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Representations and classifications of compact gauge groups for unified weak and electromagnetic interactions

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Assuming that all quarks have only electric charges 2/3 and $-1/3$, we classify all possible representations and all admissible gauge groups G underlying the unified weak and electromagnetic interactions. In particular, the group G cannot contain any exceptional Lie groups G_2, F_4, E_6, E_7, and E_8 as its factor. Moreover, the underlying irreducible representation for quark multiplets to be used must be one of fundamental representations for each component of simple Lie groups contained as a factor of G. For example, only the spinor representation is allowed for the SO($2n+1$) group, while only the basic representation is admissible for the symplectic groups. If G is semisimple in addition, then G must be a product of SU($3l$) groups.

I. INTRODUCTION AND SUMMARY OF MAIN RESULTS

The unified electromagnetic and weak-interaction theory of Weinberg and Salam[1] appears now to be on solid ground. However, parity nonviolation[2] in atomic physics as well as elastic scattering[3] of neutrinos on electrons may require enlargement of the group structure beyond the original SU(2) ⊗ U(1) group. Many larger groups have been indeed proposed by various authors. Among nonsemisimple groups we count the groups SU(2)⊗ U(1) ⊗ U(1),[4] SU(2)⊗ SU(2)⊗ U(1),[5] SU(3)⊗ U(1),[6] SU(4) ⊗ U(1),[7] and Sp(4)⊗ U(1),[8] while semisimple gauge groups so far proposed are limited only to two cases of SU(3) (Ref. 9) and SU(3)⊗ SU(3).[10] These attempts are, however, not systematic. In a previous paper[11] [hereafter referred to as (I)], I made a systematic classification of all possible semisimple groups for the unified gauge theories under some very general assumptions and showed that only products of SU($3l$) (l being integers) groups are admissible as candidates for semisimple gauge groups. In the present paper, we generalize our consideration to any compact Lie group which may not necessarily be semisimple.

Let G be any Lie group underlying the unified gauge theory of weak and electromagnetic interactions. Note that we are not including the strong interaction so that we will not discuss the color degree of freedom in this paper. Let $\underline{g}$ be the Lie algebra of G. Then our assumptions are as follows:

(1) G is a compact Lie group.

(2) Quark multiplets and lepton multiplets form separately representation spaces of the group G.

(3) The electric charge operator Q is a member of a Cartan subalgebra of $\underline{g}$.

(4) The electric charge operator Q can assume only two distinct eigenvalues x and y in any nontrivial representation of any quark or lepton multiplets.

In the last *Ansatz* (4), we particularly have in mind the fact that quarks can have only two fractional charges $\frac{2}{3}$ and $-\frac{1}{3}$, while leptons can have only two integral charges 0 and -1. Hence, we may normalize two eigenvalues x and y so that they satisfy

$$x-y=1\,. \tag{1.1}$$

We shall call the *Ansatz* (4) the two-charge condition hereafter. If leptons are assumed[12] to have three charges 1, 0, and -1, then our analysis applies only to quark multiplets as in (I).

As far as we can see, our assumptions are very general and reasonable. First of all, suppose that G is not compact. Then its nontrivial unitary representations are automatically of infinite dimension[13] so as to necessitate introduction of infinite numbers of quarks and leptons. Such a prospect would be rather unpleasant at least for the present. This justifies our *Ansatz* (1). The second condition is essentially a statement that we are considering only a unified theory of weak and electromagnetic interactions but not of all weak, electromagnetic, and strong interactions. The third condition is equivalent to the fact that any particle in a given representation of G can be classified according to eigenvalues of the electric charge operator Q together with other mutually commuting members of $\underline{g}$. Therefore, this is automatically satisfied in any model fo conventional gauge theories.[14] In comparison to the first three *Ansätze*, the fourth one (i.e., two-charge condition) is perhaps less general, since it assumes the absence of quarks with charges $-\frac{4}{3}$ or $+\frac{5}{3}$, etc. However, this condition is usually assumed and appears to be consistent with all present experimental data. We may also note that the two-charge condition is related to our desire that the intermediate gauge bosons should have electric charges of only 1, 0, and −1.

Since G is assumed to be a compact Lie group, it is essentially equivalent[15] to a product of a semisimple Lie group with U(1) groups, as far as its Lie algebra is concerned. Since we are mostly interested in the structure of the Lie algebra $\underline{g}$ rather than the Lie group G itself, we may set

$$G = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes \mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \cdots \otimes \mathrm{U}(1)\,, \quad (1.2)$$

where G_j ($j=1,2,\ldots,N$) are simple compact Lie groups and the U(1) group appears M times. If G is semisimple from the beginning, then the U(1) groups should be absent in Eq. (1.2). Corresponding to Eq. (1.2), the charge operator Q can be expressed as a direct sum

$$Q = Q_1 + Q_2 + \cdots + Q_N + R\,, \quad (1.3)$$

where Q_j ($j=1,2,\ldots,N$) are components of Q in the Lie algebra $\underline{g}_j$ of the simple group G_j and where R is a direct sum of all infinitesimal generators Z_j ($j=1,2,\ldots,M$) of U(1) groups present in Eq. (1.2) as

$$R = \sum_{j=1}^{M} C_j Z_j \quad (1.4)$$

for some constants C_j ($j=1,2,\ldots,M$). Hereafter, we consider only irreducible representations of all simple groups G_j ($j=1,2,\ldots,N$) and U(1) groups, so that R will assume only a single common eigenvalue Z in any given quark or lepton multiplet. Now our problem is reduced to the following group-theoretical question of finding all possible compact groups G and all admissible representations of G such that there exists a member Q of a Cartan subalgebra which can assume only two distinct eigenvalues x and y in the representation. This imposes a very severe constraint and we find the following results[16]:

(a) *None* of the simple groups G_j ($j=1,2,\ldots,N$) contained in G can be any of the exceptional Lie groups G_2, F_4, E_6, E_7, and E_8.

(b) If G is semisimple[12], then G must be a product[17] of SU($3l$) groups (l being integers) for the quark multiplet.

(c) Any irreducible representation $\{\lambda\}$ to be used for quark or lepton multiplets is trivial for all simple groups G_j ($j=1,2,\ldots,N$) except for one of them, say G_1, to be definite. Then the only irreducible representation to be used for G_1 must be one of the fundamental ones. More precisely, we adopt a lexiconal ordering of simple roots as in Fig. 1, following Patera and Sankoff.[18] Then the highest weight Λ of the irreducible representation $\{\lambda\}$ must be the following:

(i) A_n ($n \geq 1$), $\Lambda = \Lambda_j$ ($1 \leq j \leq n$), i.e., the SU($n+1$) group admits only completely antisymmetric tensor representations with dimensions $d(\Lambda_j) = (n+1)!/j!(n+1-j)!$ ($1 \leq j \leq n$).

(ii) B_n ($n \geq 2$), $\Lambda = \Lambda_n$, i.e., the SO($2n+1$) ($n \geq 2$) group admits only the fundamental spinor representation with dimension $d = 2^n$.

(iii) C_n ($n \geq 2$), $\Lambda = \Lambda_1$, i.e., the Sp($2n$) ($n \geq 2$) group is possible only for the basic representation with dimension $d = 2n$.

(iv) D_n ($n \geq 3$), $\Lambda = \Lambda_1$ or $\Lambda = \Lambda_{n-1}$ or Λ_n. In other words, only either the basic representation with dimension $d = 2n$ or two spinor representations with dimension $d = 2^{n-1}$ is possible for the group SO($2n$) ($n \geq 3$).

(d) The form of the electric charge operator Q is more or less uniquely determined. Its explicit form will be given in Sec. III. If none of G_j ($j=1,2,\ldots,N$) is of type A_n ($n \geq 2$) and if the multiplet belongs to a nontrivial representation of at least one simple group G_j, then R must have eigenvalues

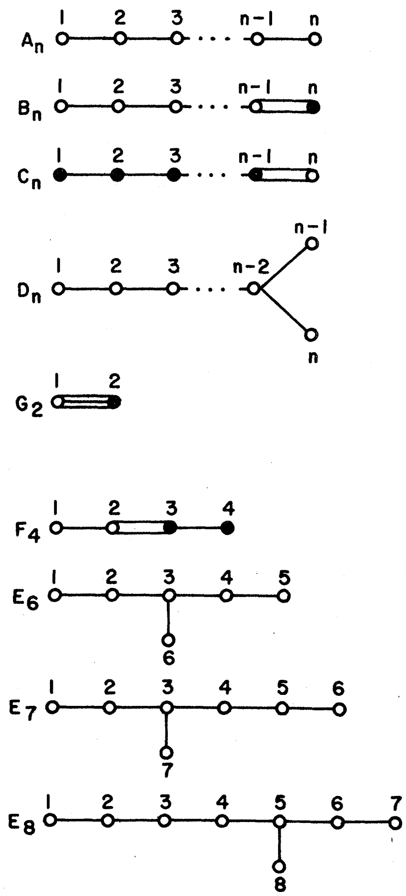

FIG. 1. Numbering of simple roots of simple Lie algebras. Black dots represent shorter roots, so that (○,○)/(●,●) = 2 for B_n, C_n, and F_4, while the ratio is 3 for G_2.

$R=\frac{1}{6}$ for the quark multiplet,

$R=-\frac{1}{2}$ for the lepton multiplet.

The case involving the algebra A_n ($n \geq 2$) is more complicated and will be discussed in Sec. III. If the multiplet belongs to the trivial representation of all simple groups G_j, then R is equivalent to Q, i.e.,

$R=\frac{2}{3}$ or $-\frac{1}{3}$ for quarks,

$R=0$ or -1 for leptons

for such cases.

Finally, a number of positively charged intermediate gauge vector bosons as well as a number of weak isotopic-spin doublets contained in the representation $\{\lambda\}$ when we reduce the group into the SU(2) subgroup are computed in Sec. IV.

II. DETAIL OF PROOF

We first note the following consequence of our two-charge *Ansatz* (4) of the Introduction. Only one of Q_j ($j=1,2,\ldots,N$), say Q_1 to be definite, can assume two distinct eigenvalues, while the rest of Q_j ($j\neq 1$) must assume only zero eigenvalues in any given quark (or lepton) multiplet. To prove it, we rewrite (1.3) as a direct sum

$$Q=Q_1+\tilde{Q}, \tag{2.1}$$

$$\tilde{Q}=Q_2+Q_3+\cdots+Q_N+R. \tag{2.2}$$

Suppose now that each of Q_1 and $\tilde{Q}$ can assume at least two distinct eigenvalues, say u and v for Q_1 and a and b for $\tilde{Q}$. Then the direct sum Q can assume at least four eigenvalues $u+a$, $u+b$, $v+a$, and $v+b$. However, since we have $u\neq v$ and $a\neq b$ by assumption, we can easily see that at least three out of these four eigenvalues must be mutually distinct. But this contradicts our two-charge *Ansatz*. If Q_1 can take only one common eigenvalue q_1 in the irreducible representation under consideration, then q_1 must be identically zero since the simplicity of G_j demands[19]

$$\mathrm{Tr}Q_j=0, \quad j=1,2,\ldots,N \tag{2.3}$$

in the representation. Moreover, if Q_1 assumes only a zero eigenvalue, then the representation must be a trivial one because of the simplicity of G_1, as we shall prove shortly. Now in the case that Q_1 is zero identically, we consider $\tilde{Q}$ instead of Q and rewrite it as

$$\tilde{Q}=Q_2+\tilde{Q}'$$

and repeat the same argument for Q_2 and $\tilde{Q}'$. In this way, we can prove that only one of Q_j, say Q_1 to be definite, can assume two distinct eigenvalues u and v, while all the rest, Q_j ($j=1,2,\ldots,N$), must be identically zero in the multiplet under consideration. Of course, if we consider another multiplet, then it will be Q_2 instead of Q_1 which will be nontrivial and so on. At any rate, we can rewrite (1.2) as

$$Q=Q_1+R \tag{2.4}$$

in the representation under consideration, since all other Q_j ($j\neq 1$) must vanish. Now the Abelian generator R can assume only a single common eigenvalue Z in the irreducible multiplet. The eigenvalues x and y of Q must have the form

$$x=u+Z, \quad y=v+Z. \tag{2.5}$$

In view of normalization (1.1), this implies

$$u-v=1. \tag{2.6}$$

Let n_1 and n_2 be multiplicities of states with eigenvalues $Q_1=u$ and v, respectively. Then the dimension $d=d(\lambda)$ which is equal to the number of quarks or leptons in the multiplet is given by

$$d(\lambda)=n_1+n_2.$$

Moreover, the traceless condition (2.3) is rewritten as

$$n_1u+n_2v=0$$

so that together with (2.6) we find

$$\begin{aligned} n_1&=-vd(\lambda)=(1-u)d(\lambda), \\ n_2&=ud(\lambda). \end{aligned} \tag{2.7}$$

We note that this requires $1\geq u\geq 0$ and $0\geq v\geq -1$. In the case that the algebra g_1 is *not* of type A_n ($n\geq 2$), we can say more. In that case, it has been shown[20,11] that a trace identity

$$\mathrm{Tr}(X^3)=0 \tag{2.8}$$

must be valid for any element X of the Lie algebra g_1. In particular, for the choice $X=Q_1$, this leads to

$$n_1u^3+n_2v^3=0$$

which requires

$$\begin{aligned} u&=-v=\tfrac{1}{2}, \\ n_1&=n_2=\tfrac{1}{2}d(\lambda) \end{aligned} \tag{2.9}$$

in view of Eqs. (2.6) and (2.7). We should note that alternative solutions with $u=1$, $v=0$, $n_1=0$ or $u=0$, $v=-1$, $n_2=0$ are *not* acceptable, since then Q_1 in reality has only single eigenvalue zero instead of the assumed two distinct eigenvalues. Then the value of Z can be computed from Eq. (2.5) to be

$$Z=\tfrac{1}{2}(x+y-u-v)=\begin{cases}\frac{1}{6} \text{ for quarks},\\ -\frac{1}{2} \text{ for leptons}\end{cases} \tag{2.10a}$$

since we have $x=\frac{2}{3}$ and $y=-\frac{1}{3}$ for quarks and $x=0$, $y=-1$ for leptons. However, it could be that some quark multiplets such as purely right-handed quarks in the Weinberg-Salam theory may belong to trivial representations of all simple groups G_j, i.e., we may have $Q_j=0$ for all $j=1,2,\ldots,N$. For such cases, the value of R is equivalent to Q itself, i.e.,

$$Z=\tfrac{2}{3} \text{ or } -\tfrac{1}{3} \text{ for quarks}, \quad Z=0 \text{ or } -1 \text{ for leptons} \tag{2.10b}$$

whenever the multiplet belongs to trivial representations of *all* simple groups G_j. However, we will *not* consider this trivial case hereafter.

We notice the fact that $Z\neq 0$ implies the presence of at least one U(1) group in Eq. (1.2). In other words, the group G cannot be semisimple. Therefore, if G is semisimple from the beginning, then any of its factors G_j ($j=1,2,\ldots,N$) must be[17] of the form SU(m) ($m\geq 3$). As we shall prove in the next section, m must be, moreover, integer multiples of three.

Another consequence of Eq. (2.9) is that the dimension $d(\lambda)$ must be an even integer, if $\underline{g}_1$ is not of the type A_n ($n\geq 2$). This fact alone eliminates many irreducible representations such as the basic representation of the algebra B_n ($n\geq 2$) corresponding to the SO($2n+1$) group. To obtain further restrictions, we need to go deeper into our analysis. Hereafter we consider the multiplet in which the group G_1 alone is nontrivial. Let n be the rank of the group G_1, and let H_j ($j=1,2,\ldots,n$), E_α, and $E_{-\alpha}$ be the standard Cartan-Weyl basis[21] of the simple Lie algebra $\underline{g}_1$. By assumption then Q_1 must be expressed as a linear combination of H_j as

$$Q_1=\sum_{j=1}^{n}\xi^j H_j \equiv (\xi,H). \tag{2.11}$$

Here ξ^j ($j=1,2,\ldots,n$) are c numbers which are regarded as the jth coordinate of a contravariant vector ξ^j in the root space and we introduced the inner product (ξ,η) of two vectors in the root space by

$$(\xi,\eta)=\sum_{j,k=1}^{n} g^{jk}\xi_j\eta_k=\sum_{j=1}^{n}\xi^j\eta_j \tag{2.12}$$

in the usual notation.[21]

Let Λ be the highest weight of the irreducible representation $\{\lambda\}$ under consideration. If M is a generic weight belonging to the representation $\{\lambda\}$, then eigenvalues of Q_1 have the form

$$(\xi,M). \tag{2.13}$$

Since this must be equal to either u or v for all weights M, this imposes a severe restriction for the type of representations as well as a form of the vector ξ. Again, we shall refer to this condition as the two-charge condition. Let

$$\pi=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}, \tag{2.14}$$

$$\Omega=\{\Lambda_1,\Lambda_2,\ldots,\Lambda_n\} \tag{2.15}$$

be a simple root system and the corresponding fundamental weight system, respectively. Then we have[22,23]

$$2(\Lambda_j,\alpha_k)=\delta_{jk}(\alpha_j,\alpha_j) \tag{2.16}$$

for all $j,k=1,2,\ldots,n$. The highest weight Λ of the irreducible multiplet is written as

$$\Lambda=m_1\Lambda_1+m_2\Lambda_2+\cdots+m_n\Lambda_n, \tag{2.17}$$

where m_j ($j=1,2,\ldots,n$) are non-negative integers specifying the irreducible representation. We shall prove the following propositions under the two-charge condition:

Proposition (I). The highest weight Λ must coincide with one of the fundamental weights, i.e., we must have $\Lambda=\Lambda_l\in\Omega$. Moreover, for any positive root β, it is necessary that

$$\frac{2(\Lambda,\beta)}{(\beta,\beta)}$$

assumes only two values 0 or +1.

Proposition (II). For any nonzero root β, (ξ,β) can take only three possible values, 1 or 0 or −1. Moreover, if a positive root β satisfies a condition

$$\frac{2(\Lambda,\beta)}{(\beta,\beta)}=1,$$

then the value of (ξ,β) is restricted to

$$0 \text{ or } 1, \text{ if } (\xi,\Lambda)=u,$$

$$0 \text{ or } -1, \text{ if } (\xi,\Lambda)=v.$$

Before proving these propositions, we mention the fact that these are sufficient to derive results quoted in the Introduction, as we shall see in the next section. Moreover, proposition (II) has the following physical implication. We note that the intermediate gauge bosons must belong[14] to the adjoint representation $\{\lambda_0\}$ of the group G_1. Then the electric charge of the gauge boson W_α corresponding to a root α is precisely given by (ξ,α) since the eigenvalue of H in the adjoint representation is α itself, and Q_1 is now given by $(\xi, \text{ad } H)$. Therefore, proposition (II) is equivalent to the physical statement that all intermediate gauge bosons have charges 1 or 0 or −1. As we shall show in Sec. IV, the number N_+ of all positively charged vector bosons can be easily computed once the group and the representation space are specified.

The next task is to prove these propositions. To this end, we adopt a lexiconal ordering of simple roots as in Fig. 1, following Patera and Sankoff.[18] Let us define n special Weyl reflection operation S_j by

$$S_j \equiv S_{\alpha_j}, \quad j=1,2,\ldots,n \tag{2.18}$$

so that S_j is the reflection operation with respect to a plane perpendicular to the jth simple root vector α_j in the root space.

Now let α_l be a given simple root in the Dynkin diagram, and let α_{l+1} (and/or α_{l-1}) be the next simple root adjacent to the right (or left) side of α_l in the diagram, where $\alpha_{l\pm 1}$ may be joined to α_l by either a single, double, or triple Dynkin line. When we set

$$y_{j\pm 1} = -\frac{2(\alpha_j, \alpha_{j\pm 1})}{(\alpha_{j\pm 1}, \alpha_{j\pm 1})}, \tag{2.19}$$

$$y'_{j\pm 1} = \frac{(\alpha_{j\pm 1}, \alpha_{j\pm 1})}{(\alpha_j, \alpha_j)} y_{j\pm 1} = -\frac{2(\alpha_j, \alpha_{j\pm 1})}{(\alpha_j, \alpha_j)} \tag{2.20}$$

we know that both $y_{j\pm 1}$ and $y'_{j\pm 1}$ are positive integers,

$$y_{j\pm 1} \geq 1, \quad y'_{j\pm 1} \geq 1. \tag{2.21}$$

We may note that $y_{j\pm 1}$ and $y'_{j\pm 1}$ are equal to unity, if the two roots α_j and $\alpha_{j\pm 1}$ are connected by a single Dynkin line. Noting for example that

$$S_{l\pm 1}\alpha_l = \alpha_l + y_{l\pm 1}\alpha_{l\pm 1},$$

$$S_{l\pm 2}S_{l\pm 1}\alpha_l = \alpha_l + y_{l\pm 1}(\alpha_{l\pm 1} + y_{l\pm 2}\alpha_{l\pm 2}),$$

$$S_{l-1}S_{l+1}\alpha_l = y_{l-1}\alpha_{l-1} + \alpha_l + y_{l+1}\alpha_{l+1},$$

etc., we can generate other adjacent simple roots α_{l-1}, α_{l+1}, and α_{l+2} by operating special Weyl reflection operations S_j successively to the root α_l. We can generalize this fact as follows. For a given simple root α_l, let us consider a connected sub-Dynkin chain of simple roots, involving $q-p+1$ simple roots $\alpha_p, \ldots, \alpha_l, \ldots, \alpha_q$ for any integer pair (p,q) satisfying

$$1 \leq p \leq l \leq q \leq n. \tag{2.22}$$

An example for the case of the algebra A_n $(n \geq 2)$ is depicted in Fig. 2. Then we define a Weyl operation $S^{(l)}_{p,q}$ by

$$S^{(l)}_{p,q} = (S_p S_{p+1} \cdots S_{l-1})(S_q S_{q-1} \cdots S_{l+1}) \tag{2.23}$$

and find

$$S^{(l)}_{p,q}\alpha_l = \sum_{j=p}^{q} x_j \alpha_j, \tag{2.24}$$

where x_j are positive integers defined by

$$x_j = y_{l-1}y_{l-2}\cdots y_j, \quad l-1 \geq j$$

$$x_j = y_{l+1}y_{l+2}\cdots y_j, \quad l+1 \leq j$$

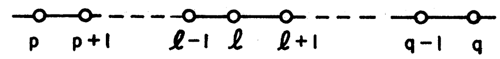

FIG. 2. A subchain of simple roots used to define the Weyl feflection operation $S^{(l)}_{p,q}$ of Eq. (2.23) for the case of the algebra A_n. We may define $S^{(l)}_{p,q}$ analogously for other cases.

$$x_l = 1, \quad l=j. \tag{2.25}$$

From Eq. (2.21), we may easily verify the fact that

$$x_j \geq 1, \tag{2.26}$$

$$\frac{(\alpha_j, \alpha_j)}{(\alpha_l, \alpha_l)} x_j \geq 1 \tag{2.27}$$

hold for all values of j, when we note that

$$\frac{(\alpha_j, \alpha_j)}{(\alpha_l, \alpha_l)} x_j = y'_{l\pm 1}y'_{l\pm 2}\cdots y'_j. \tag{2.28}$$

Although the explicit form (2.23) is, strictly speaking, applicable only to connected Dynkin subchains without any branch, the essentially same operation can be defined with suitable modifications to any connected diagram even with a branch. For example, we multiply, if necessary, the operation S_n by $S^{(l)}_{pq}$ defined by (2.23) whenever the branch chain for $l \neq n$ involves the root α_n. Also, for cases with $l=n$ where the root α_n is the end of a branch as in algebra D_n, E_6, E_7, and E_8, we may define $S^{(n)}_{p,q}$ analogously. However, for the sake of simplicity, we adhere to the same notation $S^{(l)}_{p,q}$ for all cases with an understanding of such necessary modifications whenever we are discussing a connected Dynkin subchain involving a branch, so that Eq. (2.24) with (2.26) still holds. After these preparations, we shall prove the following lemmas.

Lemma (I). For any given nonzero root β, we can find a weight M belonging to a given nontrivial representation $\{\lambda\}$ such that $(M,\beta) \neq 0$. Also, if the vector ξ satisfies $(\xi, M)=0$ for all weights M, then ξ is identically zero. In particular, for any nontrivial representation $\{\lambda\}$ and for any nonzero vector ξ, we can find a weight M such that $(\xi, M) \neq 0$.

Proof. First, let us suppose that for a given nonzero root β, we cannot find a weight M such that $(M,\beta) \neq 0$. Then we must have $(M,\beta)=0$ for all weights M belonging to the nontrivial representation $\{\lambda\}$. Now it is known[24] that any nonzero root β can be expressed in the form

$$\beta = T\alpha_l \tag{2.29}$$

for a Weyl operation T and for a simple root $\alpha_l \in \pi$. Replacing M by $TS^{-1}M$ for arbitrary Weyl reflection S, this leads to

$$(M, S\alpha_l) = 0$$

for arbitrary Weyl reflection S. When we choose

S to be $S_{p,q}^{(l)}$ defined by Eq. (2.23), this gives

$$\sum_{j=p}^{q} x_j(M, \alpha_j) = 0 .$$

We start with $p=q=l$ to find $(M, \alpha_l)=0$. Next we vary values of p and q successively and independently by lowering the value of p and increasing the value of q from the initial point $p=q=l$. Noting that x_j $(j=1,2,\ldots,n)$ are always positive, we find then

$$(M, \alpha_j)=0$$

for all $j=1,2,\ldots,n$. But since all simple roots span the whole root space, this implies $M=0$ identically for any weight M, i.e., the representation space $\{\lambda\}$ is trivial, contradicting our assumption. This proves the first part of the lemma. The second part can be proved similarly as follows. Suppose that we have no weight M such that $(\xi, M)\neq 0$. Then since both M and $S_\beta M$ for any nonzero root β are also weights, we must have

$$(\xi, M)=(\xi, S_\beta M)=0 .$$

However, noting

$$S_\beta M = M - \frac{2(M,\beta)}{(\beta,\beta)}\beta ,$$

this leads to

$$\frac{2(M,\beta)}{(\beta,\beta)}(\xi,\beta)=0 .$$

Now, for a given β, we can find a weight M such that $(M,\beta)\neq 0$ by the first part of the lemma. Therefore, this gives

$$(\xi,\beta)=0 .$$

Since the nonzero root β is arbitrary, this implies $\xi=0$ identically.

We may remark that in the proof of lemma (I), we did *not* assume the two-charge *Ansatz*. Also, if $Q_1=(\xi, H)$ assumes only zero eigenvalues and if $\xi\neq 0$, then the representation $\{\lambda\}$ must be trivial by our lemma. This fact has already been used in a previous discussion given at the beginning of this section.

Lemma (II). If for a nonzero root β, there exists a weight M satisfying

$$\left|\frac{2(M,\beta)}{(\beta,\beta)}\right| \geq 2 ,$$

then β must obey the condition

$$(\xi,\beta)=0$$

in order to satisfy the two-charge condition.

Proof. Let us consider a β series of M where M_l of the form

$$M_l = M + l\beta$$

are weights for all integer values of l in the interval $-j\leq l\leq k$ for some non-negative integers j and k. Moreover, we have

$$\frac{2(M,\beta)}{(\beta,\beta)} = j-k .$$

Therefore, if

$$\left|\frac{2(M,\beta)}{(\beta,\beta)}\right| \geq 2 ,$$

then the β series contains at least more than three distinct weights. Noting that

$$(\xi, M_l) = (\xi, M) + l(\xi,\beta) ,$$

we see that (ξ, M_l) will assume more than three distinct values unless $(\xi,\beta)=0$. By virtue of the two-charge condition, we must have $(\xi,\beta)=0$.

Lemma (III). For any weight M, and for any nonzero root β, the only possible value of

$$\frac{2(M,\beta)}{(\beta,\beta)}$$

is limited to either 1, 0, or -1.

Proof. Suppose that there exists a weight M_0 and nonzero root β_0, such that

$$\left|\frac{2(M_0,\beta_0)}{(\beta_0,\beta_0)}\right| \geq 2 .$$

Now for arbitrary Weyl operation S, we set

$$M = SM_0 ,$$

$$\beta = S\beta_0$$

and we compute

$$\left|\frac{2(M,\beta)}{(\beta,\beta)}\right| = \left|\frac{2(M_0,\beta_0)}{(\beta_0,\beta_0)}\right| \geq 2 .$$

Therefore by the lemma (II), we must have $(\xi,\beta)=0$, i.e., we have

$$(\xi, S\beta_0)=0$$

for all arbitrary Weyl operation S. Now write β_0 as

$$\beta_0 = T\alpha_l$$

as in Eq. (2.29) and replace S by $S_{p,q}^{(l)}T^{-1}$. This gives

$$(\xi, S_{p,q}^{(l)}\alpha_l) = \sum_{j=p}^{q} x_j(\xi, \alpha_j) = 0 .$$

Since p and q are arbitrary as long as $p\leq l\leq q$, we vary values of p and q successively and independently, starting from $p=q=l$. Then, as in the proof of lemma (I), this gives $(\xi,\alpha_j)=0$ for all $j=1,2,\ldots,n$ so that we have $\xi=0$ in contradiction with $Q_1\neq 0$.

Lemma (IV). The highest weight Λ must coincide with one of fundamental weight Λ_j, i.e., $\Lambda = \Lambda_l \in \Omega$.

Proof. For arbitrary integer l $(1 \leq l \leq n)$, let us set

$$M_0 = (S_{1,n}^{(l)})^{-1}\Lambda$$
$$= (S_{l+1}S_{l+2}\cdots S_n)(S_{l-1}S_{l-2}\cdots S_1)\Lambda\,.$$

Then we compute

$$\frac{2(M_0, \alpha_l)}{(\alpha_l, \alpha_l)} = \frac{2(\Lambda, S_{1,n}^{(l)}\alpha_l)}{(\alpha_l, \alpha_l)}$$
$$= \sum_{j=1}^{n} \frac{2(\Lambda, \alpha_j)}{(\alpha_l, \alpha_l)} x_j$$
$$= \sum_{j=1}^{n} \left[\frac{(\alpha_j, \alpha_j)}{(\alpha_l, \alpha_l)} x_j\right] \frac{2(\Lambda, \alpha_j)}{(\alpha_j, \alpha_j)}\,.$$

Noting now

$$\frac{2(\Lambda, \alpha_j)}{(\alpha_j, \alpha_j)} = m_j \geq 0\,,$$
$$\frac{(\alpha_j, \alpha_j)}{(\alpha_l, \alpha_l)} x_j \geq 1$$

from Eqs. (2.16), (2.17), and (2.27), this leads to

$$\frac{2(M_0, \alpha_l)}{(\alpha_l, \alpha_l)} \geq \sum_{j=1}^{n} m_j\,.$$

Hence, if we have $\sum_{j=1}^{n} m_j \geq 2$, this gives

$$\frac{2(M_0, \alpha_l)}{(\alpha_l, \alpha_l)} \geq 2$$

in contradiction to lemma (III). Therefore, we must have

$$\sum_{j=1}^{n} m_j = 0 \text{ or } 1\,,$$

which implies that Λ is either trivial or $\Lambda = \Lambda_l$ for some integer l $(1 \leq l \leq n)$.

Lemma (V). If M_0 is a dominant weight, and if α is a positive root, then we have

$$\frac{2(M_0, \alpha)}{(\alpha, \alpha)} \geq 0\,.$$

Moreover, if β_0 is a dominant root, then for any Weyl reflection S and for any weight M (which need *not* be dominant), we have

$$\frac{2(\Lambda, \beta_0)}{(\beta_0, \beta_0)} \geq \left|\frac{2(M, \beta)}{(\beta, \beta)}\right|\,,$$

where β is given by

$$\beta = S\beta_0\,.$$

Here we do *not* assume the two-charge *Ansatz* for the validity of this lemma.

Proof. The first half of this lemma is standard. Noting that

$$S_\alpha M_0 = M_0 - \frac{2(M_0, \alpha)}{(\alpha, \alpha)}\alpha\,,$$

we would have

$$S_\alpha M_0 > M_0$$

if it happens that

$$\frac{2(M_0, \alpha)}{(\alpha, \alpha)} < 0\,,$$

so that it contradicts the dominantness of M_0.

Next let M be any weight (which need *not* be dominant). It is well known that M can be written in the form

$$M = \Lambda - \sum_{j=1}^{n} p_j \alpha_j\,, \quad \alpha_j \in \pi$$

where p_j $(j = 1, 2, \ldots, n)$ are non-negative integers. If β_0 is a dominant root, then the first part of the lemma applied to the adjoint representation with $M_0 = \beta_0$ implies

$$\frac{2(\beta_0, \alpha_j)}{(\beta_0, \beta_0)} \geq 0\,, \quad j = 1, 2, \ldots, n$$

so that we find

$$\frac{2(\Lambda, \beta_0)}{(\beta_0, \beta_0)} \geq \frac{2(M, \beta_0)}{(\beta_0, \beta_0)}\,.$$

Since M is an arbitrary weight, we can replace it by $S^{-1}M$ for an arbitrary Weyl reflection S. Then the above inequality leads to

$$\frac{2(\Lambda, \beta_0)}{(\beta_0, \beta_0)} \geq \frac{2(M, \beta)}{(\beta, \beta)}\,, \quad \beta = S\beta_0\,.$$

Changing M by $S_\beta M$ and noting that $S_\beta \beta = -\beta$, this also gives

$$\frac{2(\Lambda, \beta_0)}{(\beta_0, \beta_0)} \geq -\frac{2(M, \beta)}{(\beta, \beta)}\,.$$

Combining both inequalities, we find the desired formula

$$\frac{2(\Lambda, \beta_0)}{(\beta_0, \beta_0)} \geq \left|\frac{2(M, \beta)}{(\beta, \beta)}\right|\,, \quad \beta = S\beta_0\,.$$

After these preparations, we are now in a position to prove propositions (I) and (II). First, proposition (I) is an immediate consequence of lemmas (III), (IV), and (V) with a choice of $M = \Lambda$ for lemmas (III) and (V). The proof of proposition (II) is as follows. By lemma (I), we can find a weight M such that

$$(M, \beta) \neq 0$$

for any nonzero root β. Then, by lemma (III),

$2(M,\beta)/(\beta,\beta)$ must be either +1 or −1. Now both (ξ,M) and $(\xi,S_\beta M)$ which are related by

$$(\xi,S_\beta M)=(\xi,M)-\frac{2(M,\beta)}{(\beta,\beta)}(\xi,\beta)$$

must assume either value u or v by our two-charge *Ansatz*. Noting that $u-v=1$ by Eq. (2.6), these are possible only if (ξ,β) can assume values of either 1, 0, or −1. The second half of proposition (II) can be similarly proved with choice $M=\Lambda$.

III. INDIVIDUAL ALGEBRAS

Because of proposition (I) and lemma (V) of the preceding section, we have only to consider cases where the highest weight Λ must coincide with one of the fundamental weights and that the value of

$$\frac{2(\Lambda,\beta_0)}{(\beta_0,\beta_0)}$$

for dominant roots β_0 is precisely equal to one.

First, we consider algebras other than the type A_n $(n\geq 2)$. Then we have noted that (ξ,M) must assume values only $+\frac{1}{2}$ or $-\frac{1}{2}$ for all weights M. In particular, (ξ,Λ) must also be either $+\frac{1}{2}$ or $-\frac{1}{2}$. Since these two values are interchangeable by $\xi\rightarrow-\xi$, we may normalize the sign of (ξ,Λ) to be

$$(\xi,\Lambda)=\tfrac{1}{2} \tag{3.1}$$

by changing the sign of ξ, if necessary.

Now we investigate each algebra separately.

(1) *The impossibility of exceptional Lie algebras.* By proposition (II) of the preceding section, (ξ,α_j) can assume only integer values 0 or ±1. Then, if we express the fundamental weights Λ_j in terms of simple roots α_j as in the Appendix, we see immediately without any explicit calculation that (ξ,Λ) can never assume the value $+\frac{1}{2}$ for all cases of algebras G_2, F_4, E_6, and E_8. For the algebra E_7, the same consideration precludes all cases except possibly for $\Lambda=\Lambda_4$ or Λ_6 or Λ_7. Therefore, we have only to prove the impossibility of these three cases of E_7. We first note that $\beta_0=\Lambda_1$ is the highest weight of the adjoint representation of E_7, and consequently is the dominant root of E_7. From Eq. (2.16) and from explicit forms of Λ_4 and Λ_7 as in the Appendix, we can easily compute

$$\frac{2(\Lambda_4,\beta_0)}{(\beta_0,\beta_0)}=3\,,\quad \frac{2(\Lambda_7,\beta_0)}{(\beta_0,\beta_0)}=2\,,$$

which are larger than 2. Hence, by our proposition (I), the cases $\Lambda=\Lambda_4$ and $\Lambda=\Lambda_7$ are excluded. However, for the remaining case $\Lambda=\Lambda_6$, we find

$$\frac{2(\Lambda_6,\beta_0)}{(\beta_0,\beta_0)}=1$$

and the same argument cannot be applicable. We may note that $\Lambda=\Lambda_6$ is a multiplicity-free representation with dimensionality 56. As a matter of fact, all 56 weights M satisfy by lemma (V) the condition of lemma (III), i.e.,

$$\frac{2(M,\beta)}{(\beta,\beta)}=1\,,\ 0,\ \text{or}\ -1$$

for all nonzero roots β. Therefore, we have to reinvestigate this case in detail. We express all 56 weights M as a sum of the simple root α_j. Then all 56 quantities (ξ,M) are expressed as known linear combinations of seven unknown (ξ,α_j) $(j=1,2,\ldots,7)$ which must assume only three possible values 1, 0, and −1. This must be consistent with the fact that (ξ,M) for all 56 weights M must have values $+\frac{1}{2}$ or $-\frac{1}{2}$. Actually, since all representations of E_7 are known[25] to be self-contragradient, both weights M and $-M$ belong to the same representation so that we have only to study 28 among these 56 quantities. The test for the consistency is very tedious, but nevertheless leads to a negative answer. Hence, the case $\Lambda=\Lambda_6$ is again not admissible. In conclusion, all exceptional Lie algebras are incompatible with the two-charge *Ansatz*. An alternative proof will be given in Sec. IV. A more systematic way to eliminate the exceptional Lie algebras is to utilize the following trace identity. Since algebras G_2, F_4, E_6, E_7, E_8 as well as A_1 and A_2 are known *not* to have any completely symmetric genuine fourth-order Casimir invariants,[26] we can prove[27] the trace identity

$$\mathrm{Tr}(X^4)=K(\lambda)[\mathrm{Tr}(X^2)]^2 \tag{3.2}$$

for any generic member X of these Lie algebras in any given irreducible representation $\{\lambda\}$. Here the constant $K(\lambda)$ is given by

$$K(\lambda)=\frac{1}{2[2+d(\lambda_0)]}\frac{d(\lambda_0)}{d(\lambda)}\times\left[6-\frac{I_2(\lambda_0)}{I_2(\lambda)}\right], \tag{3.3}$$

where $\{\lambda_0\}$ designates the adjoint representation and where $d(\lambda)$ and $I_2(\lambda)$ are the dimension and eigenvalue of the second-order Casimir invariant in the irreducible representation $\{\lambda\}$, respectively. Now we choose $X=Q_1$ in Eq. (3.2). Moreover, for algebras A_1, G_2, F_4, E_6, E_7, and E_8 (but *not* for A_2), we have [see Eq. (2.8)]

$$\mathrm{Tr}Q_1=\mathrm{Tr}(Q_1)^3=0\,,$$

which give for any positive integer p

$$\mathrm{Tr}(Q_1)^p=\tfrac{1}{2}d(\lambda)[(\tfrac{1}{2})^p+(-\tfrac{1}{2})^p]$$

because of $n_1=n_2=\frac{1}{2}d(\lambda)$ and $u=-v=\frac{1}{2}$ for these cases as we noted in Eq. (2.9). Therefore, the relation (3.2) reduces to

$$\frac{I_2(\lambda_0)}{I_2(\lambda)}=4\left[1-\frac{1}{d(\lambda_0)}\right]. \tag{3.4}$$

We can easily check[28] that this relation is not satisfied for any fundamental irreducible representation of any exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8, again proving the incompatibility of exceptional Lie algebras with our two-charge *Ansatz*. For the algebra A_1, we can simply find that the relation (3.4) is satisfied only for the two-dimensional representation of A_1. Therefore, we reproduced the well-known fact that only the two-dimensional representation is compatible with two-charge *Ansatz* for the SU(2) group.

(2) *Algebra* B_n ($n\geq 2$). Again we express all fundamental weights Λ_j in terms of simple roots as in the Appendix, and note the fact that (ξ,α_j) can assume only values 1, 0, and −1. Then we find that only $\Lambda=\Lambda_n$ can reproduce the value $+\frac{1}{2}$ or $-\frac{1}{2}$ for (ξ,Λ). In other words, only the spinor representation with dimension $d(\lambda)=2^n$ is permissible. Next we would like to find the explicit form of the vector ξ. Since the algebra B_2 is equivalent to C_2, we restrict ourselves to the study of B_n with $n\geq 3$ hereafter. Then the algebra B_n ($n\geq 3$) has two dominant roots Λ_1 and Λ_2. Note that $\Lambda=\Lambda_2$ corresponds to the adjoint representation for $n\geq 3$. As is expected, both dominant roots satisfy the condition $2(\Lambda_n,\beta_0)/(\beta_0,\beta_0)=1$.

Now let us consider n weights M_j ($j=1,2,\ldots,n$) given by

$$\begin{aligned}M_j&=S_jS_{j+1}\cdots S_n\Lambda_n\\&=\Lambda_n-(\alpha_j+\alpha_{j+1}+\cdots+\alpha_n)\end{aligned} \tag{3.5}$$

and set

$$\Sigma_j\equiv(\xi,\alpha_j)+(\xi,\alpha_{j+1})+\cdots+(\xi,\alpha_n). \tag{3.6}$$

Then, since (ξ,M_j) can assume only values $+\frac{1}{2}$ or $-\frac{1}{2}$, and since (ξ,Λ_n) is normalized to be $+\frac{1}{2}$ by (3.1), we see that Σ_j must be equal to either 0 or +1 for all $j=1,2,\ldots,n$. Next we express

$$\begin{aligned}(\xi,\Lambda_n)&=\tfrac{1}{2}\{(\xi,\alpha_1)+2(\xi,\alpha_2)+\cdots+n(\xi,\alpha_n)\}\\&=\tfrac{1}{2}\{\Sigma_1+\Sigma_2+\cdots+\Sigma_n\},\end{aligned}$$

which must be equal to $\frac{1}{2}$ by (3.1). But we noted that all Σ_j ($1\leq j\leq n$) are either 0 or +1. Therefore, only one of Σ_j, say Σ_l, can be equal to +1, while all other Σ_j with $j\neq l$ must be zero. Together with (3.6), this determines (ξ,α_j) to be

$$(\xi,\alpha_j)=\begin{cases}+1, & \text{if } j=l,\\ -1, & \text{if } j=l-1,\quad l\geq 2,\\ 0, & \text{if } j\neq l \text{ and } j\neq l-1.\end{cases}$$

In view of Eq. (2.16), the explicit form for ξ satisfying this condition is

$$\xi=\frac{2}{(\alpha_l,\alpha_l)}\Lambda_l-\frac{2}{(\alpha_{l-1},\alpha_{l-1})}\Lambda_{l-1}. \tag{3.7}$$

For the special case $l=1$, we omit the second term in Eq. (3.7).

So far, these conditions are necessary. But we can prove conversely that these are also sufficient as follows. To this end, it is more convenient to use a non-Cartan form for the algebra B_n corresponding to the SO($2n+1$) group [or more accurately spin($2n+1$) group]. Let A,B,C,D be indices running over $1,2,\ldots,2n+1$, and consider generators J_{AB} satisfying relations

$$J_{AB}=-J_{BA}, \tag{3.8}$$

$$[J_{AB},J_{CD}]=\delta_{BC}J_{AD}+\delta_{AD}J_{BC}-\delta_{AC}J_{BD}-\delta_{BD}J_{AC}. \tag{3.9}$$

The algebra satisfying (3.8) and (3.9) is equivalent[29] to B_n. Now the fundamental spinor representation $\Lambda=\Lambda_n$ can be constructed as follows. Let us consider $2n+1$ matrices Γ_A ($A=1,2,\ldots,2n+1$) of dimension $2^n\times 2^n$ which satisfy the anticommutation relation

$$\{\Gamma_A,\Gamma_B\}_+=2\delta_{AB}E, \tag{3.10}$$

where E is the $2^n\times 2^n$ unit matrix. Then the matrix representation of the generator J_{AB} in $\Lambda=\Lambda_n$ is given by

$$J_{AB}=\tfrac{1}{4}[\Gamma_A,\Gamma_B]. \tag{3.11}$$

For any pair (A,B), the identification

$$Q_1=iJ_{AB},\quad A\neq B \tag{3.12}$$

will give obviously

$$(Q_1)^2=\tfrac{1}{4}E \tag{3.13}$$

in view of (3.10) which guarantees that eigenvalues of Q_1 are only $+\frac{1}{2}$ and $-\frac{1}{2}$ in the spinor representation. The solution (3.12) is in fact equivalent to

$$Q_1=(\xi,H) \tag{3.14}$$

with Eq. (3.7) for ξ, if we adopt a suitable lexiconal ordering of simple roots in a suitably chosen Cartan subalgebra. Indeed, ordinarily we choose a Cartan subalgebra to consist of operators

$$H_\mu=iJ_{\mu,\mu+n}\quad(\mu=1,2,\ldots,n)$$

apart from some normalization constants which do not concern us here. Then it is not difficult to check the equivalence of (3.12) with (3.14) and (3.7), where $A=l$, and $B=l+n$, $1\leq l\leq n$.

(3) *Algebra* C_n ($n\geq 2$). We note that

$$\beta_0=\Lambda_2=\alpha_1+2(\alpha_2+\alpha_3+\cdots+\alpha_{n-1})+\alpha_n \tag{3.15}$$

is a dominant root other than the highest weight $\Lambda_0=2\Lambda_1$ for the adjoint representation. Moreover, if

$$\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + \cdots + m_n\Lambda_n ,$$

then we compute

$$\frac{2(\Lambda,\beta_0)}{(\beta_0,\beta_0)} = m_1 + 2(m_2 + m_3 + \cdots + m_n)$$

so that for $j \geq 2$ we find

$$\frac{2(\Lambda_j,\beta_0)}{(\beta_0,\beta_0)} = 2 .$$

Therefore, by our proposition (I), only the case $\Lambda = \Lambda_1$ is permissible. However, in this case we cannot determine the form of ξ uniquely. As we shall see shortly, this fact is related to many possible choices of lexiconal ordering of simple root systems.

To show that $\Lambda = \Lambda_1$ is indeed consistent with the two-charge *Ansatz*, it is convenient to write the algebra C_n in the following form:

$$[A^\mu_\nu, A^\alpha_\beta] = \delta^\mu_\beta A^\alpha_\nu - \delta^\alpha_\nu A^\mu_\beta , \tag{3.16a}$$

$$[A^\mu_\nu, R_{\alpha\beta}] = \delta^\mu_\alpha R_{\nu\beta} + \delta^\mu_\beta R_{\alpha\nu} , \tag{3.16b}$$

$$[A^\mu_\nu, R^{\alpha\beta}] = -\delta^\alpha_\nu R^{\mu\beta} - \delta^\beta_\nu R^{\alpha\mu} , \tag{3.16c}$$

$$[R_{\mu\nu}, R^{\alpha\beta}] = \delta^\alpha_\mu A^\beta_\nu + \delta^\alpha_\nu A^\beta_\mu + \delta^\beta_\mu A^\alpha_\nu + \delta^\beta_\nu A^\alpha_\mu , \tag{3.16d}$$

$$[R_{\mu\nu}, R_{\alpha\beta}] = [R^{\mu\nu}, R^{\alpha\beta}] = 0 , \tag{3.16e}$$

$$R_{\mu\nu} = R_{\nu\mu} , \tag{3.16f}$$

$$R^{\mu\nu} = R^{\nu\mu} , \tag{3.16g}$$

where indices μ, ν, α, β run from 1 to n. The member H_j of a Cartan subalgebra may be identified with A^j_j $(j = 1, 2, \ldots, n)$ apart from a constant normalization factor, while all A^j_k $(j \neq k)$, R^{jk}, and R_{jk} correspond to some E_α for nonzero root α.

The basic representation space for $\Lambda = \Lambda_1$ can be spanned by a $2n$ basis ϕ_λ and ϕ^λ $(\lambda = 1, 2, \ldots, n)$ on which A^μ_ν, $R^{\mu\nu}$, and $R_{\mu\nu}$ act as

$$A^\mu_\nu \varphi_\lambda = \delta^\mu_\lambda \varphi_\nu , \tag{3.17a}$$

$$A^\mu_\nu \varphi^\lambda = -\delta^\lambda_\nu \varphi^\mu , \tag{3.17b}$$

$$R_{\mu\nu}\varphi_\lambda = R^{\mu\nu}\varphi^\lambda = 0 , \tag{3.17c}$$

$$R_{\mu\nu}\varphi^\lambda = \delta^\lambda_\mu \varphi_\nu + \delta^\lambda_\nu \varphi_\mu , \tag{3.17d}$$

$$R^{\mu\nu}\varphi_\lambda = \delta^\mu_\lambda \varphi^\nu + \delta^\nu_\lambda \varphi^\mu . \tag{3.17e}$$

Then the desired charge operator Q_1 which is consistent with the two-charge *Ansatz* is found to be of the form

$$Q_1 = \tfrac{1}{2} \sum_{\mu=1}^{n} \epsilon_\mu A^\mu_\mu , \tag{3.18}$$

where ϵ_μ are either +1 or −1. We can check that

$$Q_1\varphi^\lambda = -\tfrac{1}{2}\epsilon_\lambda \varphi^\lambda = \mp\tfrac{1}{2}\varphi^\lambda ,$$
$$Q_1\varphi_\lambda = \tfrac{1}{2}\epsilon_\lambda \varphi_\lambda = \pm\tfrac{1}{2}\varphi_\lambda . \tag{3.19}$$

In other words, Q_1 can have only two eigenvalues $+\frac{1}{2}$ or $-\frac{1}{2}$ in the representation $\Lambda = \Lambda_1$. The arbitrariness for sign of ϵ_μ is related to nonuniqueness of solutions (ξ, α_j) in the corresponding Cartan notation. The arbitrariness can be largely eliminated if we relabel indices suitably so that Q_1 can be rewritten as

$$Q_1 = \tfrac{1}{2}\left(\sum_{\mu=1}^{l} A^\mu_\mu - \sum_{\mu=l+1}^{n} A^\mu_\mu \right) \tag{3.20}$$

for some integer l. This corresponds to a suitable choice of lexiconal ordering of simple roots. Then

$$\xi = \frac{2}{(\alpha_n, \alpha_n)} (2\Lambda_l - \Lambda_n) , \quad 1 \leq l \leq n . \tag{3.20'}$$

(4) *Algebra* D_n $(n \geq 4)$. Since the algebra D_3 is equivalent to A_3, we will consider the case D_n with $n \geq 4$ here. We note that

$$\beta_0 = \Lambda_2 = \alpha_1 + 2(\alpha_2 + \alpha_3 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \tag{3.21}$$

is the unique dominant root, since it is the highest weight of the adjoint representation of the D_n. Then we compute

$$\frac{2(\beta_0,\Lambda)}{(\beta_0,\beta_0)} = m_1 + 2(m_2 + m_3 + \cdots + m_{n-2}) + m_{n-1} + m_n$$

so that all cases $\Lambda = \Lambda_j$ with $2 \leq j \leq n-2$ lead to

$$\frac{2(\beta_0,\Lambda_j)}{(\beta_0,\beta_0)} = 2 .$$

Therefore, only cases $\Lambda = \Lambda_1$, $\Lambda = \Lambda_{n-1}$, or $\Lambda = \Lambda_n$ are permissible. To show that this condition is also sufficient, we proceed as follows. For two spinor representations $\Lambda = \Lambda_{n-1}$ or Λ_n, we use the non-Cartan form of generators J_{AB} $(A, B = 1, 2, \ldots, 2n)$ as in the case of B_n. However, the range of indices A and B in Eqs. (3.8) and (3.9) is now restricted to $A, B = 1, 2, \ldots, 2n$ but *not* $2n+1$. We still define $2^n \times 2^n$ matrices Γ_A for $A = 1, 2, \ldots, 2n + 1$ as in the case of B_n. However, Γ_{2n+1} commutes now with all J_{AB} for $A, B = 1, 2, \ldots, 2n$, and $(\Gamma_{2n+1})^2 = E$. The matrix Γ_{2n+1} is reducible when we restrict ourselves to the SO($2n$) [or spin($2n$)] subgroup of the SO($2n+1$) group [or spin($2n+1$)]. Then, the single-spinor representation Λ_n of the SO($2n+1$) group will split into two spinor representations Λ_{n-1} and Λ_n of the SO($2n$) subgroup, corresponding to eigenvalues $\Gamma_{2n+1} = +1$ or -1. Note that the dimension of the representations are

$$d(\Lambda_{n-1}) = d(\Lambda_n) = 2^{n-1} .$$

Moreover, Q_1 defined by reduced form of Eq. (3.12) still satisfies Eq. (3.13) in both $\Lambda = \Lambda_{n-1}$ and $\Lambda = \Lambda_n$.

Therefore, we conclude that our Q_1 satisfies the desired two-charge condition for these representations. Then the explicit form of ξ for both $\Lambda = \Lambda_{n-1}$ and Λ_n is found to be

$$\xi = \frac{2}{(\alpha_n, \alpha_n)} (\Lambda_l - \Lambda_{l-1}) \tag{3.22a}$$

as long as $1 \le l \le n-2$. However, for $l = n-1$ or n, we have to use

$$\xi = \frac{1}{(\alpha_n, \alpha_n)} (\alpha_{n-1} + \alpha_n) \tag{3.22b}$$

or

$$\xi = \frac{1}{(\alpha_n, \alpha_n)} (\alpha_{n-1} - \alpha_n) \tag{3.22c}$$

for the case $\Lambda = \Lambda_{n-1}$, while for $\Lambda = \Lambda_n$ we interchange the role of indices n and $n-1$, in Eqs. (3.22b) and (3.22c).

For the basic representation $\Lambda = \Lambda_1$, we rewrite the algebra D_n in a form analogous to Eqs. (3.16a)–(3.16g). The only difference now is that $R_{\mu\nu}$ and $R^{\mu\nu}$ are antisymmetric, i.e.,

$$R_{\mu\nu} = -R_{\nu\mu}, \quad R^{\mu\nu} = -R^{\nu\mu} \tag{3.23}$$

and Eq. (3.16d) must be replaced by

$$[R_{\mu\nu}, R^{\alpha\beta}] = \delta^\alpha_\nu A^\beta_\mu + \delta^\beta_\mu A^\alpha_\nu - \delta^\alpha_\mu A^\beta_\nu - \delta^\beta_\nu A^\alpha_\mu . \tag{3.24}$$

Similarly, the basic representation is spanned by $2n$ vectors φ_λ and φ^λ ($\lambda = 1, 2, \ldots, 2n$) as before. The difference is that Eqs. (3.17d) and (3.17e) must be replaced by

$$R_{\mu\nu}\varphi^\lambda = \delta^\lambda_\mu \varphi_\nu - \delta^\lambda_\nu \varphi_\mu , \tag{3.25a}$$

$$R^{\mu\nu}\varphi_\lambda = \delta^\mu_\lambda \varphi^\nu - \delta^\nu_\lambda \varphi^\mu . \tag{3.25b}$$

The form of the charge operator Q_1 is the same as (3.18). The remaining check for eigenvalues of Q_1 being always either $+\frac{1}{2}$ or $-\frac{1}{2}$ is again unchanged, so that $\Lambda = \Lambda_1$ is again compatible with the two-charge *Ansatz* also in this case. The explicit form of ξ corresponding to Eqs. (3.20) and (3.20′) is

$$\xi = \frac{2}{(\alpha_n, \alpha_n)} (\Lambda_l - \Lambda_n), \quad 1 \le l \le n-2 \tag{3.26a}$$

$$\xi = \frac{2}{(\alpha_n, \alpha_n)} \Lambda_l , \quad l = n-1 \text{ or } n . \tag{3.26b}$$

If we wish, we could replace Λ_n by Λ_{n-1} in Eq. (3.26a).

(5) *Algeba* A_n ($n \ge 1$). The only dominant root β_0 is the highest weight of the adjoint representation, i.e.,

$$\beta_0 = \Lambda_1 + \Lambda_n .$$

But we compute that for all $j = 1, 2, \ldots, n$

$$\frac{2(\Lambda_j, \beta_0)}{(\beta_0, \beta_0)} = 1 ,$$

so that all fundamental representations $\Lambda = \Lambda_j$ ($1 \le j \le n$) can be compatible with the two-charge condition. To check that they are all admissible, it is more convenient to work with the non-Cartan form of A_n:

$$\sum_{\mu=1}^{n+1} B^\mu_\mu = 0 , \tag{3.27a}$$

$$[B^\mu_\nu, B^\alpha_\beta] = \delta^\mu_\beta B^\alpha_\nu - \delta^\alpha_\nu B^\mu_\beta . \tag{3.27b}$$

The irreducible representation space of $\Lambda = \Lambda_j$ can be described by a completely antisymmetric tensor

$$\phi_{\mu_1 \mu_2 \cdots \mu_j} \tag{3.28}$$

of degree j, on which B^μ_ν acts as

$$B^\mu_\nu \phi_{\mu_1\mu_2\cdots\mu_j} = \sum_{k=1}^{j} \delta^\mu_{\mu_k} \phi_{\mu_1 \cdots \hat{\mu}_k \cdots \mu_j} - \frac{j}{n+1} \delta^\mu_\nu \phi_{\mu_1 \cdots \mu_j} . \tag{3.29}$$

In Eq. (3.29), the symbol $\hat{\mu}_k$ implies that we delete μ_k and replace it by ν. Then the most general charge operator Q_1 must have the form

$$Q_1 = \sum_{j=1}^{n} \xi_j B^j_j , \tag{3.30}$$

where ξ_j are some constants. Now we repeat essentially the same argument as in (I) to prove that Q_1 must be reduced to the form

$$Q_1 = \pm \sum_{j=1}^{p} B^j_j \tag{3.31}$$

for an integer p satisfying $1 \le p \le n$, when we relabel indices suitably. Also, again due to the two-charge *Ansatz*, we find finally

(i) $\Lambda = \Lambda_1$:

$$Q_1 = B_1{}^1 + B_2{}^2 + \cdots + B^p_p \quad (1 \le p \le n) ,$$

$$u = 1 - \frac{p}{n+1} , \quad v = -\frac{p}{n+1} , \tag{3.32a}$$

(ii) $\Lambda = \Lambda_j$ ($2 \le j \le n-1$):

$$Q_1 = B_1{}^1 , \quad u = 1 - \frac{j}{n+1} , \quad v = -\frac{j}{n+1} , \tag{3.32b}$$

(iii) $\Lambda = \Lambda_n$:

$$Q_1 = -(B_1{}^1 + B_2{}^2 + \cdots + B^p_p) \quad (1 \le p \le n) ,$$

$$u = 1 - \frac{p}{n+1} , \quad v = -\frac{p}{n+1} , \tag{3.32c}$$

where the integer p in Eqs. (3.32a) and (3.32c) is arbitrary as long as $1 \leq p \leq n$. We can easily convince ourselves that the two-charge *Ansatz* is now satisfied for all $\Lambda = \Lambda_j$ $(1 \leq j \leq n)$.

Note that for all cases, the value of Z which is the eigenvalue of the Abelian generator R is given by

$$Z = \tfrac{1}{2}(x + y - u - v)$$
$$= \tfrac{1}{2}(x + y) - \frac{1}{2}\left(1 - \frac{2m}{n+1}\right), \qquad (3.33)$$

where $m = p$ $(1 \leq p \leq n)$ for $\Lambda = \Lambda_1$ and $\Lambda = \Lambda_n$, but $m = j$ for $\Lambda = \Lambda_j$ $(2 \leq j \leq n-1)$. For the case that G is semisimple from the beginning, we require $Z = 0$. Then for $x = \frac{2}{3}$ and $y = -\frac{1}{3}$, this requires

$$n + 1 = 3m. \qquad (3.34)$$

Since m is a positive integer, G must be a product of SU($3m$) groups in agreement with the conclusion reached in (I).

The explicit form of ξ is now given by

$$\xi = \frac{2}{(\alpha_1, \alpha_1)} \Lambda_p, \text{ for } \Lambda = \Lambda_1, \qquad (3.35a)$$

$$\xi = \frac{2}{(\alpha_1, \alpha_1)} \Lambda_1, \text{ for } \Lambda = \Lambda_j \quad (2 \leq j \leq n-1), \qquad (3.35b)$$

$$\xi = \frac{2}{(\alpha_1, \alpha_1)} \Lambda_{n+1-p}, \text{ for } \Lambda = \Lambda_n \qquad (3.35c)$$

corresponding to Eqs. (3.32a)–(3.32c) with a choice of

$$(\xi, \Lambda) = u = 1 - \frac{m}{n+1}. \qquad (3.36)$$

If we wish to use another choice

$$(\xi, \Lambda) = v = -\frac{m}{n+1}, \qquad (3.37)$$

then we have only to replace Λ_k inside the expression of ξ in Eq. (3.35) by

$$\Lambda_k \to -\Lambda_{n+1-k}. \qquad (3.38)$$

We may note that apart from the sign change, Eq. (3.38) is equivalent to the inversion of the Dynkin diagram of A_n, which in turn represents effects of its corresponding outer automorphism.

IV. ADDITIONAL RESULTS

In Sec. II, we noted the fact that (ξ, α) is the electric charge of the intermediate gauge boson W_α of the type α for the gauge group G_1. Here, let us first compute the number N_+ of positively charged gauge bosons for the simple group G_1. Then N_+ is given by

$$N_+ = \tfrac{1}{2} \sum_{\beta} (\xi, \beta)^2 = \sum_{\beta > 0} (\xi, \beta)^2 \qquad (4.1)$$

since (ξ, β) can assume only three values, 1, 0, or −1, and since the number of positively charged vector bosons should be equal to the number of negatively charged bosons, in view of $T_r Q_1 = 0$.

Following Racah,[21] we adopt a normalization

$$g_{\beta, -\beta} = 1 \qquad (4.2)$$

for all nonzero roots, unless we state otherwise. Then we have also

$$g_{jk} = 2 \sum_{\beta > 0} \beta_j \beta_k, \qquad (4.3)$$

where β_j is the jth component of a positive root β in the root space. This implies

$$n = 2 \sum_{\beta > 0} (\beta, \beta) = \sum_{\beta} (\beta, \beta), \qquad (4.4)$$

where n is the rank of the group G_1. If we define δ as usual by

$$\delta = \tfrac{1}{2} \sum_{\beta > 0} \beta = \Lambda_1 + \Lambda_2 + \cdots + \Lambda_n, \qquad (4.5)$$

then the eigenvalue $I_2(\lambda)$ of the second-order Casimir invariant

$$I_2 = g^{\mu\nu} X_\mu X_\nu, \quad g_{\mu\nu} = \mathrm{Tr}(\mathrm{ad}X_\mu \mathrm{ad}X_\nu)$$

in generic irreducible representation $\{\lambda\}$ is given by

$$I_2(\lambda) = (\Lambda, \Lambda + 2\delta), \qquad (4.6a)$$
$$I_2(\lambda_0) = (\Lambda_0, \Lambda_0 + 2\delta) = 1, \qquad (4.6b)$$

where X_μ is a basis of the Lie algebra $\underline{g}_1$ and Λ_0 is the highest weight of the adjoint representation $\{\lambda_0\}$. It is sometimes more convenient to consider the second index $l_2(\lambda)$ of Dynkin[30,28] by

$$l_2(\lambda) = n I_2(\lambda) \frac{d(\lambda)}{d(\lambda_0)} \qquad (4.7)$$

apart from the normalization constant, where $d(\lambda_0)$ is the dimension of the adjoint representation $\{\lambda_0\}$. Then our starting formula is

$$\mathrm{Tr}(X_\mu X_\nu) = \frac{1}{n} l_2(\lambda) g_{\mu\nu}$$
$$= \frac{d(\lambda)}{d(\lambda_0)} I_2(\lambda) g_{\mu\nu}. \qquad (4.8)$$

Since the electric charge operator Q_1 is given by

$$Q_1 = (\xi, H), \qquad (4.9)$$

this gives

$$\mathrm{Tr}(Q_1)^2 = \frac{1}{n} l_2(\lambda)(\xi, \xi). \qquad (4.10)$$

On the other hand, (ξ, ξ) can be computed from (4.3) to be

$$(\xi,\xi)=2\sum_{\beta>0}(\xi,\beta)^2=2N_+. \tag{4.11}$$

Moreover, when we note that Q_1 can assume only two eigenvalues u and v in the representation $\{\lambda\}$, we evaluate

$$\mathrm{Tr}(Q_1)^2=n_1u^2+n_2v^2=-uvd(\lambda) \tag{4.12}$$

because of (2.7). From Eqs. (4.10), (4.11), and (4.12), we compute

$$N_+=-\tfrac{1}{2}uv\frac{d(\lambda_0)}{I_2(\lambda)}\ .$$

If we note that we have $I_2(\lambda_0)=1$ for our normalization (4.2), this can be rewritten in a form

$$N_+=-\tfrac{1}{2}uvd(\lambda_0)\frac{I_2(\lambda_0)}{I_2(\lambda)} \tag{4.13}$$

in any unspecified normalization for $g_{\mu\nu}$. For the cases that the algebra $\underline{g_1}$ is not of the type A_n ($n \geq 2$), then we know $u=-v=\frac{1}{2}$ so that this gives

$$N_+=\frac{1}{8}\frac{I_2(\lambda_0)d(\lambda_0)}{I_2(\lambda)}=\frac{1}{8}\frac{l_2(\lambda_0)}{l_2(\lambda)}d(\lambda)\,. \tag{4.14}$$

In this way, the number of positively charged vector bosons can be computed. We note that the number N_0 of neutral vector bosons is given by

$$N_0=d(\lambda_0)-2N_+. \tag{4.15}$$

Numerically, we find

(i) B_n ($n\geq 2$), $\Lambda=\Lambda_n$, $d(\lambda_0)=n(2n+1)$:

$N_+=2n-1$, $N_0=2n^2-3n+2$;

(ii) C_n ($n\geq 2$), $\Lambda=\Lambda_1$, $d(\lambda_0)=n(2n+1)$:

$N_+=\frac{1}{2}n(n+1)$, $N_0=n^2$;

(iii) D_n ($n\geq 3$), $d(\lambda_0)=n(2n-1)$:

(a) $\Lambda=\Lambda_1$,

$N_+=\frac{1}{2}n(n-1)$, $N_0=n^2$,

(b) $\Lambda=\Lambda_{n-1}$ or Λ_n,

$N_+=2(n-1)$, $N_0=2n^2-5n+4$;

(iv) A_n ($n\geq 1$), $d(\lambda_0)=n(n+2)$:

(a) $\Lambda=\Lambda_1$ or $\Lambda=\Lambda_n$,

$N_+=p(n+1-p)$, $1\leq p\leq n$,

(b) $\Lambda=\Lambda_j$ ($2\leq j\leq n-1$),

$N_+=n$, $N_0=n^2$.

Another physical quantity of some interest is the number of isotopic-spin doublets contained in a given multiplet $\{\lambda\}$. Noting the usual commutation relation[21]

$$\begin{aligned}&[E_\beta,E_{-\beta}]=(\beta,H)\,,\\&[(\beta,H),E_\beta]=(\beta,\beta)E_\beta\,,\\&[(\beta,H),E_{-\beta}]=-(\beta,\beta)E_{-\beta}\,,\end{aligned} \tag{4.16}$$

we may identify the weak isotopic-spin subgroup SU(2) of the Weinberg-Salam theory contained in G_1 to be generated by

$$\begin{aligned}&T_3=\frac{1}{(\beta,\beta)}(\beta,H)\,,\\&T_\pm=T_1\pm iT_2=\left[\frac{2}{(\beta,\beta)}\right]^{1/2}E_{\pm\beta}\,.\end{aligned} \tag{4.17}$$

for a positive root β. Lemma (III) of Sec. II guarantees then that all eigenvalues of T_3 are restricted to 0 or $\pm\frac{1}{2}$ for all weights M in the representation $\{\lambda\}$. Therefore, we conclude that only the isosinglet and isodoublet alone can appear in the reduction of the group G_1 to the SU(2) subgroup.

Let $N_{1/2}$ be the number of isotopic doublets which are contained in the representation $\{\lambda\}$. Then we have

$$\mathrm{Tr}(T_3)^2=\tfrac{1}{2}N_{1/2}. \tag{4.18}$$

On the other hand, Eq. (4.8) together with (4.17) leads to

$$\mathrm{Tr}(T_3)^2=\frac{1}{n}\frac{1}{(\beta,\beta)}l_2(\lambda)=\frac{d(\lambda)}{d(\lambda_0)}\frac{I_2(\lambda)}{(\beta,\beta)}\ ,$$

so that we find

$$N_{1/2}=\frac{2}{(\beta,\beta)}\frac{d(\lambda)}{d(\lambda_0)}\frac{I_2(\lambda)}{I_2(\lambda_0)}=\frac{2}{(\beta,\beta)}\frac{l_2(\lambda)}{l_2(\lambda_0)} \tag{4.19}$$

or

$$N_+N_{1/2}=-uvd(\lambda)\frac{1}{(\beta,\beta)}\ , \tag{4.20}$$

where we divided Eq. (4.19) by $I_2(\lambda_0)$ in the numerator since $I_2(\lambda_0)=1$ in our normalization. The value of (β,β) can be computed from normalization Eq. (4.4) or $I_2(\lambda_0)=1$ to be

(1) A_n ($n\geq 1$), D_n($n\geq 3$), E_6, E_7, and E_8:

$$(\beta,\beta)=\frac{n}{d(\lambda_0)-n}\ , \tag{4.21a}$$

(2) B_n ($n\geq 2$):

$$(\beta,\beta)=\begin{cases}\dfrac{1}{2n-1}\ , & \text{if } \beta=S\alpha_j\,,\ j\neq n\,,\\[2ex] \dfrac{1}{2(2n-1)}\ , & \text{if } \beta=S\alpha_n\,,\ j=n\,,\end{cases} \tag{4.21b}$$

(3) C_n $(n \geq 2)$:

$$(\beta,\beta)=\begin{cases}\dfrac{1}{2(n+1)}\ , & \text{if } \beta=S\alpha_j\ ,\ j\neq n\ ,\\[2ex] \dfrac{1}{n+1}\ , & \text{if } \beta=S\alpha_n\ ,\ j=n\ ,\end{cases} \tag{4.21c}$$

(4) G_2:

$$(\alpha_1,\alpha_1)=3(\alpha_2,\alpha_2)=\tfrac{1}{4}\ , \tag{4.21d}$$

(5) F_4:

$$(\alpha_1,\alpha_1)=(\alpha_2,\alpha_2)=2(\alpha_3,\alpha_3)=2(\alpha_4,\alpha_4)=\tfrac{1}{9}\ , \tag{4.21e}$$

where S in Eq. (4.21b) and (4.21c) denotes any Weyl reflection operation. From Eqs. (4.19) and (4.21), we compute $N_{1/2}$ to be

(1) A_n $(n\geq 1)$, $\Lambda=\Lambda_j$ $(1\leq j\leq n)$:

$$N_{1/2}=\frac{(n-1)!}{(j-1)!(n-j)!}\ , \tag{4.22a}$$

(2) B_n $(n\geq 2)$, $\Lambda=\Lambda_n$:

$$N_{1/2}=2^{n-2}\ ,\ \text{if } (\beta,\beta)=\frac{1}{2n-1}\ , \tag{4.22b}$$

$$N_{1/2}=2^{n-1}\ ,\ \text{if } (\beta,\beta)=\frac{1}{2(2n-1)}\ , \tag{4.22c}$$

(3) C_n $(n\geq 2)$, $\Lambda=\Lambda_1$:

$$N_{1/2}=2\ ,\ \text{if } (\beta,\beta)=\frac{1}{2(n+1)}\ , \tag{4.22d}$$

$$N_{1/2}=1\ ,\ \text{if } (\beta,\beta)=\frac{1}{n+1}\ , \tag{4.22e}$$

(4) D_n $(n\geq 3)$:

(a) $\Lambda=\Lambda_1$,

$$N_{1/2}=2\ , \tag{4.22f}$$

(b) $\Lambda=\Lambda_{n-1}$ or Λ_n,

$$N_{1/2}=2^{n-3}. \tag{4.22g}$$

The component of weak hypercharge Y in the group G_1 may be defined by

$$Q_1=T_3+\tfrac{1}{2}Y\ , \tag{4.23}$$

so that Y is given by

$$Y=(\eta,H)\ , \tag{4.24a}$$

$$\eta=2\xi-\frac{2}{(\beta,\beta)}\beta\ . \tag{4.24b}$$

The condition

$$[T_3,Y]=[T_\pm,Y]=0 \tag{4.25}$$

imposes a constraint for β of the form

$$(\xi,\beta)=1 \tag{4.26}$$

which is compatible with proposition (II) of Sec. II. Of course, these represent only a contribution from the group G_1. The real expression for T_3 and Y are direct sums of all these contributions from all simple groups G_j and the Abelian U(1) generator R for the case of Y. Then the final expression obtained in this way would correspond to the physical T_3 and Y of the Weinberg-Salam gauge subgroup $SU(2)\otimes U(1)$. Also, we may note that we defined the SU(2) subgroup by Eq. (4.17). However, it could happen that we can find many subgroups of G_1, which commute with each other. In such a case, we may choose T_3 and $T_\pm$ as a direct sum of all infinitesimal generators of all such SU(2) subgroups, if we wish to do so.

Finally, we could give another proof of incompatibility of the two-charge *Ansatz* for the case of $\Lambda=\Lambda_6$ in the group E_7 as follows.

Suppose that the representation $\Lambda=\Lambda_6$ of E_7 is compatible with the two-charge *Ansatz*. Then we compute (ξ,ξ) in the following two ways. First, because of Eqs. (4.11) and (4.14), we compute

$$(\xi,\xi)=\frac{1}{4}\frac{l_2(\lambda_1)}{l_2(\lambda_6)}d(\lambda_6)=42\ . \tag{4.27}$$

On the other hand, we may express it also as

$$(\xi,\xi)=\sum_{j=1}^{n}\frac{2(\xi,\Lambda_j)(\xi,\alpha_j)}{(\alpha_j,\alpha_j)}\ .$$

But because of proposition (II) of Sec. II, (ξ,α_j) is an integer. Then expressing Λ_j in terms of simple roots as in the Appendix, we find that $2(\xi,\Lambda_j)$ is also an integer. Moreover, noting $(\alpha_j,\alpha_j)=\frac{1}{18}$, this requires (ξ,ξ) to be an integer multiple of 18. However, this contradicts Eq. (4.27). This proves the impossibility of $\Lambda=\Lambda_6$ for E_7 in accordance with the conclusion reached in Sec. III.

V. CONCLUDING REMARKS

In the previous sections we have classified all possible gauge groups for unified weak and electromagnetic interactions which are compatible with a two-charge *Ansatz*. Although the results are found to be very restrictive, it is still not enough to determine the best possible candidate for G. To that end, we have to assume some simplicity (not in a mathematical sense) assumptions. The most reasonable way is perhaps to consider small-rank groups. Eliminating the trivial case of the pure Abelian groups, the smallest group with rank $r=1$ is the SU(2) group. However, this is semisimple and would not satisfy our criteria that only admissible semisimple groups are a product of $SU(3l)$ groups. Hence, the lowest-rank group SU(2) is not possible. The next higher-rank non-Abelian groups with rank $r=2$ are $SU(2)\otimes U(1)$, SU(3), G_2, SU(2)⊗SU(2), and Sp(4) ≃ SO(5). Again, our semisimplicity criteria eliminate all semi-simple groups except for the SU(3). Therefore, we conclude that only SU(2)⊗U(1) and SU(3) are possible candidates for the gauge group with rank $r=2$. However, models based upon the SU(3) group[9] predict a pure axial-vector neutral current which contradicts the present experimental data.[31] It appears[32] that any reasonable SU(3) model cannot reproduce the experimentally allowable linear combination of vector and axial-vector neutral currents. Therefore, we conclude that the only viable lowest-rank gauge group is precisely the Weinberg and Salam gauge group SU(2)⊗U(1). However, in case we need to go beyond this, we may have to consider groups with $r=3$. Again, the only groups compatible with the two-charge condition are SU(2)⊗U(1)⊗U(1),[4] SU(2)⊗SU(2)⊗U(1),[5] SU(3)⊗U(1),[6] and Sp(4)⊗U(1) ≃ SO(5)⊗U(1).[8] All of these groups have been indeed proposed and investigated by many authors. Note that all these groups are non-semisimple. We may remark that the first viable semi-simple group SU(3)⊗SU(3) (Ref. 10) is of rank 4. As we have emphasized in (I), the semi-simple group has many interesting predictions so that the group SU(3)⊗SU(3) may be of some physical interest.

In ending this section, we simply remark that physically we can relax our two-charge condition so as to allow three quark charges $\frac{2}{3}$, $-\frac{1}{3}$, and $-\frac{4}{3}$. In that case, the condition of lemma (III) must be relaxed to a statement that $2(M,\beta)/(\beta,\beta)$ must be restricted to values 0, ±1, and ±2. Then the highest weight Λ for an admissible multiplet must have the form

$$\Lambda=\Lambda_j \text{ or } \Lambda=\Lambda_j+\Lambda_k, \tag{5.1}$$

where Λ_j and Λ_k are fundamental weights. In such a case, even exceptional Lie algebras can now be allowed. For example, the 27-dimensional representation $\Lambda=\Lambda_1$ or $\Lambda=\Lambda_5$ of E_6 is now perfectly compatible with our three-charge condition that quarks can assume charges $\frac{2}{3}$, $-\frac{1}{3}$, or $-\frac{4}{3}$. However, for the groups

$$\text{SU}(2),\ \text{SO}(2n+1),\ \text{Sp}(2n),\ \text{SO}(4l),\ G_2,\ F_4,\ E_7,\ E_8, \tag{5.2}$$

whose representations are all self-contragredient,[25] the choice of one of these as the simple group G is not compatible with the genuine three-charge *Ansatz* by the same reason explained in (I). Moreover, if the group G is

$$G=G_1\otimes \text{U}(1) \tag{5.3}$$

and if G_1 is one of the simple groups listed in (5.2), then we have

$$Q=Q_1+R, \tag{5.4}$$

where Q_1 now satisfies[11]

$$\text{Tr}(Q_1)^p=0 \tag{5.5}$$

for all positive odd integers $p=1,3,5,\ldots$. Let n_1, n_2, and n_3 be the numbers of quarks with electric charges $\frac{2}{3}$, $-\frac{1}{3}$, and $-\frac{4}{3}$, respectively, in a given irreducible multiplet. Then if the genuine three-charge *Ansatz*[33] is obeyed for the multiplet, we can easily prove on the basis of Eq. (5.5) that Q_1 can assume only eigenvalues 1, 0, and −1 while R takes the value $-\frac{1}{3}$ in the multiplet. Moreover, we must have

$$n_1=n_3, \tag{5.6}$$

i.e., the number of the quarks with charge $-\frac{4}{3}$ must be equal to those with charge $+\frac{2}{3}$. This fact would be phenomenologically undesirable at least for the present, so that the choice of Eq. (5.3) for G_1 being one of the groups given in (5.2) will not be a good prospect. In contrast, the pure simple gauge group $G=E_6$ in the 27-dimensional representation $\Lambda=\Lambda_1$ will give $n_1=10$, $n_2=16$, and $n_3=1$ with a choice of ξ,

$$\xi=-\frac{2}{(\alpha_1,\alpha_1)}\Lambda_1. \tag{5.7}$$

We may easily check that these are consistent with Eqs. (3.2) and (4.10) for $X=Q_1=(\xi,H)$.

Here we did not discuss the cancellation of the triangular anomaly as well as Higgs mesons. These depend upon detailed dynamical considerations and upon specific assignments of representations for positive and negative chiral components of fundamental fermion multiplets.

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APPENDIX

Here we shall adopt the lexiconal ordering of simple roots as in Fig. 1. Then straightforward computations based upon Eq. (2.16) enable us to write fundamental weights in terms of simple roots[34] as follows: Here Λ_0 designates the highest weight of the adjoint representation:

(1) Algebra A_n $(n\geq 1)$, $\Lambda_0=\Lambda_1+\Lambda_n$:

$$\Lambda_j=\frac{n+1-j}{n+1}(\alpha_1+2\alpha_2+\cdots+j\alpha_j)+\frac{j}{n+1}[(n-j)\alpha_{j+1}+(n-j-1)\alpha_{j+2}+\cdots+\alpha_n]\quad(1\leq j\leq n). \tag{A1}$$

(2) Algebra B_n ($n \geq 2$), $\Lambda_0 = \Lambda_2$ ($n \geq 3$),

$\Lambda_0 = 2\Lambda_2$ ($n = 2$):

(a) Λ_j ($1 \leq j \leq n-1$),

$$\Lambda_j = \alpha_1 + 2\alpha_2 + \cdots + (j-1)\alpha_{j-1} + j(\alpha_j + \alpha_{j+1} + \cdots + \alpha_n)\,, \tag{A2a}$$

(b) Λ_n ($j = n$),

$$\Lambda_n = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)\,. \tag{A2b}$$

(3) Algebra C_n ($n \geq 2$), $\Lambda_0 = 2\Lambda_1$:

(a) Λ_j ($1 \leq j \leq n-1$),

$$\Lambda_j = \alpha_1 + 2\alpha_2 + \cdots + (j-1)\alpha_{j-1} + j(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1} + \tfrac{1}{2}\alpha_n)\,, \tag{A3a}$$

(b) Λ_n ($j = n$),

$$\Lambda_n = \alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1} + \tfrac{1}{2}n\alpha_n\,. \tag{A3b}$$

(4) Algebra D_n ($n \geq 3$), $\Lambda_0 = \Lambda_2$ ($n \geq 4$),

$\Lambda_0 = \Lambda_2 + \Lambda_3$ ($n = 3$):

(a) Λ_j, $1 \leq j \leq n-2$,

$$\Lambda_j = \alpha_1 + 2\alpha_2 + \cdots + (j-1)\alpha_{j-1} + j(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-2}) + \tfrac{1}{2}j(\alpha_{n-1} + \alpha_n)\,, \tag{A4a}$$

(b) Λ_{n-1}, $j = n-1$,

$$\Lambda_{n-1} = \tfrac{1}{2}[\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + (n-2)\alpha_{n-2}] + \tfrac{1}{4}[n\alpha_{n-1} + (n-2)\alpha_n]\,, \tag{A4b}$$

(c) Λ_n, $j = n$,

$$\Lambda_n = \tfrac{1}{2}[\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2}] + \tfrac{1}{4}[(n-2)\alpha_{n-1} + n\alpha_n]\,. \tag{A4c}$$

(5) Algebra G_2, $\Lambda_0 = \Lambda_1$:

$$\Lambda_1 = 2\alpha_1 + 3\alpha_2\,, \quad \Lambda_2 = \alpha_1 + 2\alpha_2\,. \tag{A5}$$

(6) Algebra F_4, $\Lambda_0 = \Lambda_1$:

$$\Lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\,, \tag{A6a}$$

$$\Lambda_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4\,, \tag{A6b}$$

$$\Lambda_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4\,, \tag{A6c}$$

$$\Lambda_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\,. \tag{A6d}$$

(7) Algebra E_6, $\Lambda_0 = \Lambda_6$:

$$\Lambda_1 = \tfrac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6)\,, \tag{A7a}$$

$$\Lambda_2 = \tfrac{1}{3}(5\alpha_1 + 10\alpha_2 + 12\alpha_3 + 8\alpha_4 + 4\alpha_5 + 6\alpha_6)\,, \tag{A7b}$$

$$\Lambda_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6\,, \tag{A7c}$$

$$\Lambda_4 = \tfrac{1}{3}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 10\alpha_4 + 5\alpha_5 + 6\alpha_6)\,, \tag{A7d}$$

$$\Lambda_5 = \tfrac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6)\,, \tag{A7e}$$

$$\Lambda_6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\,. \tag{A7f}$$

(8) Algebra E_7, $\Lambda_0 = \Lambda_1$:

$$\Lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7\,, \tag{A8a}$$

$$\Lambda_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + 4\alpha_7 \tag{A8b}$$

$$\Lambda_3 = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 6\alpha_7\,, \tag{A8c}$$

$$\Lambda_4 = \tfrac{1}{2}(6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 15\alpha_4 + 10\alpha_5 + 5\alpha_6 + 9\alpha_7)\,, \tag{A8d}$$

$$\Lambda_5 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 2\alpha_6 + 3\alpha_7\,, \tag{A8e}$$

$$\Lambda_6 = \tfrac{1}{2}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 3\alpha_7)\,, \tag{A8f}$$

$$\Lambda_7 = \tfrac{1}{2}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 7\alpha_7)\,. \tag{A8g}$$

(9) Algebra E_8, $\Lambda_0 = \Lambda_1$:

$$\Lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8\,, \tag{A9a}$$

$$\Lambda_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 10\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7 + 6\alpha_8\,, \tag{A9b}$$

$$\Lambda_3 = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 15\alpha_4 + 18\alpha_5 + 12\alpha_6 + 6\alpha_7 + 9\alpha_8\,, \tag{A9c}$$

$$\Lambda_4 = 5\alpha_1 + 10\alpha_2 + 15\alpha_3 + 20\alpha_4 + 24\alpha_5 + 16\alpha_6 + 8\alpha_7 + 12\alpha_8\,, \tag{A9d}$$

$$\Lambda_5 = 6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 24\alpha_4 + 30\alpha_5 + 20\alpha_6 + 10\alpha_7 + 15\alpha_8\,, \tag{A9e}$$

$$\Lambda_6 = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 16\alpha_4 + 20\alpha_5 + 14\alpha_6 + 7\alpha_7 + 10\alpha_8\,, \tag{A9f}$$

$$\Lambda_7 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 10\alpha_5 + 7\alpha_6 + 4\alpha_7 + 5\alpha_8\,, \tag{A9g}$$

$$\Lambda_8 = 3\alpha_1 + 6\alpha_2 + 9\alpha_3 + 12\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7 + 8\alpha_8\,. \tag{A9h}$$

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