Fermions and bosons in a unified framework. II. Interacting models

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(Received 13 December 1977)

In this paper we construct some fully interacting field theories. The first model has a colored, curved Weinberg-Salam-type action. It is formed by taking the Hilbert product of the (generalized) curvature of a (given) superalgebra with an auxiliary (generalized) curvature. Note that pieces of simple superalgebras are gauged; the effective superalgebra of gauge fields is not simple. The auxiliary curvature was needed to obtain the linear pieces of the action, and it thus appears to be somewhat *ad hoc*. In contrast we show how to construct an action using only the curvature of a local superalgebra without the auxiliary curvature (it is therefore quadratic). Nonetheless, linear terms arise as crossterms between pieces of the curvature. In fact, since we have chosen to use a special-unitary flavor algebra and four-component spinors, we discover we have already specified a unique simple supergroup whose other Bose gauge fields are in U(2,2), the Lie algebra formed by all the Dirac matrices. These fields gauge the spin structure of the fermions. Color causes certain complications discussed in the paper. The tensor piece of the U(2,2) curvature consists of the usual curvature plus a term identifiable as the old auxiliary tensor. Thus both linear and quadratic terms for the space-time curvature arise when the full curvature is squared. The field associated with the identity generator is electromagnetism; with the vector, torsion; with the tensor, curvature and auxiliary terms. We call the fields associated with the axial generators axial torsion and axial electromagnetism. When the fields which couple to Dirac spinors are assumed proportional to their scalar counterparts, an experimental value for a conserved axial electromagnetic coupling is $10^{-3}e$. We present a qualitative argument for the renormalizability of this action, since it is almost that of a standard Yang-Mills gauge theory, based on preservation of recoordinatization invariance by the quantization procedure.

I. INTRODUCTION

Having described the foundations of the theory of moving frames in the first paper,[1] we proceed to apply these ideas to the construction of physically plausible models. In Sec. II we show how to construct a model which is essentially the Weinberg-Salam model[2] generalized to include color and curvature. We show that this model can be viewed as gauging pieces of a local superalgebra.[3] It can be derived directly from a nonsimple superalgebra; however, we do not prove this (cf. Sec. VI of paper III). For a discussion of superalgebras see Sec. I of paper I.[1,3]

There are two ways of unifying color in these models, since there are two Bose subsectors to the superalgebra. It can be joined with the spin covariant derivative or with the flavor covariant derivative.[4,5] In view of the early relationship between spin and color,[4] we give the details for that combination. The details of the other approach are easily worked out.

In order to achieve an action with the standard linear kinetic terms for gravitation and fermions but with quadratic terms for the gauge and scalar fields, we are obliged to introduce a two-form which lacks the immediate geometric interpretation that the curvature has.[6] In view of the recent work of 't Hooft, Deser, and others[7] in which terms quadratic in the curvature were shown necessary at the one-loop level and out of a desire to have a single prescription (with geometric interpretation) for the action, we suggest in Sec. III that an action which is the Hilbert square of the curvature[1,8,9] be considered. We discover that such an action has terms like those found in the renormalizability studies.[7]

Of course the Fermi contribution is quadratic also. Several years ago Feynman, Gell-Mann, and others had considered similar quadratic actions.[10] They point out that then one obtains Gaussian functionals in the Fermi sector.[10] But these Fermi fields cannot be the usual Dirac fields. They are related to the Dirac fields as the electromagnetic potential is related to the field tensor. We therefore call them Fermi potentials. It was through use of the equations for Fermi potentials that Feynman and Gell-Mann discovered $V-A$.[10]

We briefly discuss the short-distance (ultraviolet) behavior of this quadratic action. And we suggest that, because of the requirement that recoordinatization invariance be preserved in the quantum theory, the short-distance behavior of the theory should be no worse than that of the quadratic gauge theory which it is near. That theory is renormalizable. Thus it may be hoped that this theory is.[7]

In closing the superalgebra structure[3] we find not only the generators of the Poincaré (or de Sitter) group but actually the entire set of generators for the conformal covering group SU(2,2) together with a U(1) (and some internal symmetries). Setting U(1)

$\times SU(2,2) \approx U(2,2)$ we can write the 16 generators as

$$I, \quad P_i = \tfrac{1}{2}(I+\gamma_5)\gamma_i, \quad M_{ij} = \sigma_{ij}, \quad K_i = \tfrac{1}{2}(I-\gamma_5)\gamma_i, \quad D = i\gamma_5 \tag{1}$$

or as

$$I, \quad \gamma_i, \quad \sigma_{ij}, \quad \gamma_5\gamma_i, \quad i\gamma_5 . \tag{2}$$

By assuming that the gauge fields associated with the axial-vector and axial-scalar generators (e.g.) are proportional to the vector and to the scalar generators (respectively), we are led to a situation considered by Wolfenstein and Herczeg.[11] They indicate that (experimentally) the constant of proportionality for the ratio of axial-scalar to scalar (axial-electromagnetism to electromagnetism) couplings could be as large as 10^{-3} (this is for the fields coupling to Dirac spinors) when the currents are conserved. When spinors are massive the axial-current divergence is proportional to (naively) $m\bar{\psi}i\gamma_5\psi$.

We conclude Sec. III with the classical equations of motion for this quadratic action. And we write out the classical discrete invariances C, P, T, and $i\gamma_5$.[12]

II. COLORED CURVED INTERACTING FERMI-BOSE STRUCTURES

In this section we will construct a moving frame capable of describing fermions and Higgs-type bosons with color, curvature, and flavor interactions.[2] The kinetic terms for each piece are the usual. The interactions have minimal coupling.

Since we expect to make contact with a unified superalgebraic scheme having both fermions and bosons in the algebra, we will use a frame whose connection can be related to pieces of an underlying superalgebra connection. Since fermions in a superalgebra such as $SL(m|n)$ (Ref. 3) (see Sec. I for a description), or one of its subalgebras, couple to two Lie algebras $SL(m)$ and $SL(n)$, we must decide on an assignment of gravitation, color, and flavor interactions. Obviously one either lumps gravitation (spin covariant derivatives) and color together or flavor and color together. We have chosen the first approach for a number of reasons. First, color was initially related to parastatistics and spin (via spin and statistics). Thus it belongs with the spin covariant derivative.[4] Second, color is massless. Lumping all the massless fields into one of the Bose subsectors is economical, since only one mass scale (weak interactions) needs to be introduced. In other schemes, color is massless, diatons (gauge fields with both color and flavor) are super massive (or nonexistent) and weak fields are massive. A more complicated symmetry-breaking scheme is required to arrange the color-flavor breakups.[4,5,13] Of course, the construction of the standard color-flavor scheme instead of the color-spin scheme which we will describe can be made by simple modifications on our procedure.

The spinor basis transformation law is $\delta\bar{s} = i\bar{s}\cdot\Omega(\delta)$. The coefficients of the connection $\Omega(\delta)$ may be required to form an algebra which is typically either Lie or graded Lie (see Sec. IV of paper III). We will start with a connection having coefficients in a graded Lie algebra and, by making certain restrictions, obtain an action which is essentially the colored, curved, and flavored Weinberg-Salam model.[2]

When we first considered forming the vector basis e_i from spinor bases, $e_i = s_\alpha \gamma_{i\beta}^{\alpha} s^\beta$, we examined only the simplest case. Two generalizations are possible. Additional vertical or flavor bases $\bar{s}_a$ may be included such that $\bar{e}_A = \bar{s}_a \Lambda_{A\,b}^{a} s^b$. This can be called stretching the basis. Or multiplets of bases $\bar{s}_{\alpha t}$ can be introduced such that $e_i = \bar{s}_{\alpha t}\gamma_{i\beta}^{\alpha} s^{\beta t}$. This can be called thickening the bases. The bases $\bar{s}_{\alpha t}$ carry an extra label, t. We will interpret these labels as those of color. We remark that they are directly related to the spin labels α and that they correspond to using a larger Dirac algebra. In this view the vector bases are constructed from a triplet of four component complex spinor bases. To include leptons one might consider a quadruplet of flavored spinors.

Since the fact that there are four of these complex spinor bases is related to the four dimensionality of space-time, in this view of color the answer to the question why three colors is as mysterious as why four dimensions? The only hint of an answer currently available is that this triplet of eight objects ($s^{\alpha a}$ and $\bar{s}_{\alpha a}$) may somehow be related to a triplet of octonions.[14] But that relationship is highly speculative (if intriguing), and we will not discuss it. Color and dimensionality will be taken to have their usual values.

The bases $\bar{s}_{\alpha t}$ will be required to be an SU(3) triplet of Dirac spinors. The Dirac algebra as a Lie algebra is U(2,2). The piece of the connection which represents the Lorentz group action is the spin (3,1) subgroup of U(2,2). Spin (3,1) is the twofold cover of SO(3,1) just as SU(2,2) covers SO(4,2). The Euclidean Dirac algebra is U(4). We can consider (in Euclidean space) the unified group U(12). An arbitrary U(12) matrix M can be decomposed as follows:

$$M = AI\otimes I + B^J\Gamma_J\otimes I + C^A I\otimes\Lambda_A + D^{JA}\Gamma_J\otimes\Lambda_A , \tag{3}$$

where the left matrices are 4×4 and the right are 3×3. Γ_J is a basis for SU(4) and Λ_A is a basis for SU(3). The subalgebra we want is obtained by taking those matrices for which D^{JA} vanishes and such that $B^J\Gamma_J = B^{ij}(\frac{1}{8})[\gamma_i,\gamma_j] = B^{ij}\sigma_{ij}$. The spin (4) matrices γ_i satisfy $\{\gamma_i,\gamma_j\} = 2\delta_{ij}I$. The spin (3,1) matrices satisfy $\{\gamma_i,\gamma_j\} = 2\eta_{ij}I$.

We will want masses to be given to the weak gauge fields. A technique for producing general potentials for scalar fields from curvature calculations is not known. However, quartic interactions *can* be derived.[4] But first we suggest another approach to giving boson masses. The scalars are coupled to a complex, norm-preserving group [SU(n)]. Indeed SU(n) can be defined as the group of transformations of a complex n-vector ϕ^a which preserves its length, $\|\phi\|^2 = m^2$. In previously considered actions ϕ^a was allowed to have any norm. Now let ϕ^a have a fixed norm. ϕ^a can either be viewed as proportional to a unit vector ($\phi^a = mu^a$) having $2n-1$ real independent components. Or we can include a specific constraint term, e.g., $V(\|\phi\|) = \lambda(\|\phi\|^2 - m^2)$, in the action and view ϕ^a as having $2n$ real independent components. In either case ϕ^a can be used to break the symmetry and give rise to effective boson masses. Thus the same end as the usual Higgs potential achieves maybe available in this fashion.[2,24] We have not examined all the consequences of using such a constraint in a quantum field theory.[24] But it appears to lead to a renormalizable theory at least in Abelian theories.

The technique for including potential terms can be quickly summarized. Introduce a connection Ω on a (4,3) (Bose, Fermi) manifold with coordinates (x^μ, θ^i),

$$\Omega = \begin{pmatrix} iB^A(\Lambda_A) + iM'(I) & \Phi \\ \overline{\Phi} & -iM'' \end{pmatrix}, \tag{4}$$

with

$$\begin{aligned} B^A &= B^A_\mu(x)dx^\mu , \\ \Phi &= \phi(x)[\tfrac{1}{2}(d\!\!\!/\theta^1 + i d\!\!\!/\theta^2)] , \\ \overline{\Phi} &= \overline{\phi}(x)[\tfrac{1}{2}(d\!\!\!/\theta^1 - i d\!\!\!/\theta^2)] , \\ M' &= m' d\!\!\!/\theta^3 , \\ M'' &= m'' d\!\!\!/\theta^3 , \\ M &= (m' + m'')d\!\!\!/\theta^3 \equiv m d\!\!\!/\theta^3 , \\ m'' &= [\mathrm{tr}(I)]m' . \end{aligned} \tag{5}$$

Define $d\!\!\!/\theta^i = \theta^i d\theta^i$ if $d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i$ or $d\!\!\!/\theta^i = d\theta^i$ if $d\theta^i \wedge d\theta^j = -d\theta^j \wedge d\theta^i$. The curvature $P = d\Omega + \Omega \wedge \Omega$ is easily calculated. $dd\!\!\!/\theta$ terms are omitted since they will be projected out,

$$P = \begin{pmatrix} F + \Phi\wedge\overline{\Phi} & D\Phi + iM\wedge\Phi \\ \overline{D\Phi} - iM\wedge\overline{\Phi} & \overline{\Phi}\wedge\Phi \end{pmatrix} \tag{6}$$

with

$$\begin{aligned} F &= F^A(i\Lambda_A) = (dB^A + \tfrac{1}{2}\epsilon^A_{BC}B^B\wedge B^C)(i\Lambda_A) , \\ D\Phi &= d\Phi + iB\wedge\Phi . \end{aligned} \tag{7}$$

The dual must be extended to include the Grassmann differentials. Since we will compute $\overline{P}{}^*\wedge P$, we give the following products (we will use bars on indices to indicate those which are to be antisymmetrized, thus $A_{\bar\lambda\mu\bar\rho} \equiv A_{\lambda\mu\rho} - A_{\rho\mu\lambda}$)

$$\begin{aligned} &[*(dx^\nu\wedge d\!\!\!/\theta^i)]\wedge(dx^\mu\wedge d\!\!\!/\theta^j) \equiv (g^{ij}\eta^{\nu\mu})\theta^1 d\theta^1\theta^2 d\theta^2\theta^3 d\theta^3 d^4x , \\ &[*(d\!\!\!/\theta^i\wedge d\!\!\!/\theta^j)]\wedge(d\!\!\!/\theta^k\wedge d\!\!\!/\theta^l) \equiv (g^{i\bar k}g^{j\bar l})\theta^1 d\theta^1\theta^2 d\theta^2\theta^3 d\theta^3 d^4x , \\ &[*(dx^\nu\wedge dx^\mu)]\wedge(dx^\sigma\wedge dx^\lambda) \equiv (\eta^{\nu\bar\sigma}\eta^{\mu\bar\lambda})\theta^1 d\theta^1\theta^2 d\theta^2\theta^3 d\theta^3 d^4x , \\ &*(dd\!\!\!/\theta^i) \equiv 0 . \end{aligned} \tag{8}$$

The metric g^{ij} is taken to be diagonal (+1, +1, $-\lambda$). To produce the Higgs potential discussion for the motivation behind choosing such a duality operation is found in paper I[1] and Ref. 23. Remember that $\int d\theta^i = 0$ and $\int \theta^i d\theta^j = \delta^{ij}$. This dual is nonvanishing when integrated,

$$\overline{P}{}^* = \begin{bmatrix} \overline{F}{}^* + i\overline{((\Phi\wedge\overline{\Phi})^*)} & D\phi^* + i\overline{((M\wedge\overline{\Phi})^*)} \\ \overline{D\overline{\phi}{}^*} - i\overline{((M\wedge\Phi)^*)} & \overline{((\Phi\wedge\Phi)^*)} \end{bmatrix} \tag{9}$$

We introduce $\overline{(K)} \equiv \overline{K}$ for typographical reasons. Thus

$$\begin{aligned} \mathfrak{a} &= -\tfrac{1}{2}\int \mathrm{tr}(\overline{P}{}^*\wedge P) \\ &= \int\left(-\tfrac{1}{4}F^2 + \overline{D_\mu\phi}D^\mu\phi + m^2\overline{\phi}\phi - \frac{\lambda^2}{2}\overline{\phi}\phi\overline{\phi}\phi\right)d^4x . \end{aligned} \tag{10}$$

Grassmann integrals have been evaluated. $\mathrm{tr}(\Lambda_A\Lambda_B) = \delta_{AB}$ is our normalization.

We will consider a connection with components in a subsupergroup of GL($12|f_L+1$) (Refs. 1 and 3) which we will denote by D($12|f_L+1$). $f = f_L$ is the dimension of the left-handed flavor group which we will take to be 2. Define a matrix Γ with components $\gamma_0\otimes I_3$ in the upper left quadrant and $-I_{f+1}$ in the lower right quadrant (where the subscripts on the I matrices give their dimension). Then consider only those matrices in GL($12|f+1$) satisfying $\overline{M} \equiv \Gamma M^\dagger\Gamma = -M$. A dagger indicates Hermitian conjugation. Call these skew-Dirac. They can be written as follows:

$$\begin{array}{ccc|c} 12 & f & 1 & \\ iD & L & R & 12 \\ \overline{L} & iB & \phi & f \\ \overline{R} & -\phi^\dagger & iA & 1 \end{array} , \tag{11}$$

where $(\gamma_0\otimes I)D^\dagger(\gamma_0\otimes I) = D$, $\overline{\phi} \equiv \phi^\dagger$,

$$\begin{pmatrix} iB & \phi \\ -\phi^\dagger & iA \end{pmatrix}^\dagger = \begin{pmatrix} -iB & -\phi \\ \phi^\dagger & -iA \end{pmatrix},$$
$$\overline{L} = L^\dagger \gamma_0 \otimes I\,, \tag{12}$$
$$\overline{R} = R^\dagger \gamma_0 \otimes I\,.$$

L and R are fermionic; the others are bosonic. The L and R fermions will have only left and right components, respectively. The other components will be projected out in the action.

Note that for forms dx^μ and $\not{d}\theta^i$ can be made to satisfy a Grassmann algebra.[15] Introduce the differentials of four Bose and three real Fermi coordinates: dx^0, dx^1, dx^2, dx^3, $\not{d}\theta^1$, $\not{d}\theta^2$, $\not{d}\theta^3$. Let the connection have the following special form:

$$\Omega = \begin{pmatrix} -iD_\mu dx^\mu & L\not{d}\theta^1 & R\not{d}\theta^2 \\ \overline{L}\not{d}\theta^1 & -iB_\mu dx^\mu & \phi\not{d}\theta^3 \\ \overline{R}\not{d}\theta^2 & -\phi^\dagger \not{d}\theta^3 & -iA_\mu dx^\mu \end{pmatrix}. \tag{13}$$

A supertrace[3] condition holds, $\mathrm{tr}D_\mu - \mathrm{tr}B_\mu - \mathrm{tr}A_\mu = 0$. The supertrace, Tr, is the trace of the upper Bose sector minus the trace of the lower Bose sector. In components,

$$D_\mu \to \Gamma^i_{\mu j}\sigma^{j\alpha}_{i\beta}\otimes I^t_r - C^J_\mu I^\alpha_A \otimes \Lambda^t_{Jr}\,,$$
$$B_\mu \to \tfrac{1}{2} n A_\mu I^a_b\,, \quad -B^A_\mu \Lambda^a_{Ab}$$
$$\phi \to \phi^a\,, \tag{14}$$
$$L \to L^{\alpha t}_b\,,$$
$$R \to R^{\alpha t}$$

and the adjoints. All coefficients are functions of x only. By temporarily suppressing the differentials (write L for $L\not{d}\theta$, e.g.) we can express the graded curvature $P = d\Omega + \Omega\wedge\Omega$ as follows:

$$P = \begin{pmatrix} idD + i^2 D\wedge D & dL + iD\wedge L + iL\wedge B + R\wedge\overline{\phi} & dR + iD\wedge R + iR\wedge A + L\wedge\phi \\ d\overline{L} + i\overline{L}\wedge D + iB\wedge\overline{L} + \phi\wedge\overline{R} & idB + i^2 B\wedge B & d\phi + iB\wedge\phi + \overline{L}\wedge R + \phi\wedge iA \\ d\overline{R} + i\overline{R}\wedge D + iA\wedge\overline{R} + \overline{\phi}\wedge\overline{L} & d\overline{\phi} + i\overline{\phi}\wedge B + \overline{R}\wedge L + iA\wedge\overline{\phi} & idA \end{pmatrix}, \tag{15}$$

$d\not{d}\theta$ terms have been dropped since their dual vanishes.

As in earlier sections we will introduce another two-form K whose Hilbert product with P will yield the action. Despite the fact that we have not introduced the most general possible Ω on the seven-dimensional manifold, we could almost reduce the terms in the action to those indicated through judicious use of the arbitrariness of K. That is, take K to have appropriate projection operators in its various sectors. E.g., left and right projectors for the spinors. Even so, the restriction of the connection D_μ to include only a spin (3,1) piece from the U(2,2) group is not equivalent to the use of a projection operator; although it is a closed subalgebra, it is not simple (cf. Sec. VI of paper III). In the next section we will consider the consequences of using a less restricted connection. The components of P are given as

$$P^{\alpha t}_{\beta r} = [\, iR^i_{j\mu\nu}\sigma^{j\alpha}_{i\beta} I^t_r$$
$$+ i(C^J_{\mu|\nu} + \tfrac{1}{2} i\epsilon^J_{KL} C^K_\nu C^L_\mu) I^\alpha_\beta \Lambda^t_{Jr}] dx^\nu \wedge dx^\mu\,,$$
$$P^{\alpha t}_b = [L^{\alpha t}_{b|\nu} + (i\Gamma^i_{\nu j}\sigma^{j\alpha}_{i\beta} I^t_r - i\tfrac{1}{2} A_\nu I^\alpha_\beta I^t_r) L^{\beta r}_b$$
$$+ iB^A_\nu \Lambda^a_{Ab} L^{\alpha t}_a] dx^\nu \wedge \not{d}\theta^1 + [R^{\alpha t}\overline{\phi}_b]\not{d}\theta^2 \wedge \not{d}\theta^3\,,$$
$$P^{\alpha t} = [R^{\alpha t}_{|\nu} + (i\Gamma^i_{\nu j}\sigma^{j\alpha}_{i\beta} I^t_r - ieA_\nu I^\alpha_\beta I^t_r) R^{\beta r}] \tag{16}$$
$$\times dx^\nu \wedge \not{d}\theta^2 + [L^{\alpha t}_a \phi^a]\not{d}\theta^1 \wedge \not{d}\theta^3\,,$$
$$P^a_b = [\tfrac{1}{2} iA_{\mu|\nu} I^a_b + i(B^A_{\mu|\nu} + \tfrac{1}{2} i\epsilon^A_{BC} B^B_\nu B^C_\mu)\Lambda_A]\, dx^\nu \wedge dx^\mu\,,$$
$$P^a = (\phi^a_{|\nu} - i\tfrac{1}{2} A_\mu \phi^a - iB^A_\mu \Lambda^a_{Ab}\phi^b)\, dx^\nu \wedge \not{d}\theta^3$$
$$+ (\overline{L}^a_{\beta r} R^{\beta r})\not{d}\theta^1 \wedge \not{d}\theta^2\,,$$
$$P' = (iA_{\mu|\nu}) dx^\nu \wedge dx^\mu\,.$$

The terms $P^b_{\alpha t}$, $P_{\alpha t}$, and P_a are adjoints of $P^{\alpha t}_b$, $P^{\alpha t}$, and P^a. The structure constants ϵ^J_{KL} and ϵ^A_{BC} arise, respectively, from the commutators $[\Lambda_K, \Lambda_L]$ and $[\Lambda_B, \Lambda_C]$. The components of K are given by

$$K^{\alpha t}_{\beta r} = [\, -iY^i_\mu Y^j_\nu \sigma^\alpha_{ij\beta} I^t_r$$
$$+ i(C^J_{\mu|\nu} + \tfrac{1}{2} i\epsilon^J_{KL} C^K_\nu C^L_\mu) I^\alpha_\beta \Lambda^t_{Jr}] dx^\nu \wedge dx^\mu\,,$$
$$K^{\alpha t}_b = (Y^j_\nu \gamma^\alpha_{j\beta} L^{\beta t}_b) dx^\nu \wedge \not{d}\theta^1 + (L^{\alpha t}_b)\not{d}\theta^2 \wedge \not{d}\theta^3\,,$$
$$K^{\alpha t} = (Y^j_\nu \gamma^\alpha_{j\beta} R^{\beta t}) dx^\nu \wedge \not{d}\theta^2 + (R^{\alpha t})\not{d}\theta^1 \wedge \not{d}\theta^3\,, \tag{17}$$
$$K^a_b = [\tfrac{1}{2} iA_{\mu|\nu} I^a_b + i(B^a_{\mu|\nu} + \tfrac{1}{2} i\epsilon^A_{BC} B^B_\nu B^C_\mu)\Lambda^a_{Ab}] dx^\nu \wedge dx^\mu\,,$$
$$K^a = (\phi^a_{|\nu} - i\tfrac{1}{2} A_\mu \phi^a - iB^A_\mu \Lambda^a_{Ab}\phi^b) dx^\nu \wedge \not{d}\theta^3 + (\phi^a)\not{d}\theta^1 \wedge \not{d}\theta^2\,,$$
$$K' = (iA_{\mu|\nu}) dx^\nu \wedge dx^\mu\,.$$

The terms $K^b_{\alpha t}$, $K_{\alpha t}$, and K_a are adjoints of $K^{\alpha t}_b$, $K^{\alpha t}$, and K^a. Define

$$F_{\mu\nu} = \partial_{\underline{\mu}} A_{\underline{\nu}}\,,$$
$$G^K_{\mu\nu} = \partial_{\underline{\mu}} C^K_{\underline{\nu}} + i\epsilon^K_{LM} C^L_\mu C^M_\nu\,,$$
$$H^A_{\mu\nu} = \partial_{\underline{\mu}} B^A_{\underline{\nu}} + i\epsilon^A_{BC} B^B_\mu B^C_\nu\,, \tag{18}$$
$$R^i_{jkl} = Y^\mu_k Y^\nu_l R^i_{j\mu\nu}$$
$$= \Gamma^i_{j\underline{l}|\underline{k}} + \Gamma^i_{mk}\Gamma^m_{j\underline{l}} + \Gamma^i_{jm}(\Gamma^m_{\underline{kl}} - T^m_{\underline{kl}})\,.$$

Remember to include the potential (or constraint) terms for ϕ^a in the action. We will avoid the complexities of dynamical symmetry breaking by rescaling K. Recall the definition of the operation * (see Sec. II of paper I). We write the action as

$\mathcal{A} = (-1/2)\,\mathrm{tr}(\bar{K}^* \wedge P)$ where,

$$\bar{K}^* = \begin{bmatrix} -(iC + iG)^* & (\theta L + L)^* & (\theta R + R)^* \\ \overline{(\theta L + L)^*} & -(i\tfrac{1}{2}F + iH)^* & (D\phi + \phi)^* \\ \overline{(\theta R + R)^*} & [(D\phi^\dagger + \phi^\dagger]^* & -iF^* \end{bmatrix} \tag{19}$$

[we use $\overline{(A)} \equiv \bar{A}$ for typographical reasons] and

$$P = \begin{pmatrix} iR + iG & \mathfrak{D}L + R\bar{\phi} & \mathfrak{D}R + L\phi \\ \overline{\mathfrak{D}L} + \phi\bar{R} & i\tfrac{1}{2}F + iH & D\phi + \bar{L}R \\ \overline{\mathfrak{D}R} + \bar{\phi}\bar{L} & D\phi^\dagger + \bar{R}L & iF \end{pmatrix}.$$

The gauge-covariant derivatives have been symbolized as $\mathfrak{D}L$, $\mathfrak{D}R$, and D. C is the previously described two-form whose product with R gives Einstein's action. Rescale the Bose fields by introducing multipliers κ, k, k', and k'' into K:

$$\begin{aligned} \mathcal{A} = \int \Big\{ & \Big[-\frac{k}{4}F_{\mu\nu}F^{\mu\nu} - \frac{k'}{4}G^{K}_{\mu\nu}G^{\mu\nu}_{K} - \frac{k''}{4}H^{A}_{\mu\nu}H^{\mu\nu}_{A} + \kappa R_S \Big] (\bar{\phi}_{|\nu} - i\bar{\phi}B^{A}_{\nu}\Lambda_A - i\bar{\phi}A_\nu\tfrac{1}{2})(\phi^{|\nu} + iB^{A\nu}\Lambda_A\phi + iA^{\nu}\tfrac{1}{2}\phi) \\ & + \lambda(\|\phi\|^2 - m^2)^2 + \bar{L}\Big[\Big(\gamma^j \frac{i}{2}\overleftrightarrow{\partial}_j\Big) + (\tfrac{1}{4}\Gamma^{*}_{m}\gamma_5\gamma^m) + i(C^{J}_{j}\gamma^j\Lambda_J) - i(\tfrac{1}{2}A_j\gamma^j) + i(B^{A}_{j}\gamma^j\Lambda_A)\Big]L \\ & + \bar{R}\Big[\Big(\gamma^j \frac{i}{2}\overleftrightarrow{\partial}_j\Big) + (\tfrac{1}{4}\Gamma^{*}_{m}\gamma_5\gamma^m) + i(C^{J}_{j}\gamma^j\Lambda_j) - i(A_j\gamma^j)\Big]R + (\bar{L}\bar{\phi}R + \bar{R}\phi L)\Big\} \sqrt{-g}\, d^4x \,. \end{aligned} \tag{20}$$

Obviously, rescalings permit one to absorb numerical coefficients and obtain a standard form for the coefficients of the various pieces.

We have seen that a colored, curved Weinberg-Salam-type action[2] can be obtained by gauging pieces of an underlying supergroup. In the next section we will formulate a gauge theory for particular supergroups which have connection components in the entire superalgebra. We will try to motivate these more complicated connections by considering the Dirac algebra of the spinors. We will see that the action one obtains by gauging the Dirac algebra (with spinors) can be reinterpreted as a gauging of a certain supergroup. This identification allows an extension to include flavor in the manner discussed in this section.

III. DIRAC MANIFOLDS AND QUADRATIC ACTIONS

In this section we will speculate on a different approach to manifolds with spinor structure. The motive for doing this is twofold. First, we needed to introduce the form C whose Hilbert product with R, the Riemann curvature form, yielded the usual Einstein action $\int \sqrt{-g}\kappa R_S d^4x$, but this form C was not derived from a curvature. By examining the Clifford or Dirac algebra structure we will discover that C is part of the curvature. Indeed the tensor sector of the Clifford curvature is $R + C$. This suggests the Hilbert square of the Clifford or Dirac curvature would contain the Einstein curvature. Then the action for space-time and for internal symmetry would arise from a single principle,[13] each being the Hilbert square of the curvature, $\langle P|P\rangle$ and $\langle F|F\rangle$, with P the extended space-time curvature and F the internal symmetry curvature. A further hope is that this parallelism will allow us to infer renormalizability.

To come to a clearer understanding of the idea we are discussing, let us return to an examination of the effect of replacing vectors

$$x = x^\mu e_\mu = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}, \tag{21}$$

with Dirac matrices,

$$x = x^\mu \gamma_\mu = \begin{pmatrix} x^0 & \vec{x}\cdot\vec{\sigma} \\ -\vec{x}\cdot\vec{\sigma} & -x^0 \end{pmatrix}. \tag{22}$$

Under displacements a point changes by

$$\delta P = \delta x^\mu e_\mu = (\delta x^\mu Y^{i}_{\mu})e_i \,. \tag{23}$$

The basis, e_i (or $e_\mu = Y^{i}_{\mu}e_i$), transforms also,

$$\delta e_i = e_j \omega^{j}_{i} = e_j \omega^{j}_{\mu i}\delta x^\mu \,. \tag{24}$$

Under a general displacement the matrices, x, could vary in a more general fashion so that

$$\delta P = (\delta x^\mu Y^{I}_{\mu})\gamma_I = \theta^I \gamma_I \,, \tag{25}$$

where I runs over all 16 matrices instead of just 4,

$$\delta P = (\delta x^\mu Y^{i}_{\mu})\gamma_i = \theta^i \gamma_i \,. \tag{26}$$

Still, there are only 4 independent displacements, not 16. A displacement of the bases γ_I is given as follows:

$$\delta\gamma_I = \omega^{J}_{I}\gamma^J = \omega^{J}_{\mu I}\gamma_J \delta x^\mu \,. \tag{27}$$

A special class of connections ω^{J}_{I} are those which are induced from the vectors on which the matrices γ_I act (these are called spinors). That is, we can represent γ_I as $\bar{s}_\alpha(\gamma_I)^{\alpha}_{\beta}s^\beta$ with $(\gamma_I)^{\alpha}_{\beta}$ constant coefficients, where $\{\gamma_i, \gamma_j\} = 2\eta_{ij}I$, etc. Any Lorentz-transformed γ_i's would do just as well but the effects of such a transformation can be absorbed by spinor-Lorentz transforming the basis $\bar{s}_\alpha$. We have suppressed the tensor product symbol in $E^{\alpha}_{\beta} = \bar{s}^\alpha \otimes s_\beta$. E^{α}_{β} has one in the $\alpha\beta$th place, zeros elsewhere at a point, and is the position (and path) dependent basis for $\gamma_I = (\gamma_I)^{\alpha}_{\beta}E^{\beta}_{\alpha}$ in general. Now we can find the $\omega^J{}_I$ from a law of transformation for the basis $\bar{s}_\alpha$. That is, taking

$$\delta\bar{s}_\alpha=\bar{s}_\beta\hat{\Omega}^I_\mu\gamma^\beta_{I\alpha}\delta x^\mu\ , \tag{28}$$

we can find a relationship between $\omega^J{}_I$ and $\hat{\Omega}^K=\hat{\Omega}^K_\mu\delta x^\mu$. Assume $\bar{\gamma}_I=\gamma_0\gamma_I^\dagger\gamma_0=\gamma_I$ is a self-Dirac adjoint basis. The relationship is found as follows:

$$\delta\gamma_I=\delta\bar{s}\cdot\gamma_I\cdot s+\bar{s}\cdot\gamma_I\cdot\delta s$$
$$=\mathrm{Re}\hat{\Omega}^K\bar{s}\cdot\{\gamma_K,\gamma_I\}\cdot s+i\,\mathrm{Im}\hat{\Omega}^K\bar{s}\cdot[\gamma_K,\gamma_I]\cdot s\ . \tag{29}$$

Using $\{\gamma_K,\gamma_I\}=d^L_{KI}\gamma_L$ and $[\gamma_K,\gamma_I]=\epsilon^L_{KI}\gamma_L$ we discover

$$\delta\gamma_K=\omega^L{}_K\gamma_L \tag{30}$$

with

$$\omega^L{}_I=(\mathrm{Re}\hat{\Omega}^K)d^L_{KI}+i(\mathrm{Im}\hat{\Omega}^K)\epsilon^L_{KI}\ . \tag{31}$$

We can now find the torsion and the curvature of this manifold. $T^I=d\theta^I+\omega^I{}_K\wedge\theta^K$ is the torsion and $P^I{}_J=d\omega^I{}_J+\omega^I{}_K\wedge\omega^K{}_J$ is the curvature. Evaluating θ^I on a four-dimensional slice gives $\theta^I=Y^I_\nu dx^\nu$ and $\omega^I{}_J=\omega^I_{\nu J}dx^\nu=\Gamma^I_{J\nu}dx^\nu$. Define $\eta_{IJ}=\mathrm{tr}(\gamma_I\gamma_J)$; global constancy requires that $\omega_{IJ}=-\omega_{JI}$. But

$$\omega_{IJ}=(\mathrm{Re}\hat{\Omega}^K)d^L_{KI}\eta_{LJ}+i(\mathrm{Im}\hat{\Omega}^K)\epsilon^L_{KI}\eta_{LJ}\ . \tag{32}$$

Up to possible cancellations in $d^L_{KI}\eta_{LJ}+d^L_{KJ}\eta_{LI}$ due to indefiniteness in η_{LJ} the symmetry condition picks out $\omega^L{}_I=i(\mathrm{Im}\hat{\Omega}^K)\epsilon^L_{KI}$. Writing $(-\Omega^K)$ for $\mathrm{Im}\hat{\Omega}^K$ we can set $\delta\bar{s}_\alpha=\bar{s}_\beta(-i\Omega^K_\nu\gamma^\beta_{K\alpha})\delta x^\nu$ and $\omega^L{}_I=-\Omega^K\epsilon^L_{KI}$. The $\omega^L{}_I$ are coefficients for SO(4,2) or SU(2,2). But Ω^L can include all of U(2,2). We consider the U(2,2) theory. A fiber bundle interpretation of these ideas is in Sec. IV of paper I.[1] When functionals are defined carefully, the metric η^{ij}, becomes Euclidean δ^{ij}. In so doing the U(2,2) structure is replaced by a U(4) structure.

Now we can find the curvature and torsion in terms of Ω^K,

$$P^I{}_K=d\omega^I{}_K+\omega^I{}_N\wedge\omega^N{}_K=-i\epsilon^I_{LK}d\Omega^L+(-i)^2(\epsilon^I_{\underline{M}N}\epsilon^N_{\underline{J}K}\tfrac{1}{2})\Omega^M\wedge\Omega^J\ . \tag{33}$$

But

$$\tfrac{1}{2}\epsilon^I_{\underline{M}N}\epsilon^N_{\underline{J}K}=\tfrac{1}{2}\epsilon^I_{LK}\epsilon^L_{MJ} \tag{34}$$

by the Jacobi identity. Thus define $P^I{}_K$ by $P^I{}_K=-i\epsilon^I_{LK}P^L$. This can be inverted for P^L in terms of $P^I{}_K$ since $P^Lg_{LM}=i\epsilon^K_{MI}P^I{}_K$ with $g_{LM}\equiv\epsilon^K_{MI}\epsilon^I_{LR}$, the Cartan-Killing metric, when P^L is restricted to SU(2,2):

$$P^I{}_K=-i\epsilon^I_{LK}P^L=-i\epsilon^L_{LK}\left(d\Omega^L-\frac{i}{2}\epsilon^L_{MJ}\Omega^M\wedge\Omega^J\right) \tag{35}$$

or

$$P^L=d\Omega^L-\tfrac{1}{2}i\epsilon^L_{MJ}\Omega^M\wedge\Omega^J\ . \tag{36}$$

And $T^I=d\theta^I-i\epsilon^I_{KM}\Omega^K\wedge\theta^M$.

When $\Omega=\omega I+\omega^i\gamma_i+\omega^{ij}\sigma_{ij}$, we can identify ω^i with the usual vierbein field by taking $\theta=Y^i\gamma_i$, for example. Then $P^i=d\omega^i+\omega^i{}_k\ \ \omega^k$ and $T^i=dY^i+\omega^i{}_k\wedge Y^k$. Since ω^i has dimensions of mass and Y^i is scale zero, $P^i=mT^i$ if $\omega^i=mY^i$. Here $Y^i\equiv Y^i_\mu dx^\mu$,

The basis

$$\gamma_I=\{I,\gamma_j,\sigma_{ij},\gamma_5\gamma_j,(i\gamma_5)\} \tag{37}$$

with

$$\sigma_{ij}=\frac{i}{8}[\gamma_i,\gamma_j]\ \text{ and }\ \gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{38}$$

can be Dirac self-adjoint,

$$\bar{\gamma}_I=\gamma_0\gamma_I^\dagger\gamma_0=\gamma_I\ . \tag{39}$$

For example,

$$Y^j\gamma_j=\begin{pmatrix}Y^0 & \vec{Y}\cdot\vec{\sigma}\\ -\vec{Y}\cdot\vec{\sigma} & -Y^0\end{pmatrix}\text{and }\ \gamma_5=\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{40}$$

works fine. Here $\eta_{ij}=\mathrm{diag}(1,-1,-1,-1)$.

The curvature P^I can also be found from $\frac{1}{2}[d,\delta]\bar{s}_\alpha=iP^I\delta^\beta_{I\alpha}\bar{s}_\beta$ as well as $\frac{1}{2}[d,\delta]\gamma_L=-i\epsilon^K_{IL}P^I\gamma_K$. In either case $P^K=d\Omega^K-i\epsilon^K_{LM}\Omega^L\wedge\Omega^M$. Indeed when

$$\Omega=\omega^SI+\omega^i\gamma_i+\omega^{ij}\sigma_{ij}+\omega^{*\ell}\gamma_5\gamma_\ell+\omega^*(i\gamma_5)\ , \tag{41}$$

we find

$$\begin{aligned}P^S&=d\omega^S\ ,\\ P^m&=d\omega^m+\omega^m{}_k\wedge\omega^k+2\omega^*\wedge\omega^{*m}\ ,\\ P^{mn}&=d\omega^{mn}+\omega^m{}_k\wedge\omega^{kn}-8\omega^m\wedge\omega^n+8\omega^{*m}\wedge\omega^{*n}\ ,\\ P^{*m}&=d\omega^{*m}+\omega^m{}_k\wedge\omega^{*k}+2\omega^*\wedge\omega^m\ ,\\ P^*&=d\omega^*-2\eta_{ij}\omega^{*i}\wedge\omega^j\ .\end{aligned} \tag{42}$$

Note the term $2\omega^*\wedge\omega^{*m}$ in P^m. To identify mY^m with ω^m we can set $\theta=Y^m\gamma_m+Y^{*m}\gamma_5\gamma_m$. Now $dY^m+\omega^m_k\wedge Y^k+2\omega^*\wedge Y^{*m}=T^m$ and $dY^{*m}+\omega^m_k\wedge Y^{*k}+2\omega^*\wedge Y^m=T^{*m}$.

Augmentation yields

$$\tfrac{1}{2}[d,\delta][\bar{s}_\alpha,\bar{s}]=[\bar{s}_\beta,\bar{s}]\begin{pmatrix}iP_I\gamma^\beta_{I\alpha} & d\Psi^\beta-i\Omega^K\gamma^\beta_{K\alpha}\Psi^\alpha\\ 0 & 0\end{pmatrix}\ , \tag{43}$$

when $d\bar{s}=\bar{s}_\alpha\Psi^\alpha$. We intend to use $\mathrm{tr}\int\bar{P}^*\wedge P$ for the action for Ω^I. Thus let us examine the gauge transformation properties of the connections. We will consider three kinds of transformations. The first is recoordinatizations; thus $\theta^I\gamma_I=Y^I_\mu\gamma_Idx^\mu=Z^I_\lambda\gamma_Idy^\lambda$ when $Y^I_\mu=Z^I_\lambda\partial y^\lambda/\partial x^\mu$. Expression of the relevant quantities in terms of the forms θ^I will be recoordinatization invariant. The second is local Lorentz transformations. Since $\{\gamma_i,\gamma_j\}=2\eta_{ij}I$ admits a Lorentz group action. Set $\gamma_a=\Lambda^i_a\gamma_i$ and note $\Lambda^i_a\eta_{ij}\Lambda^j_b=\eta_{ab}$ when $\Lambda^i_a(x)=\{\exp[\omega^{mn}(x)\Sigma_{mn}]\}^i_a$ with $\omega^{mn}(x)$ six arbitrary and antisymmetric functions $(\omega^{mn}=-\omega^{nm})$. Σ_{mn} is the vector matrix representative of the Lorentz generators with components

$(\Sigma_{mn})_{k\ell}=\frac{1}{2}(\eta_{mk}\eta_{n\ell}-\eta_{m\ell}\eta_{nk})$. Since all 16 Dirac matrices can be found from products of γ_i; the other local transformations are easily found, e.g.,

$$\sigma_{ab}=\Lambda_a^i\Lambda_b^j\frac{i}{8}[\gamma_i,\gamma_j]=\Lambda_a^i\Lambda_b^j\sigma_{ij}\,. \tag{44}$$

By spinor Lorentz transformations on the $\bar{s}_\alpha$'s and s^β's, the effect of the Λ_a^i's can be absorbed into the $\bar{s}_\alpha$'s. Thus constant coefficients $\gamma_{i\beta}^\alpha$ can be used in $\gamma_i=\bar{s}_\alpha\gamma_{i\beta}^\alpha s^\beta$.

The third kind of transformation is the familiar gauge transformation. When $S=\exp[iY^I(x)\gamma_I]$, set $\psi'=S^{-1}\psi$ and $[d\psi'-i\Omega'\psi']=S^{-1}[d\psi-i\Omega\psi]$. Then $\Omega'=iS^{-1}dS+S^{-1}\Omega S$. Now $P=d\Omega-i\Omega\wedge\Omega$; so P' is seen to be

$$\begin{aligned}d\Omega'=&-iS^{-1}dSS^{-1}dS-S^{-1}dS^{-1}\Omega S\\&+S^{-1}d\Omega S-S^{-1}\Omega dS\end{aligned} \tag{45}$$

plus

$$\begin{aligned}-i\Omega'\wedge\Omega'=&\,iS^{-1}dSS^{-1}dS+S^{-1}dSS^{-1}\Omega S\\&-iS^{-1}\Omega\wedge\Omega S+S^{-1}\Omega dS\end{aligned}$$

which cancel but for

$$S^{-1}(d\Omega-i\Omega\wedge\Omega)S=P'\,. \tag{46}$$

Note that $(P^*)'=(P')^*$ since S^{-1} and S are not affected by $*$. Thus the action is indeed invariant since $\mathrm{tr}\int P^{*\prime}\wedge P'=\mathrm{tr}\int P^*\wedge P$. Replacing P^* by $\bar{P}^*$ only changes the sign of both integrals. They are still equal.

The action for Ψ insofar as it is coupled to Ω^I must occur in the gauge-covariant derivative combination

$$d\Psi^\alpha-i\Omega^K\gamma_{K\beta}^\alpha\Psi^\beta\equiv(D\Psi)^\alpha\,. \tag{47}$$

We will now discuss our choice for the fermion action.

Before writing the action for this theory, let us consider a somewhat unusual possibility, that the fermions introduced here are actually fermion potentials, not the usual fermions. A somewhat different application of this idea is already well known from the papers of Feynman and Gell-Mann and Brown.[10] The introduction of fermion potentials ϕ such that $\psi=(\not{P}+m)\phi$ and thus $(P^2-m^2)\phi=0$ was what led them to $V-A$ theory. Feynman states that he has always had a prediliction for considering integrals which are quadratic rather than linear in the fields' gauge-covariant derivatives. There are two other reasons for considering fermion potentials. One, there is a beautiful simplicity in the prescription that the action be the Hilbert square of the (augmented) curvature. Nonetheless, lack of beauty cannot stop one from considering a quadratic action for gauge terms and a linear term for the matter fields. However, in that case one has to introduce a secondary structure lacking geometrical interpretation or at least lacking geometrical immediacy. Two, we saw that gravity and gauge fields coupled to fermions led to a problem with anomalies. If fermion potentials can be sensibly quantized, their quadratic propagators lead to convergent anomaly diagrams. We will not consider the consequences for partial conservation of axial-vector currents (PCAC) here. Remember too that in solving Dirac's equations one frequently uses the second-order formulation.

We have referred to these objects we are about to introduce as fermion potentials. Obviously we are thinking of some parallelism with gauge theories in using this terminology. To show how extensive such a parallelism is, we will begin by writing an action for electromagnetism entirely in terms of the field tensor. This will lead to a first-order formalism akin to that for Dirac spinors. Maxwell's equations include

$$\begin{pmatrix}I_a^a\partial_0 & -\epsilon_{dc}^a\partial^c\\ \epsilon_{dc}^a\partial^c & I_d^a\partial_0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\epsilon_a^{bc}f_{bc}\\ f_{0a}\end{pmatrix}=0 \tag{48}$$

or equivalently,

$$(M_E^{\mu A}\partial_\mu)(F_A)=0\,, \tag{49}$$

where $A,B\in(1,\ldots,6)$ and $a,b\in(1,2,3)$ and $\mu\in(0,1,2,3)$.

We can therefore employ the Lagrangian $\mathcal{L}=F^EM_{\mu E}^A\partial^\mu F_A$. $M_{\mu E}^A$ plays the role of $(\gamma_\mu)_\epsilon^\alpha$ in the Dirac Lagrangian. Indeed, setting $(\not{\partial})_E^A=M_{\mu E}^A\partial^\mu$ and $\bar{F}=$ transpose (F), $\mathcal{L}=\bar{F}\not{\partial}F$. Of course if $(S^c)_{ab}\equiv\epsilon_{ab}^c$ and $\Phi_a\equiv\frac{1}{2}\epsilon_a^{bc}f_{bc}+if_{0a}$, we can write the relations compactly as $(i\vec{S}\cdot\vec{\partial}+I\partial_0)\Phi=0$; then $\mathcal{L}=\bar{\Phi}(i\vec{S}\cdot\vec{\partial}+I\partial_0)\Phi$. Introducing $\Phi^*{}_a=\frac{1}{2}\epsilon_a^{bc}f_{bc}^*+if_{0a}^*$ where $f_{\mu\nu}^*=\frac{1}{2}\epsilon_{\mu\nu}^{\lambda\rho}f_{\lambda\rho}$ allows one to find all four Maxwell relations from

$$\mathcal{L}=[\bar{\Phi}^*,\bar{\Phi}](i\vec{S}\cdot\vec{\partial}+I\partial_0)\begin{pmatrix}\Phi^*\\ \Phi\end{pmatrix} \tag{50}$$

or

$$\mathcal{L}=[\bar{F}^*,\bar{F}](\not{\partial})\begin{pmatrix}F^*\\ F\end{pmatrix}. \tag{51}$$

Here the "gamma" matrices, $M_{\mu E}^A$, $\vec{S}$, I, must be doubled along the diagonal.

Now introducing potentials as well as the fields and varying separately in the Lagrangian,

$$\mathcal{L}=\partial_{\underline{\mu}}A_{\underline{\nu}}\partial^{\bar{\mu}}A^{\bar{\nu}}-\lambda(\partial^{\bar{\mu}}A^{\bar{\nu}}-f^{\mu\nu})(\partial_{\underline{\mu}}A_{\underline{\nu}}-f_{\mu\nu})\,. \tag{52}$$

There are three equations of motion λ, A_ν, and $f_{\mu\nu}$:

$$\begin{aligned}&(\partial^{\bar{\mu}}A^{\bar{\nu}}-f^{\mu\nu})(\partial_{\underline{\mu}}A_{\underline{\nu}}-f_{\mu\nu})=0\,,\\&\partial^\mu(\partial_{\underline{\mu}}A_{\underline{\nu}})=\partial^\mu[\lambda(\partial_{\underline{\mu}}A_{\underline{\nu}}-f_{\mu\nu})]\,,\\&\text{and}\\&\lambda(\partial_{\underline{\mu}}A_{\underline{\nu}}-f_{\mu\nu})=0\,,\end{aligned} \tag{53}$$

which are equivalent to $\partial^\mu(\partial_\mu A_\nu)=0$ and $f_{\mu\nu}=\partial_\mu A_\nu$. This Lagrangian leads to the same solution set as the Lagrangians above. Now exploiting the analogy $\sqrt{m}\gamma^\alpha_{i\beta}\psi^\beta \leftrightarrow f_{\mu\nu}$ and $\Psi^\alpha \leftrightarrow A_\mu$ we consider the following action for both ψ and Ψ:

$$\mathcal{L}=\overline{\Psi}\overline{P}^i P_i\Psi - \lambda[\overline{\Psi}P^2\Psi - \tfrac{1}{2}\overline{\psi}(\not{P}+\overline{\not{P}})\psi]\,. \tag{54}$$

Here $P_i=-i\partial_i+\Omega^I_i\gamma_I$ in the vierbein basis $e_i=Y^\mu_i e_\mu$. As above the solution set can be described by $P_i(\sqrt{m}\gamma_i\psi - P_i\Psi)=0$ and $P^iP_i\Psi=P^i\sqrt{m}\gamma_i\psi=\sqrt{m}\not{P}\psi=0$. If instead of $\mathcal{L}$ we use $\mathcal{L}\sqrt{-g}$ we find

$$\frac{1}{\sqrt{-g}}\not{P}(\sqrt{-g}\psi)=\frac{1}{\sqrt{-g}}P^i(\sqrt{-g}\,P_i\Psi)=0\,. \tag{55}$$

The kinetic term agrees with that of Sec. IV of paper I in the gauge $\partial^\mu(\sqrt{-g}\gamma_\mu)=0$.

When

$$-i\Omega^I\gamma_I=-i[eA_\mu I+mY^i_\mu\gamma_i+\lambda\omega^{ij}_\mu\sigma_{ij} + m^*Y^{*i}_\mu\gamma_5\gamma_i+e^*A^*_\mu(i\gamma_5)]dx^\mu\,, \tag{56}$$

the classical equation of motion becomes (for $\sqrt{-g}$ constant),

$$\begin{aligned}\not{P}\psi=[&i\not{\partial}+e\not{A}+4mI+(-i\lambda\tilde{\omega}^m\gamma_m+\lambda\omega^{\star m}\gamma_5\gamma_m)\\ &+m^*(Y^{*i}_\mu Y^\mu_i\gamma_5-4i\gamma_5Y^{*k}_\mu Y^\mu_i\sigma^i_k)\\ &+e^*\not{A}^*(-i\gamma_5)]\psi=0\,,\end{aligned} \tag{57}$$

$$\tilde{\omega}^m\equiv\tfrac{1}{2}\omega^{km}_k \quad\text{and}\quad \omega^{\star m}\equiv\tfrac{1}{4}\omega^{k\ell}_i\epsilon^{im}_{k\ell}\,.$$

It is amusing to note that if one tries to write a linear action for the fermions, Hermiticity puts severe requirements on the terms,

$$\tfrac{1}{2}(\overline{\psi}\not{P}\psi+\overline{\psi}\overline{\not{P}}\psi)=\overline{\psi}\left(\frac{i\not{\partial}}{2}+e\not{A}+4m+\lambda\omega^{\star m}\gamma_5\gamma_m - m^*4iY^k_\mu Y^{*m}_i\sigma^i_k\right)\psi\,. \tag{58}$$

The two Lagrangians (quadratic and linear) will give the same equations of motion if m^* and e^* vanish and if $\tilde{\omega}^m=\frac{1}{2}\omega^{k\,m}_k$ vanishes or if λ vanishes, but then the matrices I and γ_i do not close. We note that I closes and γ_i and σ_{ij} close so a subalgebra I, γ_i, σ_{ij} closes. The condition $\tilde{\omega}^m=0$ might be viewed as a gauge constraint. The first-order formalism is more restrictive; possible too restrictive in that it does not even permit the full tensor interaction. We will write out the action for the full Dirac manifold and then examine the restriction to the (I,γ_i,σ_{ij}) sector.

Let

$$\tfrac{1}{2}[\delta,d][\overline{s}_\alpha,\overline{s}]=[\overline{s}_\beta,\overline{s}]\begin{pmatrix}P^\beta_\alpha & P^\beta\\ 0 & 0\end{pmatrix}\equiv[\overline{s}_\alpha,s](\mathcal{P})\,. \tag{59}$$

The action we will consider is $\mathcal{a}=\langle\mathcal{P}|\mathcal{P}\rangle$. The metric for the matrices will be $\bullet=\frac{1}{8}k$ tr, where k is constant and tr is the trace. The result is as follows [note that when functional integrals are carefully defined the Minkowski structure is replaced by a Euclidean structure; thus U(2,2) becomes U(4) and no ghost fields persist]:

$$\begin{aligned}\mathcal{a}=k\int[&-\tfrac{1}{4}f_{\nu\mu}f^{\nu\mu}-\tfrac{1}{4}T^m_{\nu\mu}T^{\nu\mu}_m-\tfrac{1}{16}P^{mn}_{\nu\mu}P^{\nu\mu}_{mn}\\ &-\tfrac{1}{4}T^{*m}_{\nu\mu}T^{*\nu\mu}_m-\tfrac{1}{4}f^*_{\nu\mu}f^{*\nu\mu}\\ &+\tfrac{1}{2}\overline{\Psi}\overline{P}^iP_i\Psi]\sqrt{-g}d^4x\,,\end{aligned} \tag{60}$$

where

$$\begin{aligned}&f_{\nu\mu}=A_{\mu|\nu}\,,\\ &T^m_{\nu\mu}=\omega^m_{\mu|\nu}+\omega^m_{\nu k}\omega^k_\mu+2A^*_\nu\omega^{*m}_\mu\,,\\ &P^{mn}_{\nu\mu}=(\omega^{mn}_{\mu|\nu}+\omega^m_{\nu k}\omega^{kn}_\mu)-(8\omega^m_\nu\omega^n_\mu)+(8\omega^{*m}_\nu\omega^{*n}_\mu)\,,\\ &T^{*m}_{\nu\mu}=\omega^{*m}_{\mu|\nu}+\omega^m_{\nu k}\omega^{*k}_\mu+2A^*_\nu\omega^m_\mu\,,\\ &f^*_{\nu\mu}=A^*_{\mu|\nu}-2\eta_{ij}\omega^{*i}_\nu\omega^j_\mu\,,\end{aligned} \tag{61}$$

$$P_i=i\partial_i+A_i(I)+m(\gamma_i)+\omega^{jk}_i(\sigma_{jk}) +Y^\mu_i\omega^{*j}_\mu(\gamma_5\gamma_j)+A^*_i(i\gamma_5)=i\partial_i+\Omega_i\,, \tag{62}$$

when

$$\begin{aligned}&Y^i_\mu\omega^\mu_i=m\delta^j_i\,,\quad g_{\mu\nu}=Y^i_\mu\eta_{ij}Y^j_\nu\,,\quad \theta^i=Y^i_\mu dx^\mu\\ &\sigma_{jk}=\frac{i}{8}[\gamma_j,\gamma_k]\,,\quad \gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3\,.\end{aligned} \tag{63}$$

The action is written in terms of potentials. The usual fermion field obeys

$$\frac{1}{\sqrt{-g}}P^i\sqrt{-g}\gamma_i\psi=0 \tag{64}$$

with

$$P^i(P_i\Psi-\sqrt{m}\gamma_i\psi)=0\,.$$

Of course a mix between quadratic action for Ω^I and linear for ψ as described above is possible. However, that lacks some of the possible interactions:

$$P^i\gamma_i=i\not{\partial}+\not{A}+4mI+i\tilde{\omega}^m\gamma_m+\omega^{\star m}\gamma_5\gamma_m +Y^\mu_i\omega^{*i}_\mu\gamma_5+4i\gamma_5Y^\mu_i\omega^{*i}\sigma^i_j+\not{A}^*(i\gamma_5)\,. \tag{65}$$

If instead of gauging the entire Clifford algebra, we gauge the closed subalgebra generated by I, γ_m, and σ_{mn}, which amounts to replacing

$$\Omega^I\gamma_I=[eA_\nu I+mY^i_\nu\gamma_i+\lambda\omega^{mn}_\nu\sigma_{mn}+\omega^{\star m}_\nu(\gamma_5\gamma_m) +e^*A^*_\nu(i\gamma_5)]dx^\nu \tag{66}$$

by

$$\Omega^I\gamma_I=(eA_\nu I+mY^i_\nu\gamma_i+\lambda\omega^{mn}_\nu\sigma_{mn})dx^\nu\,, \tag{67}$$

we discover a much simpler action. Define

$$\begin{aligned}&f_{\mu\nu}=A_{\nu|\mu}\,,\\ &T^m_{\mu\nu}=Y^m_{\nu|\mu}+\lambda\omega^m_{\mu k}Y^k_\nu\,,\\ &R^{mn}_{\mu\nu}=\omega^{mn}_{\nu|\mu}+\lambda\omega^m_{\mu k}\omega^{kn}_\nu\,.\end{aligned} \tag{68}$$

Then

$$\langle \mathcal{P}|\mathcal{P}\rangle = -(ke^2)\int\Big[\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{m^2}{4e^2}T^m_{\nu\mu}\eta_{mn}T^n_{\rho\lambda}g^{\mu\rho}g^{\nu\lambda} + \frac{\lambda^2}{16e^2}R^{mn}_{\mu\nu}R^{kl}_{\rho\lambda}g^{\mu\rho}g^{\nu\lambda}\eta_{mk}\eta_{nl} - \frac{\lambda m^2}{e^2}(R^{mn}_{\mu\nu})(Y^k_\rho Y^l_\lambda)g^{\mu\rho}g^{\nu\lambda}\eta_{mk}\eta_{nl} + \frac{4m^2}{e^2}(Y^m_\mu Y^n_\nu Y^k_\rho Y^l_\lambda)g^{\mu\rho}g^{\nu\lambda}\eta_{mk}\eta_{nl} - \tfrac{1}{2}\bar\Psi\bar P^i P_i\Psi\Big]\sqrt{-g}\,d^4x\,, \quad (69)$$

with

$$P_i = i\partial_i + eA_i + 4m\gamma_i + \lambda\omega^{mn}_i\sigma_{mn}\,. \quad (70)$$

A factor of e has been removed from Ψ and placed at the front of the integral. Take $ke^2=1$. We can rewrite the action as follows:

$$\langle \mathcal{P}|\mathcal{P}\rangle = -\int\Big(\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{m^2}{4e^2}T^\rho_{\mu\nu}T^{\mu\nu}_\rho + 96\frac{m^4}{e^2} + \frac{\lambda^2}{16e^2}R^{\rho\lambda}_{\mu\nu}R^{\mu\nu}_{\rho\lambda} - 2\frac{\lambda m^2}{e^2}R_S - \tfrac{1}{2}\bar\Psi\bar P^i P_i\Psi\Big)\sqrt{-g}\,d^4x\,. \quad (71)$$

To recover the Einstein limit of the theory we must look only at connections having zero torsion. We have already obtained the formula for

$$\omega_{ij} = \eta_{il}\eta_{jk}\omega^{lk} = [\tfrac{1}{2}(C_{ijk}+C_{ikj}-C_{jki})Y^k_\mu]\,dx^\mu = \eta_{il}\eta_{jk}(\omega^{lk}_\mu)dx^\mu\,, \quad (72)$$

where

$$C_{ijk} = C^l_{ij}\eta_{lk} = (Y^k_{\mu|\lambda}Y^\lambda_i Y^\mu_j)\eta_{lk}\,; \quad (73)$$

in that case

$$g_{\mu\nu} = Y^i_\mu\eta_{ik}Y^k_\nu\,. \quad (74)$$

The term $\int \omega^{\mu\nu}_{\nu|\mu}\sqrt{-g}\,d^4x$ can now be integrated by parts to give surface terms. This allows us to replace

$$\int\Big(\frac{2\lambda m^2}{e^2}\Big)R_S\sqrt{-g}\,d^4x \quad\text{by}\quad -\int\Big(\frac{2\lambda^2m^2}{e^2}\Big)G_S\sqrt{-g}\,d^4x \quad (75)$$

with

$$-G_S = \omega^\mu_{\mu\rho}\omega^{\rho\nu}_\nu = \Gamma^\mu_{\rho\mu}\Gamma^{\rho\nu}_\nu\,. \quad (76)$$

This leaves the action

$$\langle \mathcal{P}|\mathcal{P}\rangle = \int\Big[\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} - 96\frac{m^4}{e^2} - \frac{\lambda^2}{16e^2}\hat R^{\rho\sigma}_{\mu\nu}\hat R^{\mu\nu}_{\rho\lambda} - \Big(\frac{2\lambda^2m^2}{e^2}\Big)G_S + \tfrac{1}{2}\bar\Psi\bar P^i P_i\Psi\Big]\sqrt{-g}\,d^4x\,. \quad (77)$$

The caret on $\hat R^{\rho\sigma}_{\mu\nu}$ is a reminder that it is evaluated at λ. The multiplier of G_S is $\kappa = 1/G$ when $\hbar = c = 1$. Thus $\lambda^2 = (e^2/2m^2)\kappa = e^2/2m^2G$. $4m$ is the particle mass. The bounds on the cosmological term Λ in the equations of motion, $\Lambda < 10^{-57}$ cm^{-2} experimentally.[16] This forces

$$\Big[-\Big(\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\lambda^2}{16e^2}\hat R^{\rho\sigma}_{\mu\nu}\hat R^{\mu\nu}_{\rho\sigma} + 96\frac{m^4}{e^2}\Big) + \tfrac{1}{2}\bar\Psi\bar P^i P_i\Psi\Big]\frac{e^2}{4\lambda^2m^2} \quad (78)$$

to be $<10^{-57}$ cm^{-2}.

Since the equations of motion are $R_{\mu\nu} + (\Lambda - \frac{1}{2}R_S)\times g_{\mu\nu} \sim \kappa T_{\mu\nu}$, $T_{\mu\nu}$ includes $R^{\rho\sigma}_{\mu\nu}R^{\mu\nu}_{\rho\sigma}\sqrt{-g}$ contributions. The pieces of the action consisting of κR_S and $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$ have been suggested in other papers.[7,8] The quadratic Riemann invariant $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}$ has occurred when considering the quantum version of the theory.[7] By using the Christoffel formula for ω^{ij}_μ in the action, fourth-order equations for $g_{\mu\nu}$ are found. For this reason we will examine separate variations of mY^i_μ and ω^{ij}_μ (the so-called first-order formulation). We remark that m is related to an inverse length coming from the de Sitter group structure. The Poincaré group is a contraction of the de Sitter group. This natural length may be important for infrared problems when the theory is quantized.

We will write out the equations of motion in detail for the following Lagrangian:

$$(\Lambda_{\mu\nu\lambda\rho})(g^{\mu\lambda}g^{\nu\rho}\sqrt{-g}) \equiv \Big[\frac{m^2}{4}(Y^m_{\nu|\mu} + \omega^{mk}_\mu\eta_{kl}Y^l_\nu)\eta_{mn}T^n_{\lambda\rho} + \tfrac{1}{16}(\omega^{mn}_{\nu|\mu} + \omega^{mk}_\mu\eta_{kl}\omega^{ln}_\nu - 8m^2Y^m_\mu Y^n_\nu)\times P_{mn\lambda\rho}\Big]\Big[g^{\mu\lambda}g^{\nu\rho}\sqrt{-g}\Big], \quad (79)$$

where

$$g_{\alpha\beta} = Y^i_\alpha\eta_{ij}Y^j_\beta\,. \quad (80)$$

They are given as follows:

$$\frac{1}{\sqrt{-g}}\partial_\mu(T^{\mu\alpha}_k\sqrt{-g}) = \Big[\omega^m_{\mu k}T^{\mu\alpha}_m - 4P^{\alpha\mu}_{kl}Y^l_\mu + \frac{2}{m^2}(\Lambda^{\mu\nu}_{\mu\nu}Y^\alpha_k - 4\Lambda^{\alpha\nu}_{k\nu})\Big], \quad (81)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(P^{\mu\alpha}_{kn}\sqrt{-g}) = 2m^2\eta_{nl}Y^l_\nu T^{\alpha\nu}_k + \omega^m_{\nu n}P^{\alpha\nu}_{km} + \omega^m_{\nu k}P^{\nu\alpha}_{mn}\,.$$

In the limit of zero torsion these reduce to the relations

$$P^{\alpha\beta} = \frac{1}{2m^2}(\Lambda^{\mu\nu}_{\mu\nu}g^{\alpha\beta} - 4\Lambda^{\alpha\beta\nu}_\nu)\,,$$

with

$$P^{\alpha\beta} \equiv g^{\alpha\lambda}Y^a_\lambda P^{\beta\mu}_{ab}Y^b_\mu\,, \quad (82)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(P^{\mu\alpha}_{kn}\sqrt{-g}) = \omega^m_{\nu n}P^{\alpha\nu}_{km} + \omega^m_{\nu k}P^{\nu\alpha}_{mn}\,. \quad (83)$$

If $\Lambda^{\mu\nu}_{\mu\nu}g^{\alpha\beta}=4\Lambda^{\alpha\beta\nu}_{\nu}$ we can take the limit $m^2\to 0$, ∞ de Sitter radius, the Poincaré limit of the de Sitter group. Then $P^{\alpha\beta}\to R^{\alpha\beta}$, the Ricci tensor. And the two equations can be shown to be equivalent to

$$R_{\alpha\beta}=0 \quad\text{and}\quad R^{\mu}_{\alpha\beta\lambda\Vert\mu}=0\,. \tag{84}$$

The second equation can be related to the Weyl[3] equations by using the Bianchi identities, $R_{\alpha\beta\Vert\lambda}=0$. These are a trivial consequence of $R_{\alpha\beta}=0$.

Is it ever reasonable that $\Lambda^{\mu\nu}_{\mu\nu}g^{\alpha\beta}=4\Lambda^{\alpha\beta\nu}_{\nu}$? Let us consider the Schwarzschild solution.[16] Set

$$\omega^0=\left(1-\frac{2M}{r}\right)^{1/2}dt\,,\quad \omega^1=\left(1-\frac{2M}{r}\right)^{-1/2}dr\,,$$

$$\omega^2=rd\theta \tag{85}$$

and

$$\omega^3=r\sin\theta d\phi\,.$$

Now define $\lambda=M/r^3$ and then note that $R_{ijk\ell}$ can be found from

$$R_{0202}=R_{0303}=-R_{1212}=-R_{1313}$$
$$=-\tfrac{1}{2}R_{0101}=-\tfrac{1}{2}R_{2323}=\lambda=\frac{M}{r^3}\,. \tag{86}$$

All other components either vanish or are obtained from these by use of symmetries. Since $\eta_{k\ell}=\text{diag}(1,-1\,-1\,-1)$ is the metric for this frame we can easily see that $\Lambda_{ikj\ell}=\frac{1}{16}R_{mnik}R^{mn}_{\ j\ell}$ satisfies the identity

$$0=\Lambda_{ikj\ell}(4\eta^{k\ell}\delta^i_m\delta^j_n-\eta^{k\ell}\eta^{ij}\eta_{mn})$$
$$=4\Lambda_{mn}-\eta_{mn}\Lambda=4\,\frac{12M^2}{r^6}\,\eta_{mn}-48\,\frac{M^2}{r^6}\,\eta_{mn}\,. \tag{87}$$

Now $R_{\alpha\beta}=0$ and $R_{\alpha\beta\Vert\lambda}=0$ are consistent equations of motion. We can conclude that as the de Sitter radius $1/m$ goes to infinity ($m\to 0$) the Schwarzschild solution solves these equations for zero torsion.

In the case $m^2\neq 0$ both torsion, $T^{m}_{\mu\nu}$, and curvature, $P^{mn}_{\mu\nu}$, can be taken to be zero. Then

$$R^{mn}_{\mu\nu}-8m^2Y^m_{\mu}Y^n_{\nu}=0 \quad\text{or}\quad R^{\alpha\beta}_{\mu\ \nu}=8m^2g^{\alpha}_{\mu}g^{\beta}_{\nu}\,. \tag{88}$$

The manifold is a de Sitter space. Since that is the flattest space for the vector-tensor theory (only $\omega^m_{\mu}\gamma_m+\omega^{mn}_{\mu}\sigma_{mn}$ nonvanishing), perturbation theory should probably be done using de Sitter propagators. The operator

$$\partial^2=\partial_t^{\,2}-m^2L^2\,, \tag{89}$$

with

$$L^2\Phi=\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta\Phi)+\frac{1}{\sin^2\theta}(\partial_\theta{}^2\Phi+\partial_\phi{}^2\Phi)\,. \tag{90}$$

The operator L^2 has eigenfunctions $P^m_\ell(\cos\theta)e^{-ik\phi}\times e^{-in\phi}$ with $m=k+n$ and $P^m_\ell(\cos\theta)$ the associated Legendre polynominals. Energy eigenvalues have a discrete spectrum. Quantization would be like quantization in a box.

If one considers the full theory (all Clifford algebra elements are gauged, $\omega^I_\mu\gamma_I$) then the equations of motion are given as follows:

$$\frac{1}{\sqrt{-g}}\,\partial^\nu(P^I_{\nu\mu}\sqrt{-g})-i\epsilon^I_{JK}\,\omega^J_\rho P^K_{\mu\nu}\,g^{\rho\nu}=J^I_\mu\,, \tag{91}$$

and, unless there are external sources,

$$J^I_\mu=(\Lambda_{\sigma\nu\alpha\beta}g^{\sigma\alpha}g^{\nu\beta}\omega^i_\mu-4\Lambda_{\mu\nu\alpha\beta}g^{\nu\beta}g^{\alpha\lambda}\omega^i_\lambda)\delta^I_i \tag{92}$$

where

$$I=(s\,,i\,,[ij],i{*}P) \tag{93}$$

runs from 1 to 16 in the usual fashion. Also,

$$\delta^I_j=\begin{cases}0 & \text{if } I\neq j\\ 1 & \text{if } I=j\end{cases}\,. \tag{94}$$

Remember that

$$\mathcal{L}=(\Lambda_{\mu\nu\alpha\beta})(g^{\mu\alpha}g^{\nu\beta}\sqrt{-g})\,, \tag{95}$$

when

$$P^I_{\nu\mu}=\omega^I_{\mu|\nu}-\frac{i}{2}\epsilon^I_{JK}\omega^J_\nu\omega^K_\mu\,, \tag{96}$$

and then

$$\Lambda_{\mu\nu\alpha\beta}=k\,\text{tr}(P^I_{\mu\nu}P^J_{\alpha\beta}\gamma_I\gamma_J) \tag{97}$$

with k a constant. Here it is assumed that

$$\omega^m_\mu\eta_{mn}\omega^n_\nu=m^2Y^m_\mu\eta_{mn}Y^n_\nu=m^2g_{\mu\nu}\,. \tag{98}$$

We note that when $J^i_\mu=0$ the equations of motion are almost the same as if the Dirac algebra had been gauged on a curved background manifold except that we have constructed the metric as

$$m^2g_{\mu\nu}=m^2Y^i_\mu\eta_{ij}Y^j_\nu=\omega^i_\mu\eta_{ij}\omega^j_\nu \tag{99}$$

from the vector fields. When ω^i_μ takes values in the vector representation of the Lorentz group and when $J^i_\mu=0$, the equations have the same solutions as those on a flat manifold for then

$$m^2g_{\mu\nu}=\omega^i_\mu\eta_{ij}\omega^i_\nu=m^2\eta_{\mu\nu}\,. \tag{100}$$

We can see that this system permits flat (or asymptotically flat) solutions. In particular take

$$\omega_\mu I=\omega^{mn}_\mu\sigma_{mn}=\omega^*_\mu(i\gamma_5)=0\,, \tag{101}$$

and set

$$\omega^m_\mu=mY^m_\mu=\omega^{*m}_\mu=m\delta^m_\mu\,. \tag{102}$$

This is a solution to the equations of motion (all curvatures vanish). The solution having zero curvature ($P^I_{\mu\nu}=0$) in the case of gauging only γ_k and σ_{mn} is a de Sitter manifold which is not asymptotically flat. It may be that in some region $\omega^{*m}_\mu\approx 0$ and the manifold looks locally de Sitter but may still be asymptotically flat. Thus one could imagine a number of baglike regions contained in an asymptotically flat universe of infinite radius despite apparent finite radii in certain regions. Quantization would not be the same as box quantization since a continuum of states must be included.

There is an underlying group-theoretic reason[17] for these facts which suggests that another way of constructing the vierbein from the connections ω_μ^I may be the correct way. The Lie algebra structure of the Dirac algebra is that of $U(2,2)\approx U(1)\times SU(2,2)$. SU(2, 2) is locally SO(4, 2), the conformal group. Indeed by taking $\hat{M}_{ij}\equiv\sigma_{ij}$, $\hat{D}\equiv(i\gamma_5)$, and $\hat{P}_i\equiv\frac{1}{2}(I+\gamma_5)\gamma_i$, and $\hat{K}_i\equiv\frac{1}{2}(I-\gamma_5)\gamma_i$, one finds the usual conformal group structure. Obviously P_i and K_i can be interchanged while switching signs on a few commutators. Please do not confuse $\hat{P}_i=\frac{1}{2}(I+\gamma_5)\gamma_i$ with $P_i=iD_i$ the Fermi gauge-covariant derivative.

Performing the Cartan decomposition $g=h+m$, with $h=[\hat{M}_{ij},\hat{D}]$ and $m=[\hat{P}_i,\hat{K}_j]$, respectively, generating $SO(3,1)\times SO(1,1)$ and $G_{4,2}$ the eight-dimensional Grassmann manifold of two-planes in R^{4+2}, respectively, we can organize the conformal group commutators to reveal a symmetric space structure for this group acting on the coset space, $G_{4,2}$.[17,18,19] The commutators are $[h,h]\subset h$, $[\hat{M}_{ij},\hat{M}_{kl}]=i\epsilon_{ijkl}^{mn}\hat{M}_{mn}$, $[\hat{M}_{ij},\hat{D}]=0$, and $[\hat{D},\hat{D}]=0$; h is a subgroup; and $[h,m]\subset m$, $[\hat{M}_{ij},\hat{P}_k]=i\epsilon_{ijk}^{l}\hat{P}_l$, $[\hat{M}_{ij},\hat{K}_k]=i\epsilon_{ijk}^{l}\hat{K}_l$, $[\hat{D},\hat{P}_i]=2i\hat{P}_i$, and $[\hat{D},\hat{K}_i]=-2i\hat{K}_i$, this property is reductive homogeneity; and finally $[m,m]\subset h$ since $[\hat{P}_i,\hat{P}_j]=0=[\hat{K}_i,\hat{K}_j]$ and $[\hat{P}_j,\hat{K}_j]=4i\hat{M}_{ij}-i\eta_{ij}\hat{D}$, symmetric space property. Here $\epsilon_{ijkl}^{mn}=\frac{1}{4}(\eta_{\bar{i}\underline{k}}\eta_{\underline{l}}^{\bar{m}}\eta_{\bar{j}}^{\bar{n}})$ and $\epsilon_{ijk}^{l}=\frac{1}{2}\eta_{\bar{i}}^{l}\eta_{\bar{j}k}$. Superbarred indices (ij) are antisymmetrized together as are subbarred indices (kl). From these relations it is evident that the Cartan decomposition $h=[\hat{M}_{ij},\hat{D},\hat{K}_l]$ and $m=(\hat{P}_l)$ yields only a homogeneous space representation for $g=h+m$ on m. It is neither reductive nor symmetric.

We note that γ_5 plays a role similar to * in that

$$\begin{aligned}&\gamma_5 I=\gamma_5,\quad \gamma_5\gamma_i=\frac{1}{3!}\epsilon_i^{jkl}\gamma_j\gamma_k\gamma_l,\\&\gamma_5\gamma_{\underline{i}}\gamma_{\underline{j}}=\tfrac{1}{2}\epsilon_{ij}^{k}\gamma_{\underline{k}}\gamma_{\underline{l}},\\&\gamma_5\gamma_{\underline{i}}\gamma_{\underline{j}}\gamma_{\underline{k}}=\epsilon_{ijk}^{l}\gamma_l,\ \text{and}\ \gamma_5\gamma_5=I,\end{aligned}\tag{103}$$

please see Table III of paper III for forms. This property implies that $\hat{D}I=\hat{D}$,

$$\begin{aligned}&\hat{D}\hat{P}_i=i\hat{P}_i,\ \hat{D}\hat{M}_{ij}=\tfrac{1}{2}\epsilon_{ij}^{kl}\hat{M}_{kl},\\&\hat{D}\hat{K}_i=-i\hat{K}_i,\ \text{and}\ \hat{D}\hat{D}=-I.\end{aligned}\tag{104}$$

We can rewrite the Dirac connection as

$$\Omega_\mu=(\omega_\mu I+\omega_\mu^{+i}\hat{P}_i+\omega_\mu^{ij}\hat{M}_{ij}+\omega_\mu^{-i}\hat{K}_i+\omega_\mu^{*}\hat{D}),\tag{105}$$

where $\omega_\mu^{+i}=\omega_\mu^i+\omega_\mu^{*i}$ and $\omega_\mu^{-i}=\omega_\mu^i-\omega_\mu^{*i}$. Since $\hat{P}_l$ and $\hat{M}_{ij}$ generate the Poincaré group, it is possible to assign $mY_\mu^i=\omega_\mu^{+i}$ instead of $mY_\mu^i=\omega_\mu^i$. The action is constructed as

$$\langle P|P\rangle=\int\bar{P}^*\wedge P=-\int P^*\wedge P\tag{106}$$

when

$$\bar{\gamma}_I=\gamma_0\gamma_I^\dagger\gamma_0=\gamma_I\ \text{ and }\ P_{\mu\nu}=\partial_{\underline{\mu}}\Omega_{\underline{\nu}}-i[\Omega_\mu,\Omega_\nu]\tag{107}$$

in

$$P=\frac{-i}{2}P_{\mu\nu}dx^\mu\wedge dx^\nu.\tag{108}$$

The equations of motion are

$$\frac{1}{\sqrt{-g}}\partial^\nu(\sqrt{-g}P_{\nu\mu}^L)-i\epsilon_{MN}^L\omega^M_\rho P_{\nu\mu}^N g^{\rho\nu}=J_\mu^L,\tag{109}$$

where ϵ_{MN}^L are U(2,2) structure constants for the basis

$$\hat{M}_L=(i,\hat{P}_i,\hat{M}_{ij},\hat{K}_l,\hat{D})\tag{110}$$

and

$$\begin{aligned}J_\mu^L=\delta^L_{(+i)}(&\Lambda_{\sigma\nu\alpha\beta}g^{\sigma\alpha}g^{\nu\beta}\omega^{(+i)}_\mu\\&-4\Lambda_{\mu\nu\alpha\beta}g^{\nu\beta}g^{\alpha\lambda}\omega^{(+i)}_\lambda),\end{aligned}\tag{111}$$

with

$$\Lambda_{\mu\nu\alpha\beta}=k\,\mathrm{tr}P_{\mu\nu}P_{\alpha\beta},\tag{112}$$

unless there are further sources.

The pieces of $P_{\nu\mu}^M$ can be written out as follows:

$$\begin{aligned}&f_{\nu\mu}=A_{\underline{\mu}|\underline{\nu}},\\&T_{\nu\mu}^{\pm m}=\omega_{\underline{\mu}|\underline{\nu}}^{\pm m}+\omega_{\underline{\nu}k}^{m}\omega_{\underline{\mu}}^{\pm k}\pm 2A^*_{\underline{\nu}}\omega_{\underline{\mu}}^{\pm m},\\&P_{\nu\mu}^{mn}=(\omega_{\underline{\mu}|\underline{\nu}}^{mn}+\omega_{\underline{\nu}k}^{m}\omega_{\underline{\mu}}^{kn})-8(\omega_{\underline{\nu}}^{+m}\omega_{\underline{\mu}}^{-n}),\end{aligned}\tag{113}$$

and

$$f^*_{\nu\mu}=A^*_{\underline{\mu}|\underline{\nu}}+2\omega_{\underline{\nu}}^{+m}\omega_{\underline{\mu}}^{-n}\eta_{mn}.$$

As before, the solution $\omega_\mu^{+i}=m\delta_\mu^i$ with all other coefficients zero is possible. This is flat space. Clearly, the linear term in $R_{\nu\mu}^{mn}$ vanishes if $\omega_\mu^{-n}=0$ prior to variation; thus it should not be taken to be zero before the variation.

The Fermi gauge-covariant derivative is given as follows:

$$\begin{aligned}P_\mu\Psi=(i\partial_\mu+A_\mu I+\omega_\mu^{+m}\hat{P}_m+\omega_\mu^{mn}\hat{M}_{mn}\\+\omega_\mu^{-m}\hat{K}_m+A^*_\mu\hat{D})\Psi.\end{aligned}\tag{114}$$

Now we will see that we can gauge a subgroup of the simple supergroup[3] SL(4|1) over a (4,1) (Bose, Fermi) base manifold and find the theory we have described. First let us find the SL(4|1) subgroup corresponding to skew-Dirac conjugate elements. That is, using $\overline{(A)}\equiv\bar{A}$ for typographical reasons,

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^\dagger & c^\dagger \\ b^\dagger & d^\dagger \end{pmatrix} \begin{pmatrix} \gamma_0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma_0 a^\dagger 0 & -\gamma_0 c^\dagger \\ -b^\dagger \gamma_0 & d^\dagger \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \tag{115}$$

This can be accomplished if $a = -i\omega^K \gamma_K$, $d = -4i\omega$, $b = \Psi$, $c = \Psi^\dagger \gamma_0 = \overline{\Psi}$, since $\mathrm{tr}\, a - d = 0$, where ω is the coefficient of I. We will denote this supergroup by $D(4|1)$ for Dirac. Taking a connection with coefficients in $D(4|1)$

$$-i\Omega = \begin{pmatrix} -i\omega_\mu^K \gamma_K dx^\mu & \Psi d\theta \\ \overline{\Psi} d\theta & -4i\omega_\mu dx^\mu \end{pmatrix}, \tag{116}$$

we calculate the curvature ($d d\theta$ terms are projected out)

$$-i\mathcal{P} = d(-i\Omega) + (-i\Omega) \wedge (-i\Omega) = \begin{bmatrix} -(i/2)(\omega^K_{\underline{\mu}|\nu} - i\epsilon^K_{LM}\omega^L_\nu \omega^M_\mu) dx^\nu \wedge dx^\mu & (\Psi|_\mu - i\omega^K_\mu \gamma_K \Psi + 4i\Psi\omega_\mu) dx^\mu \wedge d\theta \\ (\overline{\Psi}|_\mu + i\overline{\Psi}\gamma_K \omega^K_\mu - 4i\overline{\Psi}\omega_\mu) dx^\mu \wedge d\theta & -(4i/2)\omega_{\underline{\mu}|\nu} dx^\nu \wedge dx^\mu \end{bmatrix}. \tag{117}$$

Here we have used the graded wedge product and a connection with base coefficients, as we did at the end of section three. $[\overline{\Psi}(x), \Psi(x)] d\theta \wedge d\theta = 0$, cf., Berezin,[15] p. 52, since $d\theta \wedge d\theta$ vanishes by symmetry and antisymmetry. This wedge product is antisymmetric for Grassmann differentials just as it is antisymmetric for Bose-Bose or Bose-Fermi two-forms. Now we can construct the action form for a Bose-Fermi manifold as in Sec. III. To recover the $(-\frac{1}{4})$ coefficient of $f_{\nu\mu} f^{\nu\mu}$ and (+1) coefficient for Ψ we take the inner product $\circ = (\frac{1}{8}k\, \mathrm{tr})$ with $k = (9e^2/5)$, set $A_\mu = (-3/e)\omega_\mu$ and reset $\Psi = n\Psi'$ with $n = (2\sqrt{5}/3)e$. Then we find

$$A = \langle \mathcal{P} | \mathcal{P} \rangle = \int \tfrac{1}{8} k\, \mathrm{tr}(\overline{\mathcal{P}}^* \wedge \mathcal{P}), \tag{118}$$

the action. We pin $g^{\mu\nu}$ to $[\omega^{+i}_\mu \eta_{ij} \omega^{+j}_\nu (1/m^2)]$ for definiteness, and in components we write

$$\mathfrak{a} = \int \Big[-\Big(\tfrac{1}{4} f_{\nu\mu} f^{\nu\mu} + \frac{k}{4} T^{+m}_{\nu\mu} T^{\nu\mu}_{+m} + \frac{k}{16} P^{mn}_{\nu\mu} P^{\nu\mu}_{mn} + \frac{k}{4} T^{-m}_{\nu\mu} T^{\nu\mu}_{-m} + \frac{k}{4} f^*_{\nu\mu} f^{*\nu\mu} \Big) + [\overline{\Psi}_{|\mu} + i\overline{\Psi}(eA_\mu I + \omega^{\overline{K}}_\mu \gamma_{\overline{K}})][\Psi^{|\mu} - i(eA^\mu I + \omega^{\mu\overline{K}} \gamma_{\overline{K}})\Psi] \Big] \sqrt{-g}\, d^4x, \tag{119}$$

where $\int \theta d\theta = 1$ has been performed and $\gamma_K = (I, \gamma_{\overline{K}})$; $\overline{K}$ runs over the 15 conformal labels. Ψ' has been written as Ψ. Clearly, we have recovered the action we were describing previously from a more intuitive standpoint.

Now the extension to internal symmetries can be accomplished, and a truly unified theory of curvature, matter, and internal symmetry is possible. In particular, define the supergroup $D(4|m)$ to be the subgroup of $\mathrm{SL}(4|m)$ which is Dirac skew-conjugate under

$$\overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \begin{bmatrix} \gamma_0 & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger \begin{bmatrix} \gamma_0 & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}. \tag{120}$$

We define $\overline{(A)} \equiv \overline{A}$ for typographical reasons. One find bosons $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ in $\mathrm{SU}(2,2) \times \mathrm{U}(1) \times \mathrm{SU}(m)$ and fermions

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Psi \\ \overline{\Psi} & 0 \end{bmatrix}, \tag{121}$$

where $\overline{\Psi} = \Psi^\dagger \gamma_0$ and Ψ is a matrix carrying both $\mathrm{SU}(2,2)$, conformal or spinor, labels and $\mathrm{SU}(m)$ labels. If one makes the requirement that the number of Bose and Fermi generators be equal, one finds $\mathrm{OSp}(2n|m)$ with $m = 2n$ or $2n+1$ and $\mathrm{SU}(n|m)$ with $m = n \pm 1$. When $n = 4$, $\mathrm{SU}(4|5)$ and $\mathrm{SU}(4|3)$ are selected. If one puts $D(4|m)$ on a (4,1) (Bose, Fermi) manifold, the action can be constructed as before. Setting $\mathrm{tr}\Lambda_A \Lambda_B = g^2 \delta_{AB}$, for Λ_A in su(m), and taking

$$k = \frac{m}{e^2} \left[\frac{(4-m)^2}{m+4} \right], \quad m \geq 1 \tag{122}$$

and $n^2 = k/4$ one finds [with $B^A \Lambda_A dx^\mu$ in su(m)]

$$\mathfrak{a} = \int \Big\{ -\tfrac{1}{4} \Big[f_{\nu\mu} f^{\nu\mu} + \frac{k}{4} T^{+m}_{\nu\mu} T^{\nu\mu}_{+m} + \frac{k}{16} P^{mn}_{\nu\mu} P^{\nu\mu}_{mn} + \frac{k}{4} T^{-m}_{\nu\mu} T^{\nu\mu}_{-m} + \frac{k}{4} f^*_{\nu\mu} f^{*\nu\mu} + \Big(\frac{kg^2}{16}\Big) f^A_{\nu\mu} f^{\nu\mu}_A \Big] + [\overline{\Psi}_{|\mu} + i\overline{\Psi}(eA_\mu I + \omega^{\overline{K}}_\mu \gamma_{\overline{K}} + B^A_\mu \Lambda_A)][\Psi^{|\mu} - i(eA^\mu I + \omega^{\mu\overline{K}} \gamma_{\overline{K}} + B^{\mu A}\Lambda_A)\Psi] \Big\} \sqrt{-g}\, d^4x. \tag{123}$$

The connection is

$$-i\Omega=\begin{bmatrix}-i(\omega_\mu I_4+\omega_\mu^K\gamma_K)dx^\mu & \Psi d\theta \\ \bar{\Psi}d\theta & -i[-(4/m)\omega_\mu I_m+B_\mu^A\Lambda_A]dx^\mu\end{bmatrix}, \tag{124}$$

and duality is defined as in Sec. II. Note that when the functional integrals are defined carefully, the Minkowski structure is replaced by a Euclidean one; in so doing the su(2,2) algebra is replaced by an su(4) algebra. This removes the problem of negative energy states.

It is clear from the underlying group theory that no parity-violating terms appear in the action; SU(m) couplings are vectorlike, and the theory will have no anomalies.[13,20]

Higgs fields can be included in the fashion described in Sec. II. Also we point out that the base manifold can be associated with a coset structure generated by $m=[\hat{P}_i]$ or by $m=[\hat{P}_i;\bar{a}_\alpha s^\alpha]$ for a (4, 1) (Bose, Fermi) manifold. Here $\bar{a}_\alpha s^\alpha$ picks out only one generator. If we want to use a (4,8) manifold we can replace $d\theta$ by $\theta_i d\theta_i+\theta_i d\bar{\theta}^i$.

Earlier in this section we noted that the canonical one-forms, $\theta=Y_\mu^i\gamma_i dx^\mu$, could be identified with a piece of the connection form

$$\Omega=(\omega_\mu I+\omega_\mu^i\gamma_i+\omega_\mu^{ij}\sigma_{ij}+\cdots)dx^\mu, \tag{125}$$

if we assumed that $\omega_\mu^i=mY_\mu^i$. Let us examine this concept more closely. Basically, three view points on this relationship are possible. First, one can view the four-manifold as a submanifold of the sixteen-dimensional Dirac manifold with the basis one-forms chosen by identification with a four-dimensional coset, e.g., $\omega_\mu^i=mY_\mu^i$ or $\omega_\mu^{*i}=mY_\mu^i$.[17,18,19] Second, one can consider the Dirac algebra to be a sort of internal symmetry (though noncompact) on a four-dimensional curved manifold. It is then an ansatz that the canonical one-forms are proportional to four of the group's generators (e.g., the γ_i sector) and the connection one-forms are proportional to six other generators (e.g., the σ_{ij} sector). Thus one might set

$$mY_\mu^i=\omega_\mu^i$$
and
$$\Gamma_\mu^{ij}=\omega_\mu^{ij}. \tag{126}$$

The factor of m is needed since Y_μ^i is dimensionless. We will see that the action for the gauge and spin-3/2 fields can be written without explicit reference to Y_μ^i. The mass scales out. Thus a third possibility presents itself. In this view a rational (rather than polynomial) action is given for the local symmetry structure. One does not attempt to interpret the theory as being that of a curved manifold. Instead it exists on a flat background but has interactions which mock-up curved space.

For simplicity let us write only the contribution due to the gauging of ω_μ^i and ω_μ^{kj}. The action can be written as

$$\mathfrak{a}=-\tfrac{1}{4}\int(T_{\mu\nu}^{\ m}\eta_{mk}T_{\rho\lambda}^{\ k}+P_{\mu\nu}^{mn}\eta_{mk}\eta_{n\ell}P_{\rho\lambda}^{k\ell})\times\left[\frac{\omega_i^\mu\,\omega_j^\nu\omega_s^\rho\omega_t^\lambda\eta^{is}\eta^{jt}}{\omega_r^\sigma\omega_u^\tau\omega_v^\eta\omega_w^\xi\epsilon^{ruvw}\epsilon_{\sigma\tau\eta\xi}}\right]d^4x, \tag{127}$$

where

$$T_{\nu\mu}^m=\omega_{\underline{\mu}|\underline{\nu}}^m-\frac{i}{2}\epsilon_i^{\ m}{}_{k\ell}\,\omega_{\underline{\nu}}^i\omega_{\underline{\mu}}^{k\ell},$$
$$T_{\nu\mu}^{mn}=\omega_{\underline{\mu}|\underline{\nu}}^{mn}-\frac{i}{2}\epsilon_{ij}{}^{mn}{}_{k\ell}\,\omega_{\underline{\nu}}^{ij}\omega_{\underline{\mu}}^{k\ell}-\frac{i}{2}\epsilon_i^{\ mn}{}_k\,\omega_{\underline{\nu}}^i\omega_{\underline{\mu}}^k. \tag{128}$$

The other terms are included in the usual way (sum of squares).

The de Sitter group can be contracted to find the Poincaré group; ϵ_{ik}^{mn} then vanishes. Or the Poincaré subgroup of U(2,2) may be employed,

$$\epsilon_i^{\ m}{}_{k\ell}=\frac{i}{2}\eta_{\underline{\ell}}^m\eta_{i\underline{k}},\quad \epsilon_{ij}^{\ mn}{}_{k\ell}=\frac{i}{4}\eta_{I\underline{k}}\eta_{\underline{\ell}}^m\eta_j^n,$$
and
$$\epsilon_i^{\ mn}{}_k=4(-i)\eta_{\underline{I}}^m\eta_{\underline{k}}^n. \tag{129}$$

We point out that explicit reference to Y_μ^i can be omitted since the bracketed expression is scale independent. Please note also that what appears in the bracket is the inverse matrix ω_i^μ, not ω_μ^i. But, of course, ω_i^μ can be expressed in terms of ω_μ^i by using cofactors and determinants,

$$\left[\frac{\omega_i^\mu\omega_j^\nu\omega_s^\rho\omega_t^\lambda\eta^{is}\eta^{jt}}{\omega_r^\sigma\omega_u^\tau\omega_v^\eta\omega_w^\xi\epsilon^{ruvw}\epsilon_{\sigma\tau\eta\xi}}\right]_{\text{CL or QM}}=\left[\frac{Y_i^\mu Y_j^\nu Y_s^\rho Y_t^\lambda\eta^{is}\eta^{jt}}{Y_r^\sigma Y_u^\tau Y_v^\eta Y_w^\xi\epsilon^{ruvw}\epsilon_{\sigma\tau\eta\xi}}\right]. \tag{130}$$

The classical solution is viewed as the lowest-order QM solution.

It is interesting to observe that if a spin-3/2 gauge field is introduced, it will behave like the other gauge fields. Specifically, let the connection be

$$d[\bar{s}_\alpha,\bar{s}]=[\bar{s}_\beta,\bar{s}]\begin{pmatrix}-i\Omega_\mu^I(\gamma_I)_\alpha^\beta & i\chi_\mu^\beta\\ 0 & 0\end{pmatrix}. \tag{131}$$

The additional contribution to the action will be

$$\mathfrak{a}_{3/2}=\int(\bar{\chi}_{\underline{\mu}}\bar{P}_{\underline{\nu}}P_{\underline{\lambda}}\chi_{\underline{\rho}})\left[\frac{\omega_i^\mu\omega_j^\nu\omega_s^\lambda\omega_t^\rho\eta^{is}\eta^{jt}}{\omega_r^\sigma\omega_u^\tau\omega_v^\eta\omega_w^\xi\epsilon^{ruvw}\epsilon_{\sigma\tau\lambda\xi}}\right]d^4x, \tag{132}$$

with

$$[P_\mu\chi_\nu]^\alpha=[(i\partial_\mu+eA_\mu)I_\beta^\alpha+\omega_\mu^i(\gamma_i)_\beta^\alpha+\lambda\omega_\mu^{ij}(\sigma_{ij})_\beta^\alpha+\omega_\mu^{*i}(\gamma_5\gamma_i)_\beta^\alpha+e^*A_\mu^*(i\gamma_5)_\beta^\alpha]\chi_\nu^\beta. \tag{133}$$

Now it is not necessary to know the specific relation of ω_μ^i to Y_μ^i. The discussion given for Ω_μ now applies to χ_μ also. We have used spin-$\frac{3}{2}$ potentials to obtain a quadratic action. However, the usual spin-$\frac{3}{2}$ superaction can also be written in terms of ω_μ^i instead of Y_μ^i. No explicit mention of the dimensionless Y_μ^i fields to be made for either gauge or spin-$\frac{3}{2}$ fields.

In the quantum version of the theory a field of fluctuations h_μ^i about the classical field exists such that

$$\omega_\mu^i = m\delta_\mu^i + h_\mu^i = m\left(\delta_\mu^i + \frac{1}{m} h_\mu^i\right) = m(Y_\mu^i)\,.$$

or (134)

$$\omega_\mu^i = mY_\mu^i + h_\mu^i = (\omega_\mu^i)_{\rm CL} + (\omega_\mu^i)_{\rm QM}$$

If either ω_μ^i or Y_μ^i is taken to be fundamental, h_μ^i still has dimensions of mass. Note that we can consider fluctuations h_μ^i about an *arbitrary* classical solution, $(\omega_\mu^i)_{\rm CL}$ but use only the classical field in Eq. (130). Then fluctuations occur in the fields but *not* in the manifold. This would amount to quantizing on a consistent, curved background manifold. It should be a less complicated theory.

Because of Ward identities required to preserve the invariances (gauge, local Lorentz, and recoordinatization), we expect that ω_μ^i can be set to $m\delta_\mu^i$ (or $g_{\mu\nu}$ set to $\eta_{\mu\nu}$) at a point. Thus after renormalization we anticipate that in the neighborhood of a point the bracketed expression will be led by $\eta^{\mu\rho}\eta^{\nu\lambda}$. So the short-distance behavior should be controlled by the coefficient of the brackets.[21,22] In this case it is the Lagrangian term for a pure gauge theory (when contracted with $\eta^{\mu\rho}\eta^{\nu\lambda}$). Thus the short-distance behavior should be that of a pure gauge theory. In other words the theory should be renormalizable. This idea could be extended to the entire Dirac Lie algebra in a similar fashion. We have not included a discussion of the effect of ghosts in this heuristic argument. The point is merely that the action is expected to be renormalizable.

Assuming that this idea can be made into a proof, we still would have difficulty with the spin-$\frac{1}{2}$ fields. For them we need the specific relationship between ω_μ^i and Y_μ^i, since the Y_μ^i are constrained fields and ω_μ^i is dynamical; because the Fermi action for the full Dirac coupling is given as

$$\mathcal{A}_{1/2} \sim \int [\overline{\Psi} P_\mu P_\nu \Psi]\left[\frac{Y_i^\mu Y_j^\nu \eta^{ij}}{\det Y}\right] d^4x\,, \tag{135}$$

$$P_\mu\psi = [i\partial_\mu + eA_\mu)I + \omega_\mu^i\delta_i + \lambda\omega_\mu^{ij}{}_{ij}\,\sigma_{ij} + \omega_\mu^{*i}\gamma_5\gamma_i + e^*A_\mu^*(i\gamma_5)]\psi\,. \tag{136}$$

There is no similar way to write the quantity in brackets in $A_{1/2}$ entirely in terms of ω_μ^i without explicit use of a scale. But typically there is some region (or at least a point) in which the metric becomes Minkowski. There, $\omega_\mu^i = m\delta_\mu^i$ so that

$$g_{\mu\nu} = \frac{1}{m^2}\,\omega_\mu^i\eta_{ij}\omega_\nu^j = \eta_{\mu\nu}\,. \tag{137}$$

Let us denote this special value for ω_μ^i by a special symbol $\epsilon_\mu^i = m\delta_\mu^i$. The existence of a local (orthonormal) Lorentz frame at a point means that there ω_μ^i can be set equal to ϵ_μ^i. The choice of location of the Lorentz site (s) is a boundary condition. One can write

$$g_{\mu\nu} = 16\left[\frac{\omega_\mu^i\eta_{ij}\omega_\nu^j}{({\rm tr}\epsilon)^2}\right] \tag{138}$$

to emphasize that the value of $m = \frac{1}{4}{\rm tr}\epsilon$ is a boundary condition. But the critical point is that the action for spin-$\frac{1}{2}$ or spin-0 matter fields requires the introduction of a scale. Still it may be hoped that the short-distance behavior of the $\mathcal{A}_{1/2}$ contribution to the action will be controlled by the coefficient of the quantity in parentheses in a similar fashion to that of the gauge and spin-$\frac{3}{2}$ fields since the recoordinatization and other invariances should be preserved by the quantum theory.[12,21] Thus at a point the bracketed quantity

$$\left[\frac{Y_i^\mu\eta^{ij}Y_j^\nu}{\det Y_k^\beta}\right] = \left[\frac{\omega_i^\mu\eta^{ij}\omega_j^\nu({\rm tr}\epsilon_i^a)^2}{16\,{\rm tr}\omega_k^\beta}\right]_{\rm CL\ or\ QM} \tag{139}$$

can be adjusted to be $\eta^{\mu\nu}$ by sending ω in to ϵ_μ^i. If in Eq. (139) $Y_\mu^i \sim (\omega_\mu^i)_{\rm CL}$ not $(\omega_\mu^i)_{\rm QM}$, then the theory should be easier to quantize.

Then the coefficient should govern the short-distance (ultraviolet) behavior. Of course since we have not worked out the details of the operators, it is not clear that the naive Ward identities will be preserved. We intend this merely to be a plausibility argument, not a proof.

In closing we will give the Lagrangian, classical equations of motion for the supergroup $D(4|n)$. By an obvious extension, the Lagrangians and equations of motion for SL$(m|n)$ can be constructed.

Let

$$\begin{aligned}
&\Omega_\mu \equiv \omega_\mu^a\gamma_a \equiv \omega_\mu^I\gamma_I + B_\mu^A\Lambda_A\,,\\
&D_\mu\Psi = (\partial_\mu - i\Omega_\mu)\Psi\,,\\
&P_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - i[\Omega_\mu,\Omega_\nu]\,,
\end{aligned} \tag{140}$$

and

$$m^2 g_{\mu\nu} = \omega_\mu^i\eta_{ij}\omega_\nu^j\,.$$

ω_μ^i comes from an appropriately chosen four-dimensional coset of $D(4|n)$ such as that generated by γ_i or by $\frac{1}{2}(I+\gamma_5)\gamma_i$. A includes both I and A labels. I takes 16 values; A takes n^2-1 values, unless $n=4$. For $n=4$ the generator with $+1$ on the upper diagonal and -1 on the lower diagonal is removed.

Define

$$\Lambda_{\mu\nu\alpha\beta}=\mathrm{tr}(\overline{P}_{\mu\nu}P_{\alpha\beta})$$

and

$$\Delta_{\mu\nu}=\mathrm{tr}(\overline{\Psi}\overline{D}_{\mu}D_{\nu}\Psi)\,. \tag{141}$$

The action is

$$A=\int(-\Lambda_{\mu\nu\alpha\beta}g^{\mu\alpha}g^{\nu\beta}+\Lambda_{\mu\nu}g^{\mu\nu})\sqrt{-g}d^4x\,. \tag{142}$$

Let us choose coupling constants such that the matrices γ_I and Λ_A are normalized to 1 with the trace as metric.

Normalization:

$$\mathrm{tr}(\overline{\gamma}_I\gamma_J)=\eta_{IJ}\,,\quad \mathrm{tr}(\overline{\Lambda}_A\Lambda_B)=\delta_{AB}\,. \tag{143}$$

Any other normalization is easily found from this. The equations of motion are (assuming the usual boundary conditions)

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}P^a_{\nu\sigma})-i\epsilon^a_{bc}\,\omega^b_\mu g^{\mu\nu}P^c_{\nu\sigma}=J^a_\sigma\,,\quad \frac{1}{\sqrt{-g}}D_\mu(g^{\mu\nu}\sqrt{-g}D_\nu\Psi)=0\,. \tag{144}$$

The current J^a_σ can be decomposed into three pieces: one external piece which is assumed to vanish and two internal pieces constructed as follows:

total:

$$J^a_\sigma=J^a_{u\sigma}+J^{\,a}_{g\sigma}+J^a_{e\sigma}\,, \tag{145}$$

external:

$$J^a_{e\sigma}=0\,, \tag{146}$$

gravitational:

$$J^a_{g\sigma}=\delta^a_i[(\Lambda_{\mu\nu\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\omega^i_\sigma-4\Lambda_{\sigma\nu\alpha\beta}g^{\alpha\lambda}g^{\nu\beta}\omega^i_\lambda)-(\Delta_{\mu\nu}g^{\mu\nu}\omega^i_\sigma-\Delta_{\sigma\nu}g^{\nu\lambda}\omega^i_\lambda-\Delta_{\nu\sigma}g^{\nu\lambda}\omega^i_\lambda)]\,, \tag{147}$$

usual:

$$J^a_{u\sigma}=i(\overline{\Psi}_{|\mu}\gamma^a\Psi-\overline{\Psi}\gamma^a\Psi_{|\mu})-2\omega^a_\mu\overline{\Psi}\Psi\,. \tag{148}$$

Assuming the usual relationship between Ψ and ψ in the Dirac basis, consider C, PT, and $(i\gamma_5)$ transformations on the Dirac equation (for $\sqrt{-g}=1$)

$$\begin{aligned}0&=[P]\psi\\&=[i\partial\!\!\!/+(eA\!\!\!/-i\lambda\tilde{\phi\!\!\!/})-(ie^*A\!\!\!/^*\gamma_5+\lambda\phi\!\!\!/^*\gamma_5)\\&\quad+4mI+m^*Y^{*i}_{\mu}Y^{\mu}_{i}\gamma_5+4m^*Y^{*i}_{\mu}Y^{\mu}_{k}(i\gamma_5\sigma^k_i)]\psi\,.\end{aligned} \tag{149}$$

The charge-conjugation operator gives $\psi_c=i\gamma^2\tilde{\psi}$ (the ~ on ψ will denote complex conjugation) and replaces e and e^* by $-e$ and $-e^*$ in P, thus defining P_c such that $P_c\psi_c=0$. The operator PT gives

$$\psi_{PT}(x)=(\gamma^0)(i\gamma^1\gamma^3)\tilde{\psi}(-x)\,, \tag{150}$$

and sends m, m^*, and e and e^* into $-m$, $-m^*$, and $-e$ and $-e^*$ defining P_{PT} such that $P_{PT}\psi_{PT}=0$. Note $\psi_P=\gamma_0\psi$. Further, acting on ψ with $D=i\gamma_5$ to give $\psi_D=i\gamma_5\psi$ has the effect of sending m and m^* into $-m$ and $-m^*$ in P defining P_D such that $P_D\psi_D=0$. D can be called γ_5 conjugation. Evidently the matrices of $CPTD=I$ since the complex conjugations of C and T cancel and the matrices yield $(i\gamma^2)(\gamma^0 i\gamma^1\gamma^3)(i\gamma^5)=I$. Of course these operations look slightly different in the second-quantized version as can be seen in Bjorken and Drell.[12]

IV. SUMMARY

In summary, we have described a general theory which relates the algebras of infinitesimal basis transformations to a number of physically interesting theories. The procedure is quite simple. First one introduces a basis for the various fields to be considered, $\overline{s}_A$. Second one gives a law for their displacement, $\delta\overline{s}_A=\overline{s}_B\Omega^B_A(\delta)$. We considered only the cases for which Ω^B_A took values in the zero- and one-form sectors of the base manifold (space-time) exterior algebra. Third, one compares the effect of making a second-order displacement first one way then the other. The result is

$$\overline{s}_BP^B_A=\tfrac{1}{2}[\delta,d]\overline{s}_A=\tfrac{1}{2}\overline{s}_B[\delta\Omega^B_A(d)-d\Omega^B_A(\delta)+\Omega^B_C(\delta)\Omega^C_A(d)-\Omega^B_C(d)\Omega^C_A(d)] \tag{151}$$

The coefficient P^B_A is called the curvature. Fourth, one introduces another curvaturelike object having a similar pattern of forms K^B_A, which may be identical with P^B_A. It leads to either linear or quadratic actions. Fifth, setting $K=\overline{s}_AK^A_Bs^B$ and $P=\overline{s}_AP^A_Bs^B$, we introduce the Hilbert product $\langle K|P\rangle=k\int\mathrm{Re}[\mathrm{tr}(\overline{K}^*\wedge P)]$. Here, the overbar denotes conjugation and the asterisk denotes Hodge duality. We require reality of the action. This Hilbert product is the action. Its minima are the classical equations of motion. Its path integral yields the quantum version of the theory. Along the way we were led to several interesting points: (a) Axial-vector terms occur in the coupling of fermions to gravity; (b) a unified theory having electromagnetism, mass gravitation, and possible axial mass and axial electromagnetism arises from gauging the four-dimensional Dirac algebra; (c) basis frames for space-time e_μ can be replaced by currents of spinor bases $\overline{s}\cdot\gamma_\mu\cdot s$ in analogy with viewing currents j^μ as $\overline{\chi}\gamma^\mu\chi$; (d) we have also found potentials for fermions analogous to the potentials for electromagnetism; (e) a unified theory of fermions, gravity, and internal fields arose from gauging the supergroups $D(4|n)\subset\mathrm{SL}(4|n)$. We discussed its renormalizability.

ACKNOWLEDGMENTS

I would like to thank the members of the theory group at LASL for help and interesting discussions. In particular I would like to thank R. Slansky, M. M. Nieto, D. H. Sharp, G. B. West, G. J. Stephenson, J. Kiskis, and T. Goldman. I would also like to thank M. Gell-Mann for encouragement and interesting discussions.

This work was supported in part by the U.S. Department of Energy.

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