_3-\sqrt{3}\,\phi_8}{2}\right)^2\right]A_{6,7}=0\ .$$

It is clear that these three solutions correspond to the three SU(2) subgroups of SU(3), I, V, and U, respectively. For a given external charge distribution, there are in general four different solutions up to global gauge transformations. The first one is the Coulomb solution obtained by setting, $A=0$, in which case (i), (ii), and (iii) are all equivalent. The other three solutions have the long-range behavior of a Coulomb field in a direction of isospin space which is in the orbit of λ_8, plus a magnetic dipole field within the SU(2) that commutes with the direction in isospin space of the Coulomb field.

In case (i), q_8 produces a Coulomb field while q_3 gets screened and produces a magnetic dipole field associated with $\mathrm{SU}(2)_I$.

In case (ii), $\frac{1}{2}(-\sqrt{3}\,q_3+q_8)$ produces a Coulomb field while $\frac{1}{2}(q_3+\sqrt{3}\,q_8)$ gets screened and produces a magnetic dipole field associated with $\mathrm{SU}(2)_V$.

In case (iii), $\frac{1}{2}(\sqrt{3}\,q_3+q_8)$ produces a Coulomb field while $\frac{1}{2}(q_3-\sqrt{3}\,q_8)$ gets screened and produces a magnetic dipole field associated with $\mathrm{SU}(2)_U$. Of course if q is completely within an SU(2) subgroup, the charge can be totally screened.

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