Fermions and bosons in a unified framework. III. Mathematical structures and physical questions

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By examining the implications of the usual physical assumptions for gauge theories we are led to a number of interesting structures which can be formulated in terms of moving frames. The first problem we consider is Faddeev's fiber bundle procedure for removal of gauge degeneracies from quantum fields. We show how the theory of moving frames can be related to the theory of fiber bundles. In the fiber bundle extra displacements and extra components of the gauge fields are included. These precisely remove the gauge degeneracies. In order to formulate this problem in the language of fiber bundles, we begin with a section detailing the algebra of forms in four dimensions and their associated operations (exterior differentiation, Hodge duality and the natural Hilbert product of forms). We then show how to form the action for the SO(3) fiber bundle. This opens the general question (implicitly answered by this example) of how to quantize in a fiber bundle. Next we examine the question of choice of local symmetry structure. First we consider a space whose local "rotations" do not form a group. By generalizing the structure equations for manifolds, we can study such a space. Second, in general relativity it is assumed that at any point one can choose an orthonormal basis and a vanishing connection. This corresponds to giving the zeroth, first, and second coefficient of a Taylor's expansion of the coordinate system. But manifolds flatter than those of general relativity do exist for which the structure is carried in the higher coefficients. We show how to give an action principle for them. Third, algebras more general than Lie algebras are possible local symmetries, e.g., superalgebras, the octonian algebra, and others violating the Jacobi relation. An action is given for a one-parameter family of quasi-Lie algebras which reduces to that of u(1) + su(2) in the zero limit of the parameter. Fourth, simple examples of actions for gauged superalgebras are given which do not include the full complexity of general relativity. Fifth, an easy derivation of the supergravity action is made; then generalized to a supergravity Weinberg-Salam-type model.

I. INTRODUCTION

The description of physics in terms of moving frames[1] has the advantage of a central position. On the one hand, by choosing a specific basis the explicit components of the field are manifested. In detailed calculations it is often necessary to use these. On the other hand, by formulating the usual physical assumptions in this general language, it becomes easier to observe and modify their consequences for local structures. By examining some of these assumptions we are led to a number of models which generalize the standard view[2] of gauge theory but can still be expressed in terms of individual components in a specific basis.

We begin with a look at Faddeev's technique for removing gauge field degeneracies. To do this we consider "cohesions in a vector stratification," i.e., "connections in a fiber bundle."[3] We show how to relate the theory of moving frames to that of fiber bundles.[4,5] Essentially a fiber bundle is an extension of the usual base manifold (space-time) to include an internal-symmetry space such as a local gauge group by "attaching" one copy of the group to each point of space-time. Both the local frame and the number of allowed displacements are increased. The "fiber" at each point is the group at each point.

Section II details some of the "natural" bundles found on a four-manifold (those associated with infinitesimal lines, areas, etc.) and some of the characteristic operations on those bundles from an intuitive viewpoint.[4] A description of the technical features of a fiber bundle is found in the Refs. 4 and 5 and is sketched in Sec. III.

The operations on the "natural" bundles which are mentioned are the exterior derivative (a generalization of the curl) and the Hodge dual (a generalization of Maxwell duality). From these the cohomology groups and the "natural" Hilbert product of n-forms can be found.[4]

We proceed to a discussion of the details of an SO(3) bundle on Minkowski space in Sec. III. There we see that if the action is formed as the Hilbert square of the Yang-Mills curvature of SO(3) up in the bundle, the gauge fields have further degrees of freedom corresponding to the group generators which must be fixed through some sensible prescription. Consideration of this problem yields the Faddeev procedure for dealing with quantum field's degeneracies, and suggests that a detailed examination of the technique of quantizing in a fiber bundle may be useful.[6,7]

One of the obvious advantages of the theory of moving frames (or of fiber bundles) is that it permits one to examine internal and space-time symmetries with a single formalism (Sec. IV).[8] In this viewpoint arbitrary manifolds are straightforward generalizations of group manifolds. To obtain them one permits the connection coefficients to become functions instead of structure constants.

The theory of moving frames (or of fiber bundles) clearly includes a wide variety of structures. In Sec. IV we indicate some of the more amusing examples of local structure.[9] We begin with an object having a local rotation manifold instead of a local rotation group. This structure can be (even locally) anisotropic. But still it is not a difficult object to describe differentially. Next we examine manifolds which can be locally flatter than those considered in general relativity.[10,11] Usually one assumes that at any given point an orthonormal vector basis with vanishing connection (and torsion) can be chosen. This amounts to picking a zeroth-, first-, and second-order structure. But what about frames of higher than second-order contact (jets)?[12] We sketch the theory of a manifold in which the third-order structure at a point can also be chosen.

Other modifications of the usual local (algebraic) structure are possible. For example, instead of restricting the local algebra to be Lie, it can have mixed Lie and Jordan structures (as a graded Lie algebra does.)[13,14] Or the Jacobi identity can fail to hold in a number of interesting ways, giving rise to alternating, Mal'cev or even more unusual algebras.[15] We provide an example of a local algebra which has a broken Jacobi relation but in the zero limit of a given parameter becomes $u(1)+su(2)$. Some of these algebras may be physically interesting because they allow a different kind of symmetry breaking than is available through the Higgs-Kibble[2,16] mechanism. Clearly, exhausting the Lie algebraic structures does not exhaust the options available to local field theories.

Since there is some reason to hope that graded Lie algebras have physical relevance,[13] we describe some moving frames related to those algebras in the manner discussed in Sec. III of the second paper in this series.[1] But in Sec. V we concentrate on some simpler versions having two- or three-component spinor fields. We discover that the $SL(2|1)$[14] super connections are not adequate to include general relativity. This occurs because $SL(2,C)$ covers only the Lorentz group. Translations are needed to obtain the full effects of general relativity as pointed out in Sec. II of the first paper in this series.[1]

In Sec. VI we show how to obtain the well-known spin-2–spin-$\frac{3}{2}$ supergravity action from a moving frame.[17,18] We point out that this is not the gauge theory of any simple supergroup (cf. Sec. II of paper II[1]). The nearest simple supergroup is $OSp(1,4)$. When it is gauged, extra pieces appear in the action (at least a cosmological constant). If simplicity is not a requirement, then many more structures are available. As an example we write down a colored supergravity Weinberg-Salam-type model.[2,19,20]

II. DIFFERENTIAL FORMS, GRASSMANN ALGEBRA, HODGE DUALITY, AND COHOMOLOGY GROUPS

Given a bose manifold (space-time) it is possible to consider oriented basis elements associated with infinitesimal points, lines, areas, etc. The coefficients of these basic elements form linear vector spaces: functions, gauge potentials, etc. One can express the bases for planes, etc., in terms of line elements by using the alternating tensor product (wedge product), $\wedge$. The infinitesimal n-plane element can be written as follows,[4,5,11]

$$dx^{\mu}\wedge\cdots\wedge dx^{\nu}=\sum_{P}\frac{1}{n!}\sigma_P\,(dx^{\mu_P}\otimes\cdots\otimes dx^{\nu_P}) \tag{1}$$

summed over all $n!$ permutations each with sign σ_P. An area element, for example, is given as follows,

$$dx^{\mu}\wedge dx^{\nu}=\tfrac{1}{2}(dx^{\mu}\otimes dx^{\nu}-dx^{\nu}\otimes dx^{\mu})\,. \tag{2}$$

The exception to the rule of alternation occurs when the basis, I, for the space of functions (points) is included. It satisfies

$$I\wedge dx^{\mu}=dx^{\mu}\wedge I=dx^{\mu}\,. \tag{3}$$

We can now tabulate the objects associated with points, lines, etc.; please see Table I.

Sometimes the collection of vector spaces,

$$\Lambda(M)=\sum_{p=-\infty}^{\infty}\Lambda^p(M)\equiv\sum_{p=0}^{n}\Lambda^p(M) \tag{4}$$

with $n=$ dimension (M) is called the exterior algebra of M. $\Lambda^p(M)$ is called the space of p-forms on M. It is also called the Grassmann algebra. Note that these are special fiber bundles on M. Compare Sec. III,

$$\Lambda^p(M)\equiv 0 \text{ for } p\notin\{0,\ldots,n\}\quad(n=4,\text{ here})\,. \tag{5}$$

It is possible to introduce[4,5,11] a map, d, called the exterior derivative from $\Lambda^p(M)$ to $\Lambda^{p+1}(M)$ which acts on one-forms like $(\vec{\nabla}\times)$ and on functions like (∇), where M is a three-manifold. We will give a table of the action of this operator for the four-space exterior algebra. See Table II.

All other actions are trivial since $\Lambda^p(M)=0$ for

$p \notin \{0, \ldots, 4\}$. We remark that the operator d satisfies $d \circ d = dd = 0$ as can readily be verified for $f \in \Lambda^0$,

$$d \circ d f = d(f_{|\nu} dx^\nu) = \tfrac{1}{2} f_{|\underline{\nu}\underline{\lambda}} dx^\lambda \wedge dx^\nu = 0 . \qquad (6)$$

Indices with bars are to be antisymmetrized. Thus, $A_{\underline{\mu}\nu\underline{\rho}} = A_{\mu\nu\rho} - A_{\rho\nu\mu}$. Further, d is an antiderivation $d(a\omega + b\tau) = ad\omega + bd\tau$ for a, b constants and $\omega, \tau \in \Lambda^p(M)$, and $d(\omega \wedge \tau) = d\omega \wedge \tau + (-)^p \omega \wedge d\tau$ when $\omega \in \Lambda^p(M)$ and $\tau \in \Lambda^k(M)$. The product form is $\omega \wedge \tau \in \Lambda^{p+k}(M)$ with coefficients defined by

$$\omega \wedge \tau = \sum_{\substack{\mu_1 < \cdots < \mu_p \\ \nu_1 < \cdots < \nu_k}} \omega_{\mu_1 \cdots \mu_p} \tau_{\nu_1 \cdots \nu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_k} , \qquad (7)$$

here we have used ordered indices $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_k$. Note that any terms having two indices repeated between ω and τ vanish.

From the spaces $\Lambda^p(M)$ one can construct[4,5] other spaces of topological interest. First, $Z^p(M) \subset \Lambda^p(M)$, the set of closed p-forms consists of those elements vanishing under d. That is if $\omega \in Z^p(M)$, then $d\omega = 0$. Second, $B^p(M) \subset \Lambda^p(M)$, the set of exact p-forms consists of those elements which are derivatives of $(p-1)$-forms. That is if $\tau \in B^p(M)$, then $\tau = d\sigma$ for some σ in $\Lambda^{p-1}(M)$. Since $d \circ d$ is zero $[d(d\sigma) = 0]$, every element of $B^p(M)$ is in $Z^p(M)$, $B^p(M) \subset Z^p(M)$. It is therefore possible to consider $H^p(M) = Z^p(M)/B^p(M)$, those elements of $Z^p(M)$ not in $B^p(M)$. This is called the pth cohomology group for M. If it vanishes, $Z^p(M) = B^p(M)$ and every closed form is exact. (If $df = 0$ there exists an A such that $f = dA$. Magnetic Maxwell equations imply a potential unless there are line charges.) The nonvanishing of $H^p(M)$ is related to the presence of p-dimensional holes. Indeed $\beta_p = \dim[H^p(M)]$ the pth Betti number is the dimension of $H^p(M)$ and counts the number of p-holes. We will not examine the topological aspects of manifolds here.

The last operation on the exterior algebra that we will describe is the Hodge dual denoted by an asterisk. It maps elements of $\Lambda^p(M_n)$ onto $\Lambda^{n-p}(M_n)$. That the map is onto is evident from Pascal's triangle. Note $\dim[\Lambda^p(M_n)] = \binom{n}{p} = n!/p!(n-p)!$. We use the fully antisymmetric tensor to relate ω_p and ω_p^* so that

$$\omega_p \wedge \omega_p^* = (\pm)\sqrt{-g} p! \, dx^1 \wedge \cdots \wedge dx^n \, \omega_{\alpha \cdots \beta} \, \omega^{\alpha \cdots \beta} , \qquad (8)$$

where $n = \dim M_n$ and $g_{\mu\nu}$ is the metric of M_n; g is its determinant. In four dimensions we use $\sqrt{-g}\, \epsilon_{\nu\mu\lambda\rho}$. See Table III.

A metric on the space of p-forms is thus induced.[4,5] Set $\sqrt{-g} = 1$; then

$$[(A_\mu dx^\mu)^* \wedge (B_\nu dx^\nu)]^* = \left(A_\mu B_\nu \frac{1}{3!} \epsilon^\mu{}_{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma\nu}\right)(d^4x)^*$$
$$= (A_\mu B_\nu g^{\mu\nu}) I = A \cdot B \qquad (9)$$

and

$$[(F_{\mu\nu} dx^\mu \wedge dx^\nu)^* \wedge (G_{\alpha\beta} dx^\alpha \wedge dx^\beta)]^*$$
$$= (-|F_{\mu\nu} G_{\alpha\beta} g^{\mu\bar{\alpha}} g^{\nu\bar{\beta}})(d^4x)^*$$
$$= -2! F_{\mu\nu} G^{\mu\nu} I , \qquad (10)$$

for example. By omitting the second operation of the asterisk, a natural Hilbert product is formed:

$$\langle A | B \rangle = \int (A_\mu B^\mu) \sqrt{-g} \, d^4x = \int A^* \wedge B , \qquad (11)$$

for one-forms or

$$\langle F | G \rangle = -\int (F_{\mu\nu} G^{\mu\nu}) \sqrt{-g} \, d^4x = \int \tfrac{1}{2} F^* \wedge G , \qquad (12)$$

for two-forms.

Two important facts about * should be noted. First, * maps p-forms into $(n-p)$-forms and thus changes the scale dimension of a form. For example $\dim(A_\mu dx^\mu) = 0$, but $\dim[A_\mu (dx^\mu)^*] = l^2$. Second, in the canonical basis, θ^i, one can naively apply * twice because $\epsilon_{ij}{}^{kl}$ is just ± 1. In a coordinate basis such that $\sqrt{-g} \neq 1$ the second application of * requires dividing by $1/\sqrt{-g}$. Using $\theta^i = Y^i_\mu dx^\mu$ and $\det Y = \sqrt{-g}$ allows one to verify the sense of these statements and to discover that $** = \pm 1$.

III. QUANTIZATION IN FIBER BUNDLES

In this section we will relate the theory of moving frames to the theory of fiber bundles.[4,5] And we will see that the Faddeev-Popov approach to the removal of gauge degeneracies can be (essentially) understood as quantization in a fiber bundle.[3,6,7]

The basic idea is to extend the manifold from just the space-time manifold, M, to admit extra degrees of freedom corresponding to the generators of the internal-symmetry group, G (called the structure group). This new manifold, B, called the bundle will have a dimension equal to the sum of the dimensions of M and G. For example if M is the usual space-time and G is the group SU(2), then the bundle will be seven dimensional. We will see that in that setting gauge degeneracies are removed naturally and that it is the natural place to do a sum of paths quantization.

We will concentrate on this example first and return to the general definitions at the end. We will consider a flat space-time. Thus if $(e_\mu, \bar{e}_A)$ denotes our moving frame we find

$$\left\{\begin{array}{c} dP = e_\mu dx^\mu \\ de_\mu = 0 \\ d\bar{e}_A = \bar{e}_B B^B_{\mu A} dx^\mu \end{array}\right\} \text{ becomes } \left\{\begin{array}{c} dP = e_\mu dx^\mu + \bar{e}_A \theta^A \\ de_\mu = 0 \\ d\bar{e}_A = \bar{e}_B (\hat{B}^B_{\mu A} dx^\mu + \hat{B}^B_{CA}\theta^C) \end{array}\right\}, \tag{13}$$

where θ^A is an su(2) algebra-valued one-form such that $d\theta^A + \frac{1}{2}\epsilon^A_{BC}\theta^B \wedge \theta^C = 0$, and $[\bar{e}_A, \bar{e}_B] = \epsilon^C_{AB}\bar{e}_C$. We will write the coordinate differentials, dx^μ, in the bundle using the same notation as in the base. We will introduce the Euler angles ϕ^a such that $\theta^A = Y^A_a d\phi^a$ (see Sec. IV) and the matrices Λ_A with components $(\Lambda_A)^C_B = \epsilon^C_{AB}$. We will rewrite the connection $(\hat{B}^B_{\mu C} dx^\mu + \hat{B}^B_{DC}\theta^D)$ as $\hat{B}^A(\Lambda_A)^B_C$ or $(B^A_\mu dx^\mu + \hat{B}^A_a d\phi^a)(\Lambda_A)^B_C$. We can readily compute the curvature P^A from $\frac{1}{2}[d,\delta]\bar{e}_C = \bar{e}_B P^A(\Lambda_A)^B_C$ or from

$$\begin{aligned} P^A &= d\hat{B}^A + \tfrac{1}{2}\epsilon^A_{BC}\hat{B}^B \wedge \hat{B}^C \\ &= \tfrac{1}{2}(\hat{B}^A_{\underline{\nu}|\underline{\mu}} + \epsilon^A_{BC}\hat{B}^B_\mu \hat{B}^C_\nu) dx^\mu \wedge dx^\nu \\ &\quad + (\hat{B}^A_{\mu|a} - \hat{B}^A_{a|\mu} + \epsilon^A_{BC}\hat{B}^B_a\hat{B}^C_\mu) d\phi^a \wedge dx^\mu \\ &\quad + \tfrac{1}{2}(\hat{B}^A_{\underline{b}|\underline{a}} + \epsilon^A_{BC}\hat{B}^B_a \hat{B}^C_b) d\phi^a \wedge d\phi^b . \end{aligned} \tag{14}$$

Recall that $A_{\underline{\nu\mu\rho}} \equiv A_{\nu\mu\rho} - A_{\rho\mu\nu}$.

In order that $\hat{B}^A$ be a "connection on a fiber bundle"[5] the coefficients of the vertical pieces ($d\phi^a \wedge dx^\mu$ and $d\phi^a \wedge d\phi^b$) must vanish. We can solve the equations obtained by setting these coefficients to zero by taking

$$\begin{aligned} \hat{B}^A\Lambda_A &= [\hat{B}^A_\mu(x,\phi)dx^\mu + \hat{B}^A_a(x,\phi)d\phi^a]\Lambda_A \\ &= \Omega(x,\phi)B^A_\mu(x,0)dx^\mu \Lambda_A \Omega^{-1}(x,\phi) \\ &\quad + \Omega(x,\phi)d\Omega^{-1}(x,\phi) \end{aligned} \tag{15}$$

with $\Omega(x,\phi) = \exp[\tau^C(x,\phi)\Lambda_C]$. ϕ labels elements, g, of the fiber at x. For each x, Ω must be a diffeomorphism of the group at x into the group, G. $\tau^C(x,\phi)$ is a parametrization of this diffeomorphism. Identifying the origin in the fiber with the group identity fixes all points in the fiber. Thus if $p \in B$ and $g \in G$, then $\Omega(pg) = \Omega(p)g$. We find that $P \equiv P^A\Lambda_A = \Omega R^A \Lambda_A \Omega^{-1} = \Omega R \Omega^{-1}$ where

$$R^A = \tfrac{1}{2}[B^A(x,0)_{\underline{\nu}|\underline{\mu}} + \epsilon^A_{BC}B^B_\mu(x,0)B^C_\nu(x,0)]dx^\mu \wedge dx^\nu \tag{16}$$

is manifestly purely horizontal (contains no $d\phi^a \wedge dx^\nu$ or $dx^\mu \wedge dx^\nu$ pieces).

Now we are confronted with two possibilities: either we construct the action on the base manifold or up in the bundle. To define the curvature in the base manifold, M, we must identify M with a particular slice through the bundle. This is analogous to identifying the x-y plane with the set of points in x-y-z space determined by $z = f(x,y)$ with f a function of x and y. Choosing such a slice is called taking a section. We will define this in a general context later in this section. Here we merely set ϕ^a equal to functions of x, $\phi^a = f^a(x^\mu)$. Then $\tau^C(x,\phi^a) = \tau^C(x, f^a(x)) \equiv \sigma^C(x)$. Define

$$(F^A\Lambda_A) = \exp[\sigma^C(x)\Lambda_C](R^A\Lambda_A)\exp[-\sigma^C(x)\Lambda_C] . \tag{17}$$

The operation of Hodge duality (cf. Sec. II) depends explicitly on the dimension of the manifold; we will denote the four-dimensional dual by * and the seven-dimensional dual by ★. One uses the appropriate Levi-Civita tensor. In the basis $dx^\mu, d\phi^a$ the seven-dimensional metric is $\text{diag}[1,-1,-1,-1;-1,-(\sin\phi^1)^2, -(\sin\phi^1\sin\phi^2)^2]$.

Now find the actions

$$\begin{aligned} \text{base } \langle F|F\rangle &= \tfrac{1}{2}\int \text{tr}(\bar{F}^* \wedge F) \\ &= -\tfrac{1}{4}\int F^A_{\mu\nu}F^{\mu\nu}_A d^4x , \\ \text{bundle } \langle P|P\rangle &= \tfrac{1}{2}\int \text{tr}(\bar{P}^\star \wedge P) \\ &= -\text{vol}[\text{SU}(2)]\langle F|F\rangle , \end{aligned} \tag{18}$$

since

$$(dx^\mu \wedge dx^\nu)^\star \wedge (dx^\alpha \wedge dx^\beta) = +\tfrac{1}{2}\eta^{\mu\bar{\alpha}}\eta^{\nu\bar{\beta}} d^4x \sin^2\phi^1 \sin\phi^2 d\phi^1 d\phi^2 d\phi^3$$

and

$$(dx^\mu \wedge dx^\nu)^* \wedge (dx^\alpha \wedge dx^\beta) = -\tfrac{1}{2}\eta^{\mu\bar{\alpha}}\eta^{\nu\bar{\beta}} d^4x .$$

The SU(2) matrices have been normalized to 1, $\text{tr}(\Lambda_A\Lambda_B) \equiv \delta_{AB}$. If $\text{tr}(\Lambda_A\Lambda_B) = N\delta_{AB}$, then use tr/$N$ in defining $\langle F|F\rangle$. Note that in defining $\langle P|P\rangle$ we can either rescale the connection, $B \to gB$ with $(g^2)\,\text{vol}[\text{SU}(2)] = 1$ or replace the trace by trace/vol[SU(2)] or generally, trace/(N vol[SU(2)]).

TABLE I. Forms in four dimensions.

Name	Description	Basis	Dimension of basis	Vector space of n-forms
0-form	points (functions)	I	one I	$\Lambda^0(M)=\{fI \mid f=\text{function of } M\}$
1-form	line element	dt, dx, dy, dz	four dx^μ	$\Lambda^1(M)=\{A_\mu dx^\mu \mid A_\mu=4 \text{ functions of } M\}$
2-form	oriented area element	$dt\wedge dx,\ dx\wedge dy,\ dt\wedge dy,\ dy\wedge dz,\ dt\wedge dz,\ dz\wedge dx$	six $dx^\mu\wedge dx^\nu$	$\Lambda^2(M)=\{f_{\mu\nu}dx^\mu\wedge dx^\nu \mid f_{\mu\nu}=6 \text{ functions of } M\}$
3-form	oriented three-volume element	$dx\wedge dy\wedge dz,\ dy\wedge dz\wedge dt,\ dz\wedge dt\wedge dy,\ dt\wedge dx\wedge dz$	four $dx^\mu\wedge dx^\nu\wedge dx^\rho$	$\Lambda^3(M)=\{B_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho \mid B_{\mu\nu\rho}=4 \text{ functions of } M\}$
4-form	oriented space-time volume element	$d^4x\equiv dt\wedge dx\wedge dy\wedge dz$	one d^4x	$\Lambda^4(M)=\{P_{\mu\nu\rho\lambda}dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\lambda \mid P_{\mu\nu\rho\lambda}=\text{function of } M\}$ functions are antisymmetric on indices

To form a quantum field theory we will introduce an integral over paths. But, as Faddeev has pointed out,[3] field theories having a geometric interpretation lead to singular Lagrangians. By geometric interpretation he means field theories of "cohesions in a vector stratification," i.e., "connections in a fiber bundle." He goes on to exploit gauge invariance to resolve the singularities. But since these problems arise for theories of connections in a fiber bundle and since the connection in the bundle explicitly includes the gauge term Ω $(B=\Omega A\Omega^{-1}+\Omega d\Omega^{-1})$, it is interesting to examine the concept of quantization in the fiber bundle. There, a sum over paths must involve all of B, including Ω. Of course to uniquely split B into A and Ω we should have an unambiguous procedure for choosing A. Picking a gauge condition is not always unambiguous. However, some ambiguities are canceled out. For example in U(1) with $\partial\cdot A=0$ gauge transformations with $\partial^2\lambda=0$ preserve the gauge condition. Nonetheless the zeros in the ghost kinetic term precisely cancel the zeros in the gauge-field kinetic terms. Gribov has shown that for non-Abelian groups extra zeros may arise.[21] These occur for finite gauge transformations. No problems arise if all fields (including gauge transformations) are perturbatively small. Abelian theories appear to be without problem in gauges such as $\partial\cdot A=\alpha$.

Another minor problem presents itself. A zero-mass field should have only two on-shell or three off-shell components. A gauge field in $4+n$ dimensions will have $4+n$ components. Therefore, there should be $1+n$ constraints on each gauge field. From the formula for B the additional components are in the form of a pure gauge term, $\Omega\partial_a\Omega^{-1}$. By taking the n additional constraints to be $\phi_a=f_a(x)$, we define B on a section. This reduces the dimension of the independent variables from seven to four. All ϕ^a dependence can be completely integrated out of the theory leaving a theory depending effectively on the x^ν variables only. This is essential for renormalizability.

We will write a class of gauge conditions for U(n) as follows, $\gamma^D=\alpha\partial^\mu A^D_\mu+\beta d^D_{AB}A^A\cdot A^B$. We define d^A_{BC} by the U(n) equations $\frac{1}{2}\{\Lambda_B,\Lambda_C\}=d^A_{BC}\Lambda_A$. This constraint includes many of the usual gauge conditions for A^D_μ, Landau, Feynman, and Dirac-Nambu (quadratic), by selecting various values of α, β, and γ^D. We note that $G=G^D\Lambda_D=\star(\alpha d+\beta A\wedge)\star A=\gamma^D\Lambda_D$ is the dual of the gauge-covariant derivative of the dual of A when $\alpha=\beta(=1)$. For U(1), $\gamma=\alpha\partial\cdot A+\beta A^2$. A calculation of the Coulomb force requires the introduction of ghosts in such a gauge.

Now we can define an action functional by

TABLE II. Action of exterior derivative.

Spaces	Elements
$\Lambda^0 \to \Lambda^1$	$df = f_{\mid\nu} dx^\nu = \partial_\nu f dx^\nu \quad ; \quad f_{\mid\nu} \equiv \partial_\nu f$
$\Lambda^1 \to \Lambda^2$	$dA = d(A_\mu dx^\mu) = A_{\mu\mid\nu} dx^\nu \wedge dx^\mu = \frac{1}{2} A_{\underline{\mu}\mid\underline{\nu}} dx^\nu \wedge dx^\mu ;\ d(dx^\nu) = 0$
$\Lambda^2 \to \Lambda^3$	$dF = \frac{1}{2} d(F_{\mu\nu} dx^\mu \wedge dx^\nu) = \frac{1}{2} F_{\mu\nu\mid\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu = \frac{1}{12} F_{\underline{\mu\nu}\mid\underline{\rho}} dx^\rho \wedge dx^\mu \wedge dx^\nu$
$\Lambda^3 \to \Lambda^4$	$dB = \frac{1}{6} d(B_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{6} B_{\mu\nu\rho\mid\lambda} dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{144} B_{\underline{\mu\nu\rho}\mid\underline{\lambda}} dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho$
$\Lambda^4 \to \Lambda^5 = 0$	$dP = d(P d^4x) = P_{\mid\lambda} dx^\lambda \wedge dt \wedge dx \wedge dy \wedge dz = 0$ by antisymmetry and symmetry

$$W(J) = \int [dB] \tilde{\Delta}_G \left(\prod \delta(\phi_a - f_a(x)) \right) \exp\left\{ \frac{-i}{32\pi^2} \int \mathrm{tr}[\overline{P}^\star \wedge P + (\overline{G-\gamma})^\star \wedge (G-\gamma) + \overline{J}^\star \wedge B^\Omega] \right\}$$
$$= \int [d\Omega] \int [dA] \Delta_G \exp\left\{ \frac{-i}{4} \int [F^A_{\mu\nu} F^{\mu\nu}_A + (G^A - \gamma^A)(G_A - \gamma_A) + J^A_\mu A^\mu_A] d^4x \right\}. \tag{19}$$

We have taken $\mathrm{vol}[\mathrm{SU}(2)] = 8\pi^2$ and $\mathrm{tr}\Lambda_A \Lambda_B = 2\delta_{AB}$. F and $\star G$ are (up to coupling constants) the gauge-covariant derivatives of A and of its dual. P is the gauge-covariant derivative of B. We have defined

$$\tilde{\Delta}_G{}^{-1} = \int [d\Omega(x,\phi)] \prod \delta(G-\gamma) \prod_{a=1}^{3} \delta(\phi_a - f_a(x)),$$
$$\Delta_G{}^{-1} \simeq \int [d\Omega(x)] \exp\left\{ \frac{-i}{4} d^4x [(G^A - \gamma^A)(G_A - \gamma_A)] \right\}. \tag{20}$$

In going from product to Gaussian form extra factors are absorbed in the $W(J)$'s norm (which factors out of matrix elements).

Under a gauge transformation $\tilde{A} = \tilde{\Omega}(x) A \tilde{\Omega}^{-1}(x) + \tilde{\Omega}(x) d\tilde{\Omega}(x)^{-1}$, and

$$\tilde{G}^A \Lambda_A = \tilde{\Omega}\{G^A \Lambda_A + \beta[2\star (A \wedge \star d) + \star(d \wedge \star d)]\} \tilde{\Omega}^{-1} + (\alpha - \beta) \star (d\tilde{\Omega} \wedge \star d\tilde{\Omega}^{-1}). \tag{21}$$

Infinitesimally $(\tilde{\Omega}^{-1}) = I + \tau^C \Lambda_C$, and

$$\tilde{G}^A = G^A + \epsilon^A_{BC} G^B \tau^C + 2\beta d^A_{BC} A^B \cdot \partial \tau^C + \beta \partial^2 \tau^A. \tag{22}$$

The conventional ghost action is therefore

$$\Delta_G = \int [dc][d\bar{c}] \exp\left\{ i \int d^4x [\beta(\partial^\mu \bar{c} \partial_\mu c - 2\bar{c}_A d^A_{BE} A^B \cdot \partial c^E) - \bar{c}_A \epsilon^A_{BE} G^B c^E] \right\}. \tag{23}$$

It may be that in spite of the difficulties mentioned by Gribov[21] for small field values (perturbation theory) the Feynman rules are still accurate (if only small gauge transformations are allowed).

Mathematically, a connection is a splitting of the tangent space of the bundle into horizontal and vertical. The one-form $B = B^A_\mu \Lambda_A dx^\mu + B^A_a \Lambda_A d\phi^a = B^A \Lambda_A$ can be related to such a choice as follows. View $B^A = B^A_\mu dx^\mu + B^A_a d\phi^a$ and $B^\mu \equiv dx^\mu$ as a basis for the one-forms in the bundle. The B^A's are the vertical basis and the B^μ's are the horizontal. Choosing a basis amounts to choosing a vertical sub-

TABLE III. Action of Hodge dual.

Spaces	Canonical basis $\theta^i = Y^i_\mu dx^\mu$	First application on a coordinate basis (dx^μ)	Second application on dx^μ
$\Lambda^0 \xrightarrow{*} \Lambda^4$	$I^* = \frac{1}{4!} \epsilon_{ijkl} \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l$	$I^* = \frac{1}{4!} \sqrt{-g}\, \epsilon_{\mu\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \sqrt{-g} d^4x$	$I^{**} = -I$
$\Lambda^1 \xrightarrow{*} \Lambda^3$	$(\theta^i)^* = \frac{1}{3!} \epsilon^i_{jkl} \theta^j \wedge \theta^k \wedge \theta^l$	$(dx^\mu)^* = \frac{1}{3!} \sqrt{-g}\, \epsilon^\mu_{\nu\lambda\rho} dx^\nu \wedge dx^\lambda \wedge dx^\rho$	$(dx^\mu)^{**} = (dx^\mu)$
$\Lambda^2 \xrightarrow{*} \Lambda^2$	$(\theta^i \wedge \theta^j)^* = \frac{1}{2!} \epsilon^{ij}_{kl} \theta^k \wedge \theta^l$	$(dx^\mu \wedge dx^\nu)^* = \frac{1}{2!} \sqrt{-g}\, \epsilon^{\mu\nu}_{\lambda\rho} dx^\lambda \wedge dx^\rho$	$(dx^\mu \wedge dx^\nu)^{**} = -(dx^\mu \wedge dx^\nu)$
$\Lambda^3 \xrightarrow{*} \Lambda^1$	$(\theta^i \wedge \theta^j \wedge \theta^k)^* = \epsilon^{ijk}_l \theta^l$	$(dx^\mu \wedge dx^\nu \wedge dx^\rho)^* = \sqrt{-g}\, \epsilon^{\mu\nu\rho}_\lambda dx^\lambda$	$(dx^\mu \wedge dx^\nu \wedge dx^\rho)^{**} = (dx^\mu \wedge dx^\nu \wedge dx^\rho)$
$\Lambda^4 \xrightarrow{*} \Lambda^0$	$(\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3)^* = I$	$(d^4x)^* = \sqrt{-g} I$	$(d^4x)^{**} = -d^4x$

space. We remark that since $\langle dx^\mu | \partial_\mu\rangle = \delta^\mu_\nu$, etc., B^A annihilates the "horizontal" vector fields $X = X^\mu \partial_\mu + X^a \partial_a \equiv X^\mu \partial_\mu - B^a_A B^A_\mu X^\mu \partial_a$ with $B^a_A B^A_b = \delta^a_b$ since $\langle B^A | X\rangle = B^A_\mu X^\mu - B^A_a X^a = 0$. There is a four-parameter family, determined by X^μ, of such horizontal fields for each choice of B^A. The others, which do not vanish, have "vertical" components.

We will return to the technical properties of fiber bundles briefly. Let B be the bundle, M the base, and G the fiber. B has a right multiplication by elements in G. If $b \in B$ and $g \in G$, then $bg \in B$. This allows us to write $M = B/G$. The map π: $B \to M = B/G$ is called canonical projection. It is differentiable. The inverse image of $x \in M$, $\pi^{-1}(x)$ is called the fiber at x. The union of these fibers is the bundle. It is not always true that if $M = B/G$, then $B = M \times G$; when that is true, B is called trivial. What is true is that for every $x \in M$ there exists a set S_α such that $M = \cup_{\alpha\in I} S_\alpha$ (with I an indexing set) and such that τ_α: $\pi^{-1}(S_\alpha) \to S_\alpha \times G$ is a diffeomorphism. τ_α is called the trivializing map. It says that around each point there exists a "trivial" neighborhood (a cross product of S_α and G). $\tau_\alpha(b) \equiv (\pi(b), \phi_\alpha(b))$. ϕ_α maps $\pi^{-1}(S_\alpha)$ into G and satisfies $\phi_\alpha(b)g = \phi_\alpha(bg)$ for $b \in \pi^{-1}(S_\alpha)$ and $g \in G$. On intersects of the open cover $\{S_\alpha\}$ of M one defines $\tau_{\beta\alpha}(\pi(b)) = \phi_\beta(b)\phi_\alpha(b)^{-1}$, called the transition function. It depends only on $x = \pi(b)$. It satisfies $\tau_{\gamma\alpha} = \tau_{\gamma\beta} \circ \tau_{\beta\alpha}$ on triple intersects. Note, $\tau_\beta(b) = (\pi(b), \tau_{\beta\alpha}\phi_\alpha(b)) = (\pi(b), \phi_\beta(b))$. Under such a change what happens to the connection? First, define a section to be a map $\sigma_\alpha : S_\alpha \to B$ such that $\pi \circ \sigma_\alpha = 1$. Thus σ_α picks a unique element in $\pi^{-1}(x)$ for each x. Take $\sigma_\alpha = \tau_\alpha{}^{-1}(x, I)$ where $x \in S_\alpha$ and I is the identity of G. τ_α is the trivializing map. Let θ be the left-invariant, algebra of G-valued one-form such that $\theta(A) = A$ for A the algebra of G. If Λ_A is a basis for the algebra, then $\theta = \theta^A \Lambda_A$ and $d\theta + \theta \wedge \theta = \Lambda_A(d\theta^A + \frac{1}{2}\epsilon^A_{BC}\theta^B \wedge \theta^C) = 0$. It is dual to θ_A, a vector field satisfying $[\theta_A, \theta_B] = \epsilon^C_{AB}\theta_C$. Duality means that $\langle \theta^A | \theta_B\rangle = \delta^A_B$.

Any map, f, from a manifold M into N induces a linear mapping, f_*, of their tangent bundles. Thus if X is a vector field in $T(M)$ acting on a function g, then $(f_* X)g = X(g \circ f)$ defines f_*; $Xf = X^\nu \partial_\nu f$. Similarly if ω is a p-form then $\langle f^*\omega | X_1, \ldots, X_p\rangle = \langle \omega | f_* X_1, \ldots, f_* X_p\rangle$ defines $f^*\omega$.[5] Let B be a connection in the bundle. Consider the intersect $U_\alpha \cap U_\beta$. Set $B_\alpha = \sigma^*_\alpha B$ and $\theta_{\alpha\beta} = \tau^*_{\alpha\beta}\theta$. Then $B_\beta = ad(\tau^{-1}_{\alpha\beta})B_\alpha + \theta_{\alpha\beta}$.[5] Remember that α and β refer to sets, not components. This formula can be translated to read

$$(B^A_\mu \Lambda_A)_\beta = \tau^{-1}_{\alpha\beta}(B^B_\mu \Lambda_B)_\alpha \tau_{\alpha\beta} + \tau^{-1}_{\alpha\beta}\partial_\mu \tau_{\alpha\beta} . \tag{24}$$

Changing from U_α to U_β is just a gauge transformation with the transition function $\tau^{-1}_{\alpha\beta}[\pi(b)] = \phi_\beta(b)\phi_\alpha(b)^{-1}$ as the gauge map.

In the language of moving frames we see that $\bar{e}_i = V^\mu_i \bar{e}_\mu$ together with $d\bar{e}_i = \bar{e}_j B^j_i(d)$ allows us to infer that

$$\begin{aligned} &d\bar{e}_\mu = \bar{e}_\nu B^\nu_\mu(d), \quad \text{with} \\ &B^\nu_\mu(d) = V^\nu_i dV^i_\mu + B^j_i(d) V^i_\mu = B^\nu_{\lambda\mu}\, dx^\lambda . \end{aligned} \tag{25}$$

In the theory of fiber bundles, going from one frame to the next occurs in going from one element of the open cover $\{S_\alpha\}$ of M to another. When the theory of frames was made precise in terms of bundles, it was this property which determined the transition behavior. Of course, in most gauge theories the change of frames, V^i_μ, is required to be a function taking values in some representation of the group. An exception to this is the theory of general relativity, which can be viewed as an SO(3, 1) theory disguised by V^i_μ's taking GL(4) values. This is obvious since $V^i_\mu g^{\mu\nu} V^j_\nu = \eta^{ij}$ and $\omega^i_{\mu j} = V^i_\alpha \partial_\mu V^\alpha_j + V^i_\alpha \Gamma^\alpha_{\beta\mu} V^\beta_j$ is an SO(3, 1)-valued connection. If one introduces ISO(3, 1) the V^i_μ's can be viewed as the other pieces of the connection, rather than mysterious, external objects. We note that an action such as $\int \mathrm{tr}(\bar{F}^* \wedge F)$ is invariant under extended gauge transformations $B \to \Omega A \Omega^{-1} + \Omega\, d\Omega^{-1}$ where $\Omega = \exp[\tau^C_{(x)}\Sigma_C]$ and Σ_C generates *any group* whatsoever in *any representation*. General relativity is the only example in which such a transformation is commonly employed. But there, because $dx^\mu \to Y^i_\mu dx^\mu$ under the transformation, the requirement that coordinates be used as a basis selects the usual procedure.

Use of this technique (of extended gauge transformations) allows one to introduce extra fields to which the gauge field can be coupled. Let $\tilde{\Omega}$ be an extended gauge transformation. Introduce a fixed vector u normalized to m, $\bar{u}u = m^2$. Set $\Phi = \tilde{\Omega}^{-1}u$. Note that $|\partial_\mu \Phi - B_\mu \Phi|^2$ is proportional to $m^2\,\mathrm{tr}(\bar{B}_\mu B_\nu \eta^{\mu\nu})$ if $\tilde{\Omega}$ is unitary. An example of this idea is to start with a U(2) bundle, restrict $A_\mu dx^\mu$ in B to U(1) and view the extra "extended" gauge transformations as matter fields. $W(0)$ is given as

$$\begin{aligned} W(0) &= \int [d\Omega][dA]\Delta_F \exp\left[\frac{i}{2}\int d^4x (A_\mu \partial^2 A^\mu + m^2\,\mathrm{tr} A^\Omega_\mu A^{\mu\,\Omega})\right] \\ &= \int [d\bar{\Phi}][d\Phi][dA]\,\delta(\bar{\Phi}\Phi - m^2)\Delta_F \exp\left[\frac{i}{2}\int d^4x (A_\mu \partial^2 A^\mu + |\partial_\mu \Phi - A_\mu \Phi|^2)\right] . \end{aligned} \tag{26}$$

Δ_F is the determinant for the gauge $\partial \cdot A = 0$. See Sec. II and Ref. 24 of the second paper in this series.[1] Actually the notation in (26) may be somewhat confusing. Two measures are possible $[\mathfrak{D}(\Omega^{-1} d\Omega)]$ or $[\mathfrak{D}(\bar{\Omega}u)][\mathfrak{D}(\Omega u)]$. The first is the usual Haar measure; the second is what we have used in (26). The second measure is similar to that used in the Weinberg-Salam model.

IV. LOCAL ALGEBRAIC STRUCTURES, LIE AND OTHER ALGEBRAS AND GENERALIZED MANIFOLDS

In this section we will examine the vector field formalism we have been using somewhat more closely. The first point we wish to emphasize is that the use of vector fields (or covector fields, one-forms) shows manifolds to be "almost" Lie groups. To study a Lie group one considers its Lie algebra. Usually one considers the space of tangents satisfying $[\theta_i, \theta_j] = \epsilon^k_{ij}\theta_k$ where $\theta_k = Y^\mu_k \partial_\mu = \vec{\partial}_k$. ϵ^k_{ij} are the structure constants and i and μ both run over n = dimension of the algebra. However, a formalism in terms of cotangents $\theta^k = Y^k_\mu dx^\mu$ can also be considered[4,11]; then

$$d\theta^k + \tfrac{1}{2}\epsilon^k_{ij}\theta^i \wedge \theta^j = 0 . \tag{27}$$

θ^k is dual to θ_l in that

$$\langle \theta^k | \theta_l \rangle = Y^k_\mu Y^\nu_l \langle dx^\mu | \partial_\nu \rangle = Y^k_\mu Y^\nu_l \delta^\mu_\nu = \delta^k_l . \tag{28}$$

Let us give a specific example: SO(3) tangents and cotangents are as follows,

$$\text{tangent} \qquad\qquad \text{cotangent}$$

$$[\theta_i, \theta_j] = \epsilon^k_{ij}\theta_k , \quad d\theta^k + \tfrac{1}{2}\epsilon^k_{ij}\theta^i \wedge \theta^j = 0 . \tag{29}$$

One can verify that these are satisfied by the following vectors and one-forms:

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = -\begin{bmatrix}\cos\psi & \dfrac{\sin\psi}{\sin\theta} & -\sin\psi\cot\theta\\ \sin\psi & -\dfrac{\cos\psi}{\sin\theta} & \cos\phi\cot\theta\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\partial_\theta\\ \partial_\phi\\ \partial_\psi\end{bmatrix} \text{ tangents (vectors),} \tag{30}$$

$$\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} = -\begin{bmatrix}\cos\psi & \sin\theta\sin\psi & 0\\ \sin\psi & -\sin\theta\cos\psi & 0\\ 0 & \cos\theta & 1\end{bmatrix}\begin{bmatrix}d\theta\\ d\phi\\ d\psi\end{bmatrix} \text{ cotangents (one-forms).} \tag{31}$$

Here $\partial_\mu = [\partial_1, \partial_2, \partial_3] = [\partial_\theta, \partial_\phi, \partial_\psi]$ are Euler-angle derivatives and $dx^\mu = [dx^1, dx^2, dx^3] = [d\theta, d\phi, d\psi]$ are Euler-angle differentials. The matrices Y^μ_i and Y^i_μ are the coefficient matrices given above.

In what sense are manifolds almost Lie algebras? The answer is found by introducing the concept of first and second structural equations. First define

$$\omega^k_j = \tfrac{1}{2}\epsilon^k_{ij}\theta^i . \tag{32}$$

Then consider (1)

$$T^k = d\theta^k + \omega^k_j \wedge \theta^j , \tag{33}$$

the torsion two-form, and (2)

$$R^k_l = d\omega^k_l + \omega^k_j \wedge \omega^j_l , \tag{34}$$

the curvature two-form. When ϵ^k_{ij} is a constant, $T^k = 0$ and $R^k_l = \frac{1}{4}(-\epsilon^k_{il}\epsilon^i_{mn} + \epsilon^k_{mj}\epsilon^j_{nl})\theta^m \wedge \theta^n$. We have the usual structural equations of a Lie algebra. We hope context will distinguish the curvature two-form from the Ricci tensor. Please note that we have not explicitly used the Ricci tensor here.

The coefficients are the components of the Riemann tensor of the group manifold

$$R^k_{\ lmn} = \tfrac{1}{8}(-\epsilon^k_{jl}\epsilon^j_{\underline{mn}} + \epsilon^k_{\underline{m}j}\epsilon^j_{\underline{nl}}) = \tfrac{1}{8}(\epsilon^k_{\underline{n}j}\epsilon^j_{\underline{ml}}) \tag{35}$$

which are constant.

A manifold is obtained by replacing the constants $\frac{1}{2}\epsilon^k_{ij}$ by functions $\Gamma^k_{ij}(x)$. The two structural equations are the same but now the equation for R^k_l is slightly more complicated,

$$\begin{aligned} R^k_l &= d\omega^k_l + \omega^k_j \wedge \omega^j_l \\ &= \tfrac{1}{2}(\Gamma^k_{\underline{nl}|\underline{m}} + \Gamma^k_{\underline{m}j}\Gamma^j_{\underline{nl}} - \Gamma^k_{il}\Gamma^i_{\underline{mn}})\theta^m \wedge \theta^n , \end{aligned} \tag{36}$$

when $T^k = 0$, because of the derivative of Γ^k_{nl}. Nonetheless we see that this formulation is well suited to both Lie groups and manifolds; one is just a special case of the other, or manifolds are "almost" Lie groups having varying structure, Γ^k_{lm}, instead of constant, ϵ^k_{lm}.

This leads to an amusing speculation on "almost" manifolds having higher structural equations. For example

$$\begin{aligned}&(1)\ T^k = d\theta^k + \omega^k_j \wedge \theta^j, \quad \omega^{kj} = \Gamma^{kj}_i \theta^i,\\&(2)\ R^{kl} = d\omega^{kl} + \Omega^{kl}_{mn} \wedge \omega^{mn},\\&\quad \Omega^{kl}_{mn} = \Gamma^{kl}_{ij\ mn}\,\omega^{ij} = \Gamma^{kl}_{ij\ mn}\,\Gamma^{ij}_p \theta^p,\\&(3)\ P^{kl}_{mn} = d\Omega^{kl}_{mn} + \Omega^{kl}_{ij} \wedge \Omega^{ij}_{mn}.\end{aligned} \tag{37}$$

For a manifold take $\Omega^{kl}_{ij} = \frac{1}{2}\epsilon^{kl}_{ij\,mn}\,\omega^{mn}$ with $\epsilon_{ij}{}^{kl}{}_{mn}$ the structure constants of the rotation group. When these become functions the local rotation group becomes a local "rotation" manifold. One could introduce $K^{ab}{}_{cd} = \omega^{ab} \wedge \omega_{cd}$ and then minimize the Hilbert product $\langle K|P\rangle$. Fourth and higher structural equations can be considered. These are looser structures than manifolds just as manifolds are looser than groups.

We have described the idea of "almost" manifolds to illustrate the structural similarities of general manifolds and Lie group manifolds. Fundamental to the construction of general relativity[10,11] is the set of prinicples that, one, an orthonormal Lorentz frame (or basis) can be chosen at a point (or that the metric can be chosen Minkowski), and, two, the symmetric part of the connection can be chosen to be zero (torsion is assumed zero, usually). "Almost" manifolds allow one to deal with the idea that the local rotations may not form a group.[9] But another generalization is possible. Instead of defining the local structure by choosing the basis and connection forms, it is possible to pick a flatter structure by placing requirements on the higher derivatives of the local coordinate functions.[12]

Specifically, let $x^1, \ldots, x^n$ be a local coordinate system in the manifold, M, and let $z^1, \ldots z^n$ be a coordinate system in R^n, then we can take

$$x^\mu(z) = Y^\mu + \sum_j Y^\mu_j z^j + \sum_{jk} Y^\mu_{jk} z^j z^k + \sum_{jkl} Y^\mu_{jkl} z^j z^k z^l\ . \tag{38}$$

The set $(Y^\mu, Y^\mu_j, Y^\mu_{jk}, Y^\mu_{jkl})$ defines a local coordinatization of the three-frame bundle, $P^3(M)$. The Y's are symmetric on latin indices. In general the bundle of r-frames, $P^r(M)$, is given by the equivalence classes of functions whose Taylor's series agree (pointwise) up to the rth derivatives. These are also referred to as r-jets.[12]

One can proceed to introduce one forms on (e.g.) $P^3(M)$ in the following way:

$$\begin{aligned}&\omega^i = Y^i_\mu dy^\mu,\\&\omega^i_k = Y^i_\mu[dY^\mu_k - 2Y^\mu_{kj} Y^j_\nu dY^\nu] = Y^i_\mu[dY^\mu_k - 2Y^\mu_{kj}\omega^j],\\&\omega^i_{jk} = Y^i_\mu[dY^\mu_{jk} - 2Y^\mu_{jl}\omega^l_k - 3Y^\mu_{jkl}\omega^l],\end{aligned} \tag{39}$$

etc.

By watching the indices the generalization is obvious. These forms satisfy the following relations:

$$\begin{aligned}&d\omega^i = -\omega^i_j \wedge \omega^j,\\&d\omega^i_j = -\omega^i_k \wedge \omega^k_j - 2\omega^i_{jk} \wedge \omega^k,\\&d\omega^i_{jk} = -\omega^i_{lk} \wedge \omega^l_j - \omega^i_{jl} \wedge \omega^l_k - 3\omega^i_{jkl} \wedge \omega^l,\end{aligned} \tag{40}$$

etc. The ω's are symmetric on their lower indices. For r-frames only $r-2$ of these relations can be proved from the formulas for ω.

Let us consider a space for which not just the point, the orthonormal frame and the connection, Y^μ, Y^μ_i, and Y^μ_{kj} can be chosen, but also the "three" connection Y^μ_{ijk}. We remind the reader that (up to equivalences) the usual choice is $Y^\mu = 0$, $Y^\mu_i = \delta^\mu_i$, and $Y^\mu_{ij} = 0$ at the given point (equivalence principle). Let us consider a manifold for which Y^μ_{ijk} can also be chosen (for definiteness set Y^μ_{ijk} equal to a constant matrix). From the frame structure we can immediately infer

$$0 = T^i = d\omega^i + \omega^i_j \wedge \omega^j$$

and

$$\begin{aligned}0 &= [R^i_j] - [C^i_j]\\&= [d\omega^i_j + \omega^i_k \wedge \omega^k_j] - [2\omega^i_{jk} \wedge \omega^k].\end{aligned} \tag{41}$$

Under gauge transformations (which preserve these equations)

$$\begin{aligned}&\omega^\alpha = Y^\alpha_i \omega^i,\\&\omega^\alpha_\beta = Y^\alpha_k[dY^k_\beta + \omega^k_j Y^j_\beta],\\&\omega^\alpha_{\beta\gamma} = Y^\alpha_k Y^l_\beta Y^m_\gamma \omega^k_{lm}.\end{aligned} \tag{42}$$

The gauge-covariant curvature of ω^i_{jk} can be constructed as

$$P^i_{jk} = d\omega^i_{jk} + \omega^i_{jl} \wedge \omega^l_k + \omega^i_{lk} \wedge \omega^l_j + \omega^l_{jk} \wedge \omega^i_l\ . \tag{43}$$

This format arises from requiring gauge covariance of P^i_{jk}. There is no obvious analog (in four dimension) of Einstein's linear action (η_{ij} is the Minkowski metric)

$$\mathfrak{A}_E = \int [\omega^i \wedge \omega^j]^* \wedge R^l_m \eta^m_i \eta_{jl}\,, \tag{44}$$

for P^i_{jk} because of the gauge properties of the connections. However, a quadratic is available

$$\mathfrak{A}_Q = \int [P^i_{jk}]^* \wedge P^l_{mn}\,\eta^{jm}\eta^{kn}\eta_{il}\,. \tag{45}$$

Appropriate dimensional constants have been omitted. A more complicated structure would be the sum of squares of rth-order curvature.

To see a concrete example of the various connections and forms involved here let ω^i be the three one-forms for SO(3) (formulas in terms of Euler angles were given at the beginning of this section, i.e., set $\omega^i = \theta^i$). Then take $\omega^i_j = \epsilon^i_{jk}\,\omega^k$; calculate R^i_j. Since $R^i_j = 2\omega^i_{jk} \wedge \omega^k$ a formula for the coefficients $\omega^i_{jk} = \epsilon^i_{jkl}\omega^l$ can be found (symmetrize on jk) in terms of the structure constants (or metric in this case). Then calculate P^i_{jk}.

From the jet bundle standpoint the usual theory of curved manifolds is not the simplest structure; a theory of pure torsion is. Einstein's is the second simplest nontrivial structure (see Sec. I-II of paper I).[1] The simplest theory would just gauge translations $\tau^i = d\omega^i$ with an action $\mathfrak{A} = \int d\omega^{i*} \wedge d\omega_i$.

Other, more commonly applied, local structures are certainly available. Next we will describe some of the simple generalizations[9,15] of Lie algebras which can be easily employed in the frame language.

There are a number of algebraic structures which are almost Lie algebras but have a nonzero Jacobi relation. Let e_a be a basis for the local algebra. Introduce a product, $e_a \cdot e_b = M^c_{ab} e_c$. If $M^c_{ab} = -M^c_{ab}$, then $e_a \cdot e_b = -e_b \circ e_a$. The product is antisymmetric and is sometimes said to be a Lie product. If it is entirely symmetric it may be called Jordan. Supersymmetry algebras have a mixture of Lie and Jordan products. If the product is antisymmetric and satisfies the Jacobi identity, that is, if

$$J(e_a, e_b, e_c) \equiv e_a \cdot (e_b \cdot e_c) + e_b \cdot (e_c \cdot e_a) + e_c \cdot (e_a \cdot e_b) \tag{46}$$

vanishes, then the e_a's form a basis for some Lie algebra. If $X = X^a e_a$, $Y = Y^a e_a m$, etc., and if

$$J(X, Y, XZ) = J(X, Y, Z)X\,, \tag{47}$$

the algebra is Mal'cev.[15] If $J(e_a, e_b, e_c)$ is cyclicly alternative on abc the algebra is alternative. The octonions are alternative, but not Lie; indeed, all alternative algebras can be constructed over the octonions.

Break M^c_{ab} into symmetric (Jordan), d^c_{ab}, and antisymmetric (Lie) f^c_{ab}, terms. Since $dx^\nu \wedge dx^\mu = -dx^\mu \wedge dx^\nu$, only the f^c_{ab} terms play a role in a gauge theory of these algebras, in the following sense:

$$\text{connection:}\quad B = B^a_\mu dx^\mu e_a\,, \tag{48}$$

$$\begin{aligned}\text{curvature:}\quad P &= dB + B \wedge B \\ &= \tfrac{1}{2}[\partial_\nu B^a_\mu + f^a_{bc} B^b_\nu B^c_\mu] e_a dx^\nu \wedge dx^\mu \\ &\equiv \tfrac{1}{2} P^a_{\nu\mu} e_a dx^\nu \wedge dx^\mu\,,\end{aligned} \tag{49}$$

$$\begin{aligned}\text{current:}\quad j &= j_\mu dx^\mu \\ &= \partial^\nu [P^a_{\nu\mu}] + f^a_{bc} B^b_\lambda \eta^{\lambda\nu} P^c_{\nu\mu} \\ &= *[d{*P} + B \wedge {*P} - {*P} \wedge B]\,.\end{aligned} \tag{50}$$

A simple example is given by an algebra near to u(1) + su(2) whose product is entirely antisymmetric. e_0 and e_i, $i \in \{1, 2, 3\}$, are the bases $e_0 \cdot e_i = -e_i \cdot e_0 = \lambda e_i$ and $e_i \cdot e_j = \epsilon^k_{ij} e_k$. λ going to zero recovers the u(1) + su(2) algebra and $e_i \cdot e_j = [e_i, e_j]$, etc. Define $B \wedge B = B^I \wedge B^J e_I \cdot e_J = B^I_\mu B^J_\nu e_I \cdot e_J dx^\mu \wedge dx^\nu$ with $I = \{0, i\}$ and for the connection, $B = B^I_\mu dx^\mu e_I$, one discovers the curvature

$$P = \tfrac{1}{2}[(B^0_{\underline{\nu}|\underline{\mu}})e_0 + (B^i_{\underline{\nu}|\underline{\mu}} + \epsilon^i_{jk} B^j_\mu B^k_\nu + \lambda B^0_{\underline{\mu}} B^j_{\underline{\nu}})e_i] dx^\mu \wedge dx^\nu\,. \tag{51}$$

By normalizing e_I's square to minus one (extend the product), one can form the action as usual. These algebras are interesting since they provide broken Lie algebras and could therefore have interesting experimental consequences.

V. SIMPLE EXAMPLES OF MOVING SPINOR FRAMES

In this section we take a closer look at some of the effects of viewing the space-time basis elements e_μ as composed of spinor structures so that $e_\mu = \bar{s}_a (J_\mu)^a_b s^b$ where $(J_\mu)^b_a$ are the components of some matrices such as γ_μ and the basis elements $\bar{s}_b$ transform under some group which at least contains the Lorentz group. We say at least because if $\bar{s}_b$ transforms only under the Lorentz group then one is not led to gravity since $g_{\mu\nu} = e_\mu \circ e_\nu$ can be chosen globally constant. Remember that $\Lambda^T g \Lambda = g$ is a way of finding the Lorentz group. We will consider this point in somewhat more detail when we look at two-component spinors $\bar{s}_a$ in a short while. We will thus find that there must be more generators J_A than just those J_μ giving rise to e_μ. For example in the case $e_\mu = \bar{s}_\alpha (\gamma_\mu)^\alpha_\beta s^\beta$ there are also $e_{\mu\nu} = \bar{s}_\alpha (\sigma_{\mu\nu})^\alpha_\beta s^\beta$. In general there are sixteen $e_I = \bar{s}_\alpha (\gamma_I)^\alpha_\beta s^\beta$. There is therefore a sixteen-dimensional manifold (possibly complex) of which only a certain four-dimensional slice is to be identified as space-time. Displacing a point gives $dP = dY^{\tilde{I}} e_{\tilde{I}}$ (with $\tilde{I}$ referring to the μ, ν basis) that is

$$dP = dY^S e_S + dY^\mu e_\mu + dY^{\mu\nu} e_{\mu\nu} + dY^{*\mu*} e_\mu + dY^* e^*\,. \tag{52}$$

By setting $dY^S = dY^{\mu\nu} = dY^{*\mu} = dY^* = 0$ we restrict displacements to a four-dimensional subspace. Clearly, taking $Y^{\tilde{I}} = Y^{\tilde{I}}(x^\mu)$ selects a general four-manifold, $dY^{\tilde{I}} = Y^{\tilde{I}}_{|\mu} dx^\mu$. Of course if $Y^{\tilde{I}} = \bar{\chi}\gamma^{\tilde{I}}\chi$ and $\chi = \chi(x)$ and $\bar{\chi} = \bar{\chi}(x)$, then $dY^{\tilde{I}} = d\bar{\chi}\gamma^{\tilde{I}}\chi + \bar{\chi} d\gamma^{\tilde{I}}\chi + \bar{\chi}\gamma^{\tilde{I}} d\chi$ can be written as

$$\begin{aligned} dY^{\tilde{I}} &= (\bar{\chi}_{|\mu}\gamma^{\tilde{I}}\chi + \bar{\chi}\gamma^{\tilde{I}}_{|\mu}\chi + \bar{\chi}\gamma^{\tilde{I}}\chi_\mu) dx^\mu \\ &= (\bar{\chi}\gamma^{\tilde{I}}\chi)_{|\mu} dx^\mu = Y^{\tilde{I}}_{|\mu} dx^\mu\,. \end{aligned} \tag{53}$$

We can examine the effect of an infinitesimal transformation on the current basis e_I by computation from knowing the effect on the spinor basis

$\bar{s}_\alpha$. Thus if $\delta s^\alpha = [\Omega^I(\gamma_I)^\alpha_\beta \delta x^\mu] s^\beta$ and $\delta \bar{s}_\alpha = \bar{s}_\beta[\Omega^I_\mu(\gamma_I)^\beta_\alpha \delta x^\mu]$, where $\bar{\gamma}_I = \gamma_I = \gamma_0 \gamma_I^\dagger \gamma_0$ (a factor of i must be included with γ_5 if $\gamma_5^\dagger = \gamma_5$), we find

$$\delta e_I = \bar{s}(\delta\bar{\Omega}^J \gamma_J \gamma_I + \gamma_I \gamma_J \delta\Omega^J) s . \quad (54)$$

And setting $\delta\bar{\Omega} = \delta\alpha + i\delta\beta$ allows us to write

$$\delta e_I = \bar{s}\cdot\{\gamma_J, \gamma_I\}\cdot s\, \delta\alpha^J + i\bar{s}\cdot[\gamma_J, \gamma_I]\cdot s\delta\beta^J . \quad (55)$$

These can be found from the following list of nonvanishing relations. Take

$$\gamma_I = \left\{ I, \gamma_i, \frac{i}{8}[\gamma_k, \gamma_l] = \sigma_{kl}, \gamma_5\gamma_j, i\gamma_5 \right\},$$
$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 , \quad (56)$$
$$\gamma_5^2 = 1 .$$

The anticommutators are

$$\{I, \gamma_J\} = 2\gamma_J ,$$
$$\{\gamma_i, \gamma_j\} = 2\eta_{ij} I ,$$
$$\{\gamma_i, \sigma_{kl}\} = \tfrac{1}{2}\epsilon_{iklj}\gamma_5\gamma^j ,$$
$$\{\gamma_i, \gamma_5\gamma_j\} = 8(i\gamma_5)\sigma_{ij} ,$$
$$\{\sigma_{ij}, \sigma_{kl}\} = \tfrac{1}{8}[(\eta_{jl}\eta_{ik} - \eta_{jk}\eta_{il})I + i\epsilon_{ijkl}\gamma_5] , \quad (57)$$
$$\{\sigma_{ij}, \gamma_5\gamma_k\} = \tfrac{1}{2}\epsilon_{ijkl}\gamma^l ,$$
$$\{\sigma_{ij}, i\gamma_5\} = 2(i\gamma_5)\sigma_{ij} ,$$
$$\{\gamma_5\gamma_i, \gamma_5\gamma_j\} = -2\eta_{ij} I .$$

Note $(i\gamma_5)\sigma_{ij} = \frac{1}{2}\epsilon_{ij}{}^{kl}\sigma_{kl}$. The commutators are

$$[\gamma_i, \gamma_j] = -8i\sigma_{ij} ,$$
$$[\gamma_i, \sigma_{kl}] = \tfrac{1}{2}i\eta^m_{\underline{l}}\eta_{i\underline{k}}\gamma_m ,$$
$$[\gamma_i, \gamma_5\gamma_j] = 2i\eta_{ij}(i\gamma_5) ,$$
$$[\gamma_i, (i\gamma_5)] = -2i\gamma_5\gamma_i ,$$
$$[\sigma_{ij}, \sigma_{kl}] = \tfrac{1}{4}i\eta_{\bar{i}\underline{k}}\eta^m_{\underline{l}}\eta^n_{\bar{j}}\sigma_{mn} , \quad (58)$$
$$[\sigma_{ij}, \gamma_5\gamma_k] = \tfrac{1}{2}i\eta^m_{\underline{i}}\eta_{k\underline{j}}\gamma_5\gamma_m ,$$
$$[\gamma_5\gamma_j, \gamma_5\gamma_k] = 8i\sigma_{jk} ,$$
$$[\gamma_5\gamma_j, (i\gamma_5)] = -2i\gamma_j .$$

Overbarred indices are antisymmetrized together as are underlined indices.

We can write $\{\gamma_J, \gamma_I\} = d^K_{JI}\gamma_K$ and $[\gamma_J, \gamma_I] = -if^K_{JI}\gamma_K$ as shorthand. Then

$$\delta e_I = e_K(\delta\alpha^J d^K_{JI} + \delta\beta^J f^K_{JI}) . \quad (59)$$

If one defines $g_{IK} = e_I \cdot e_K$, then $\delta g_{IK} = 0$ if

$$\delta\alpha^J(d_{JIK} + d_{JKI}) + \delta\beta^J(f_{JIK} + f_{JKI}) = 0 . \quad (60)$$

Thus only those $\delta\alpha^J$ associated with vanishing $d_{JIK} + d_{JKI}$ can survive by antisymmetry. We remark that we have typically used $\gamma_I \circ \gamma_K = k\,\mathrm{tr}(\gamma_I\gamma_K)$ for the metric in evaluating the action $\langle P|P\rangle$. $d_{JIK} + d_{JKI}$ can vanish.

We can see why gravitation cannot be included in a pure SO(3,1) or SL(2,C) theory and develop a model heaving some of the features of the Dirac algebra described above. Here we set $e_\mu = \bar{s}_a(\sigma_\mu)^a_b s^b$ where $\sigma_\mu = (I, \vec{\sigma})$ with $\vec{\sigma}$ the Pauli matrices. We introduce the inner product, $\circ$, between $X = x^\mu e_\mu$ and $Y = y^\mu e_\mu$ as follows,

$$X \circ Y = \tfrac{1}{2}(\mathrm{tr}X\,\mathrm{tr}Y - \mathrm{tr}XY) , \quad (61)$$

$$X \circ Y = x_0 y_0 - \vec{x}\cdot\vec{y} , \quad (62)$$

and

$$X \circ X = x_0^2 - \vec{x}^2 = \det X . \quad (63)$$

The product $X \circ X$ is invariant by GL(2, C) under adjoint action $X \to X' = GXG^{-1}$ with $G \in$ GL(2, C). This can be seen not just from $X \circ X = \det X$ but also from the characteristic polynomial of X, $\det(\lambda I - X) = \lambda^2 - \lambda\,\mathrm{tr}X + \frac{1}{2}[(\mathrm{tr}X)^2 - \mathrm{tr}(X^2)]$. Clearly

$$\det(\lambda I - X') = \det[G(\lambda I - X)G^{-1}]$$
$$= \det GG^{-1}\det(\lambda I - X)$$
$$= \det(\lambda I - X) . \quad (64)$$

Of course under Hermitian conjugate action $\bar{U}'X'U' = \bar{U}XU$ with $U' = LU$ we find $X' = LX\bar{L}$ and $\det(\lambda I - LX\bar{L}) = |\det L|^2\det(\lambda - X)$. Invariance sets $|\det L|^2 = 1$. Then $L \in$ U(1)$\otimes$SL(2, C). SL(2, C) gives rise to Lorentz transformations on the basis e_μ. However, we will be interested in the full GL(2, C) group. The transformations are given as

$$\delta s^a = [(\theta^\mu_\lambda + i\alpha^\mu_\lambda)\delta x^\lambda(\sigma_\mu)^a_b]s^b ,$$
$$\delta\bar{s}_a = \bar{s}_b[(\sigma_\mu)^a_b(\theta^\mu_\lambda + i\alpha^\mu_\lambda)\delta x^\lambda] , \quad (65)$$
$$\delta\theta^\mu = \theta^\mu_\lambda\delta x^\lambda ,$$
$$\delta\alpha^\mu = \alpha^\mu_\lambda\delta x^\lambda ,$$

and since $e_\mu = \bar{s}_a(\sigma_\mu)^a_b s^b$,

$$\delta e_\mu = e_\nu\left[\delta\theta^0 I^\nu_\mu + \begin{pmatrix} 0 & \delta\vec{\alpha} \\ \delta\vec{\theta} & (\delta\vec{\alpha})\times \end{pmatrix}^\nu_\mu\right],$$
$$\delta\theta^\mu = (\delta\theta^0, \delta\vec{\theta}) , \quad (66)$$
$$\delta\alpha^\mu = (\delta\alpha^0, \delta\vec{\alpha}) .$$

Note $\delta\alpha^0$ does not affect the e_μ's. $(\delta\vec{\alpha})\times\vec{e}$ denotes the cross product. Now in order to make connection with our earlier work we set $e_\nu = y_\nu{}^i e_i$ and set

$$\delta\omega^\nu_\mu \equiv \begin{pmatrix} 0 & \delta\vec{\theta} \\ \delta\vec{\theta} & (\delta\vec{\alpha})\times \end{pmatrix}^\nu_\mu . \quad (67)$$

then

$$\delta e_i = e_j[(\delta Y^\mu_i)Y^j_\mu + \delta\theta^0 I^j_i + Y^\mu_i\delta\omega^\nu_\mu Y^j_\nu] \equiv e_j\omega^j_i . \quad (68)$$

We want to find e_i's such that $\delta(e_i\cdot e_j) = \delta(\eta_{ij}) = 0$. This implies $\omega_{ij} + \omega_{ji} = 0$. Note that $\omega^j_i = \omega_{ik}\eta^{kj}$ can have slightly different symmetry for η^{kj} Minkowski. We see that $Y_i^\mu \delta\omega_\mu^\nu Y_\nu^j \eta_{jk} \to Y_i^\mu \delta\omega_{\mu\nu} Y_k^\nu$ is antisymmetric on the indices ik. Thus $(\delta Y_i^\mu) Y_\mu^j + \theta^0 I_i^j = 0$ for $i=j$. Further $(\delta Y_i^\mu) Y_\mu^j \eta_{jk} = Y_{i|\nu}^\mu Y_\mu^j \eta_{jk} \delta x^\nu$ must be skew-symmetric on the indices ik for $i \neq k$. If it happens that $\delta\theta^0 = \theta_\nu^0 \delta x^\nu$ can be expressed using $\theta_\nu^0 = \lambda_{|\nu}$ for some λ, then $Y_i^\mu = e^{-\lambda} I_i^\mu$ and

$$\delta e_i = e_j \begin{pmatrix} 0 & \delta\vec{\theta} \\ \delta\vec{\theta} & (\delta\vec{\alpha})\times \end{pmatrix}_i^j \qquad (69)$$

with $e_j = e^{-\lambda} I_i^\mu e_\mu$. Then

$$e_i \cdot e_j = \eta_{ij} = e^{-2\lambda} e_\mu e_\nu I_i^\mu I_j^\nu ,$$

$$g_{\mu\nu} = e^{2\lambda} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{\mu\nu} \qquad (70)$$

When $\lambda = 0$, $\theta_\nu^0 = 0$, and

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{\mu\nu}$$

can be chosen globally. In this case we restrict L to $U(1) \otimes SL(2, \mathbb{C})$. No gravitation is included in such a globally flat theory and therefore there must be more than four currents in a theory with gravity, unless extended gauge transformations are allowed (cf. Sec. III). Including two-component spinors, ϕ^a, with $\lambda \neq 0$ we find

$$\begin{aligned} \delta\bar{s}_a &= \bar{s}_b (\theta_\rho^\mu + i\alpha_\rho^\mu) \delta x^\rho \sigma_{\mu a}^b , \quad \theta_\nu^0 = \lambda_{|\nu} \\ \delta\bar{s} &= \bar{s}_a \phi^a . \end{aligned} \qquad (71)$$

The curvature is found using $\vec{\sigma} \times \vec{\sigma} = \frac{1}{2}[\vec{\sigma}, \vec{\sigma}] = i\vec{\sigma}$,

$$\tfrac{1}{2}[d,\delta]\bar{s}_a = \bar{s}_b \left(\tfrac{1}{2}\{ i\alpha_{\underline{\rho}|\underline{\sigma}}^0 I_a^b + [(\vec{\theta}_{\underline{\rho}|\underline{\sigma}} + 2i\vec{\theta}_\sigma \times \vec{\theta}_\rho) + i(\vec{\alpha}_{\underline{\rho}|\underline{\sigma}} - 2\vec{\alpha}_\sigma \times \vec{\alpha}_\rho)] \cdot \vec{\sigma}_a^b \} dx^\sigma \wedge dx^\rho \right), \qquad (72)$$

$$\tfrac{1}{2}[d,\delta]\bar{s} = \bar{s}_a [\phi_{|\nu}^a + (\lambda_{|\nu} + i\alpha_\nu^0)\phi^a + (\vec{\theta}_\nu + i\vec{\alpha}_\nu) \cdot \vec{\sigma}_b^a \phi^b] dx^\nu . \qquad (73)$$

The action is taken to be

$$\begin{aligned} \mathcal{a} = \int \{ & -\tfrac{1}{4}(\alpha_{\underline{\rho}|\underline{\sigma}}^0 \alpha^{0\overline{\rho}|\overline{\sigma}} + \vec{\bar{\theta}}_{\rho\sigma} \cdot \vec{\theta}^{\rho\sigma} + \vec{\bar{A}}_{\rho\sigma} \cdot \vec{A}^{\rho\sigma}) e^{-2\lambda} \\ & + \tfrac{1}{2}\bar{\phi}[\bar{\partial}_\nu + (\lambda_{|\nu} - i\alpha_\nu^0) + (\vec{\theta}_\nu - i\vec{\alpha}_\nu) \cdot \sigma][\partial^\nu + (\lambda^{|\nu} + i\alpha^{0\nu}) + (\vec{\theta}^\nu + i\vec{\alpha}^\nu) \cdot \sigma]\phi \} e^{2\lambda} d^4x , \end{aligned} \qquad (74)$$

with

$$\begin{aligned} &\overline{\phi\partial_\nu} \equiv \partial_\nu \bar{\phi} , \\ &\vec{\theta}_{\rho\sigma} = (\vec{\theta}_{\underline{\rho}|\underline{\sigma}} + 2i\vec{\theta}_\sigma \times \vec{\theta}_\rho) , \\ &\text{and} \\ &\vec{A}_{\rho\sigma} = (\vec{\alpha}_{\underline{\rho}|\underline{\sigma}} - 2\vec{\alpha}_\sigma \times \vec{\alpha}_\rho) . \end{aligned} \qquad (75)$$

Setting λ and $\theta_\rho = 0$ [$\lambda = 0$ amounts to restricting the group to $U(1) \times SL(2, \mathbb{C})$, $\theta_\rho = 0$ amounts to restricting the group to $U(1) \times SU(2)$], then

$$\mathcal{a} = \int \{ -\tfrac{1}{4}[\alpha_{\underline{\rho}|\underline{\sigma}}^0 \alpha^{0\overline{\rho}|\overline{\sigma}} + (\vec{\alpha}_{\underline{\rho}|\underline{\sigma}} - 2\vec{\alpha}_\sigma \times \vec{\alpha}_\rho) \cdot (\alpha^{\overline{\rho}|\overline{\sigma}} - 2\alpha^\sigma \times \alpha^\rho)] + \tfrac{1}{2}\bar{\phi}(+i\bar{\partial}_\nu + \alpha_\nu^0 + \vec{\alpha}_\nu \cdot \vec{\sigma})(-i\partial_\nu + \alpha_\nu^0 + \vec{\alpha}_\nu \cdot \vec{\sigma})\phi \} d^4x . \qquad (76)$$

When functional integrals are constructed carefully, the Minkowski metric is replaced by a Euclidean one. The covering group of SO(4) is $SU(2) \times SU(2)$. The two-dimensional representation is no longer faithful. Spontaneous breakdown is possible by including another term, $-\lambda(|\phi|^2 - m^2)^2$, in the manner described in Sec. II of paper II.[1] Remember that ϕ is still a fermion potential. This model can be related to SL(2|1) in the fashion described in Sec. III of paper II.[1]

It is amusing to consider the possibility of constructing vector-current bases from other groups than $U(1) \times SL(2, \mathbb{C})$ or $U(1) \times SL(4, \mathbb{C})$ which correspond to two- or four-component spinors. For example one might begin from a three-component object intermediate between the other two objects. Then we start by taking the generating matrices $\Lambda_{\bar{A}} = (I, \Lambda_A)$ with Λ_A the usual Gell-Mann Λ matrices. Introduce a set of complex gauge fields, $\bar{\Omega}_\mu^A = \theta_\mu^A + i\alpha_\mu^A$, with the proviso that $\theta_\mu^0 = \lambda_{|\mu}$ or zero as for the two-component case. The transformations of the basis frames is given as

$$\begin{aligned} \delta\bar{s}_a &= \bar{s}_b(\theta_\mu^{\tilde{A}} + i\alpha_\mu^{\tilde{A}})\delta x^\mu(\Lambda_{\tilde{A}})^b_a\,, \\ \delta s^a &= [(\Lambda_{\tilde{A}})^a_b(\theta_\mu^{\tilde{A}} - i\alpha_\mu^{\tilde{A}})\delta x^\mu]s^b\,, \\ \tilde{A} &= (0;1\cdots 8) = (0;A)\,, \\ \Lambda_{\tilde{A}} &= (I;\Lambda_A)\,. \end{aligned} \tag{77}$$

The current bases transform as

$$\begin{aligned} \delta(e_{\tilde{A}}) &= \delta(\bar{s}\Lambda_{\tilde{A}}s) \\ &= [(\theta_\mu^{\tilde{B}} d^{\tilde{C}}_{\tilde{B}\tilde{A}} + \alpha_\mu^{\tilde{B}} f^{\tilde{C}}_{\tilde{B}\tilde{A}})\delta x^\mu]e_{\tilde{C}}\,, \end{aligned} \tag{78}$$

where

$$\Lambda_{\tilde{A}}\Lambda_{\tilde{B}} \equiv d^{\tilde{C}}_{\tilde{A}\tilde{B}}\Lambda_{\tilde{C}} - if^{\tilde{C}}_{\tilde{A}\tilde{B}}\Lambda_{\tilde{C}} \tag{79}$$

for $\Lambda_{\tilde{A}} = (I, \Lambda_A)$ and $d^{\tilde{C}}_{\tilde{A}\tilde{B}}$ and $f^{\tilde{C}}_{\tilde{A}\tilde{B}}$ the symmetric and antisymmetric structure constants for U(3). For definiteness take $e_\mu = \bar{s}(I, \vec{\Lambda})_\mu s$, where $\Lambda = (\Lambda_1, \Lambda_4, \Lambda_6)$ with the SL(2, $\mathbb{C}$) subgroup is defined by taking these generators with complex coefficients. The action, after including matter fields $\delta\bar{s} = \bar{s}_a\Psi^a$, can be written as the Hilbert square of the curvature

$$\mathfrak{A} = \int [-\tfrac{1}{4}(\bar{F}^{\tilde{A}}_{\mu\nu}F^{\mu\nu}_{\tilde{A}}) + \tfrac{1}{2}\overline{\Psi P^\mu}P_\mu\Psi]\sqrt{-g}\; d^4x\,. \tag{80}$$

Here $g_{\mu\nu} = e_\mu\cdot e_\nu$ can have constant determinant,

$$\begin{aligned} F^{\tilde{A}}_{\mu\nu} &= (\theta^{\tilde{A}}_{\underline{\mu}|\underline{\nu}} + i\alpha^{\tilde{A}}_{\underline{\mu}|\underline{\nu}}) + f^{\tilde{A}}_{\tilde{B}\tilde{C}}(\theta^{\tilde{B}}_\nu + i\alpha^{\tilde{B}}_\nu)(\theta^{\tilde{C}}_\mu + i\alpha^{\tilde{C}}_\mu) \\ &= \text{complex conjugate } (\bar{F}^{\tilde{A}}_{\mu\nu})\,, \end{aligned} \tag{81}$$

$$P_\mu\Psi \equiv -i\partial_\mu\Psi^a + (\theta^{\tilde{A}}_\mu + i\alpha^{\tilde{A}}_\mu)\Lambda^a_{\tilde{A}b}\Psi^b \quad (\theta^0_\mu = 0)\,. \tag{82}$$

VI. SUPERGRAVITY FROM THE VIEWPOINT OF MOVING FRAMES

In this section we will see how to obtain the spin-2–spin-$\frac{3}{2}$ supergravity Lagrangian using frames.[17,18] We then extend this theory to include gauge fields and spin-$\frac{1}{2}$ particles. Consider the following connection for spinors:

$$\delta[\bar{S}_\alpha, \bar{S}_{3/2}] = [\bar{S}_\beta, \bar{S}_{3/2}]\begin{pmatrix} -i\Gamma^i_{\mu j}(\sigma^j_i)^\beta_\alpha\delta x^\mu & \chi^\beta_\mu\delta x^\mu \\ 0 & 0 \end{pmatrix}, \quad \sigma^{ij} = \tfrac{1}{8}i[\gamma_i, \gamma_j]\,. \tag{83}$$

The connection is not in any of the simple superalgebras. It is in an inhomogeneous subalgebra of OSp(1|4). The curvature is readily computed by considering the difference in two displacements. Y^i_μ is used to switch indices, $\theta^i = Y^i_\mu dx^\mu$,

$$\begin{aligned} [\bar{S}, \bar{S}_{3/2}][P] &= \tfrac{1}{2}[d, \delta][\bar{S}, \bar{S}_{3/2}] \\ &= [\bar{S}, \bar{S}_{3/2}]\begin{bmatrix} \tfrac{1}{2}(R^i_{jkl}\theta^k\wedge\theta^l)(-i\sigma^j_i)^\beta_\alpha & \tfrac{1}{2}[\chi^\beta_{\underline{m}|\underline{k}} + \Sigma^n_{\underline{k}\,\underline{m}}\chi^\beta_n - i\Gamma^n_{\underline{k}l}(\sigma^l_n)^\beta_\delta\chi^\delta_{\underline{m}}]\,\theta^k\wedge\theta^m \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{84}$$

Since

$$d(\chi^\alpha_\mu\theta^\mu\bar{S}_k) = (d\chi^\alpha_m\theta^m + \chi^\alpha_m d\theta^m)\bar{S}_\alpha + \chi^\alpha_\mu\theta^\mu d\bar{S}_\alpha \tag{85}$$

and

$$d\theta^m = (T^m_{jl} - \Gamma^m_{jl})\theta^j\wedge\theta^l \equiv \Sigma^m_{jl}\theta^j\wedge\theta^l\,. \tag{86}$$

In a coordinate base $\Sigma^\rho_{\mu\nu}$ vanishes. We now introduce a K structure as we did for Einstein's action but include an index saturating term for χ^α_m which is a two-form

$$K = \begin{pmatrix} -\tfrac{1}{4}\kappa\gamma_i\gamma_j\theta^i\wedge\theta^j & -\tfrac{1}{8}i\{\gamma^k, \gamma_i\gamma_j\}\chi\theta^i\wedge\theta^j \\ 0 & 0 \end{pmatrix}. \tag{87}$$

The action is the Hilbert product $\langle K|P\rangle$,

$$\mathfrak{A} = \int \{\kappa R_S + \tfrac{1}{2}\epsilon^{klnm}\bar{\psi}_k\gamma_5\gamma_l[\partial_m\delta^i_n + \Sigma^i_{mn} + \Gamma_{mp}(-i\sigma^p_j)\delta^i_n]\chi_i\}\sqrt{-g}\, d^4x\,. \tag{88}$$

Here $\gamma_5 = (i/4!)\epsilon^{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$ in the μ basis $\gamma^{(\mu)}_5 = (\det Y^i_\mu)\gamma^{(k)}_5$, $(|\det g|)^{1/2} = \det Y$. In a coordinate basis the spin-$\frac{3}{2}$ term becomes

$$\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\chi}_\mu\gamma_5\gamma_\nu[\partial_\lambda + \Gamma^i_{\lambda j}(-i\sigma^j_i)]\chi_\rho\,. \tag{89}$$

If one required the connection to be in the simple superalgebra OSp(1|4) additional terms would arise; four of these can be associated with torsion (as described in Sec. III of paper II).[1] Or at the least a cosmological constant arises from those generators.[18] Note that if $\chi_k = \gamma_k\chi$ the action reduces to a standard form for the spin-$\frac{1}{2}$ field, χ. This is not true for the equation of motion.

For the case of both spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields on a curved manifold the augmented connection can be given as

$$\delta[\overline{S}_\alpha, \overline{S}_{3/2}, \overline{S}_{1/2}] = [\overline{S}_\beta, \overline{S}_{3/2}, \overline{S}_{1/2}] \begin{bmatrix} -i\Gamma^i_{\mu j}(\sigma^j_i)^\beta_\alpha \delta x^\mu & \chi^\beta_\mu \delta x^\mu & \psi^\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{90}$$

The curvature as calculated from $\frac{1}{2}[d, \delta]$ has the following matrix:

$$P = \begin{bmatrix} \frac{1}{2} R^i_{jkm}\theta^k \wedge \theta^m(-i\sigma^j_i)^\beta_\alpha & \frac{1}{2}[\chi^\beta_{\underline{m}|\underline{k}} + \Sigma^n_{\underline{k}\,\underline{m}}\chi^\beta_n + \Gamma^n_{\underline{kl}}(-i\sigma^l_n)^\alpha_\beta \chi^\beta_{\underline{m}}]\,\theta^k \wedge \theta^m & [\psi^\beta_{|k} - i\Gamma^n_{kl}(\sigma^l_n)^\beta_\alpha \psi^\alpha]\theta^k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{91}$$

The K structure (from index saturation) is given by

$$K = \begin{bmatrix} -\frac{1}{4}\kappa\gamma_i\gamma_j\theta^i \wedge \theta^j & -\frac{1}{8}i\{\gamma^k, \gamma_i\gamma_j\}\chi_k\,\theta^i \wedge \theta^j & -i\gamma_l\psi\theta^l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{92}$$

The action is $\mathfrak{a} = \langle K|P\rangle$ (appropriately symmetrized),

$$\mathfrak{a} = \int \{\kappa R_S + \tfrac{1}{2}\epsilon^{klmn}\overline{\chi}_k\gamma_5\gamma_l[\partial_m\delta^i_n + \Sigma^i_{mn} + \Gamma^i_{mp}(-i\sigma^p_j)\delta^i_n]\chi_i + \overline{\psi}(i\tfrac{1}{2}\overleftrightarrow{\partial\!\!\!/} + \tfrac{1}{4}\Gamma^{*m}\gamma_5\gamma m)\psi\}\sqrt{-g}\,d^4x\,, \tag{93}$$

where $\Gamma^*_m \equiv \Gamma_i{}^j{}_k\epsilon^i{}_j{}^k{}_m$ and $\theta^i = Y^i_\mu dx^\mu$ and $g_{\mu\nu} = Y^i_\mu \eta_{ij} Y^j_\nu$. If the connection is not in a simple superalgebra, then classically there is nothing to stop one from considering a colored supergravity Weinberg-Salam-type model having gauge group $SU_{N_L} \times U_1 \times SU_{M_C}$.[19,20] See also Sec. II of paper II.[1] It is amusing to write out the action

$$\begin{aligned} \mathfrak{a} = \int \Big[&\kappa R_S - \tfrac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A - \tfrac{1}{4}F^I_{\mu\nu}F^{\mu\nu}_I - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}^L[\tfrac{1}{2}i\overleftrightarrow{\partial\!\!\!/} + \tfrac{1}{4}\Gamma^{*m}\gamma_5\gamma_m + (B^A_j\Lambda^A + C^I_j\Lambda_I + \tfrac{1}{2}A_j)\gamma^j]\psi^L \\ &+ \overline{\psi}^R[\tfrac{1}{2}i\overleftrightarrow{\partial\!\!\!/} + \tfrac{1}{4}\Gamma^{*m}\gamma_5\gamma_m + (C^I_j\Lambda_I + A_j)\gamma^j]\psi^R \\ &+ \tfrac{1}{2}\epsilon^{klmn}(\overline{\chi}^L_k\gamma_5\gamma_l\{[\partial_m + \Gamma^j_{mp}(-i\sigma^p_j) - i(B^A_m\Lambda_A + C^I_m\Lambda_I + \tfrac{1}{2}A_m)]\,\delta^i_n + \Sigma^i_{mn}\}\,\chi^L_i) \\ &+ \tfrac{1}{2}\epsilon^{klmn}(\overline{\chi}^R_k\gamma_5\gamma_l\{[\partial_m + \Gamma^j_{mp}(-i\sigma^p_j) - i(C^I_m\Lambda_I + A_m)\delta^i_n + \Sigma^i_{mn}]\,\chi^R_i\}) \\ &+ \tfrac{1}{2}[\overline{\phi}_{|\mu} - i\overline{\phi}(B^A_\mu\Lambda_A + \tfrac{1}{2}A_\mu)][\phi^{|\mu} + i(B^{\mu A}\Lambda_A + \tfrac{1}{2}A^\mu)\phi] + \lambda(\|\phi\|^2 - m^2)^2 \\ &+ \delta(\overline{\psi}^L\phi\psi^R + \overline{\psi}^R\overline{\phi}\psi^L) + \beta(\overline{\chi}^L\phi\chi^R + \overline{\chi}^R\phi\chi^L)\Big]\sqrt{-g}\,d^4x\,. \end{aligned} \tag{94}$$

Λ_A forms a representation for SU_N, Λ_I forms a representation for SU_M. The fields ϕ are color independent,

$$\chi^R = \tfrac{1}{2}(I + \gamma_5)\chi, \quad \chi^L = \tfrac{1}{2}(I - \gamma_5)\chi, \quad \text{etc.} \tag{95}$$

VII. SUMMARY

The purpose of this paper has been to provide more detail on the mathematical and theoretical boundaries of the theory of moving frames and to suggest possible avenues of future development. Motivated by Faddeev's treatment of gauge degeneracies,[3] we have shown the relationship between the theory of frames[1,8] and the theory of fiber bundles,[4,5] since it is in a fiber bundle that the gauge degeneracies are naturally removed. Some of the details of this relationship are given in detail for the fiber group SO(3) in Sec. III. To provide the setting for this discussion (as well as that for a number of the detailed calculations in papers I and II) we have tabulated a number of the natural structures on a four-dimensional manifold such as p forms, duality, exterior differentiation, etc.

By formulating the usual physical assumptions in a more general framework, we have shown how they can be extended to produce a finer control over the local structure. Numerous detailed examples were given: First, a space whose local "rotations" could be elements of a manifold which was not necessarily a group (generalizing the Lorentz structure) was considered. Second, a space was given whose local coordinates could be flatter than general relativity's[10,11] (jet bundles generalize the usual tangency assumptions of relativity).[12] Third, a space with a vertical structure which was more general than a Lie algebra yet "almost" one (up to a parameter) was considered (by violating the Jacobi relation, nearly Lie structures can be created which, if the violation is soft enough, may produce useful generalizations of Lie algebras).[13–15] Fourth, theories for graded[13,14] Lie algebras which have some of the features described in paper II without the full complexities of general relativity, are described (these models may provide useful insight into the more complicated structures). Fifth, an easy derivation of the supergravity action was presented (showing its relation to the gauging of a local superalgebra[17,18]) and from that a generalization was constructed.[19,20]

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