Fermions and bosons in a unified framework. I. Physical foundations

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We begin with a general introduction to the theory of moving frames. A moving frame is a tangent vector basis equipped with a connection (gauge field). We then construct extended moving frames by adding extra, normal basis vectors to a basis for the tangents. The coefficients of transformation of the basis under infinitesimal displacement are the components of the connection (on tangents) and gauge field (on normals). The action is constructed from a generalized Hilbert product of the curvatures, the coefficients of the bases' transformation for a closed, infinitesimal (second-order) loop. We give the Einstein-Yang-Mills action as an example. Bose matter fields are included by first adding on at least one more basis element. Then one can either introduce an auxiliary coordinate differential, $d\theta$, which may be Bose or Fermi, with which the field is associated. Or one can extend the concept of a connection to one which takes values which are functions (as well as coefficients of differentials). This technique is actually related to the use of graded Lie algebras. A description of the simple superalgebras which we will use is included in the Introduction. From the generalized curvature the action for a scalar matter field coupled to its gauge bosons is formed. Fermi fields are introduced by establishing spinor bases which transform as the spin-1/2 Lorentz representation of the local orthonormal vector bases. The Dirac-Einstein action is given as an example. The connection acts as the gauge field for the fermions. Interestingly, they are coupled to the connection via an axial-vector term. This leads to a problem with anomalies. When spinor bases were introduced, the vector bases appeared as composites (currents). We discuss the idea that vector coordinates may be composites of underlying spinor coordinates.

I. INTRODUCTION

The idea that all the fields of physics, gravitational, internal, and matter, are manifestations of a single underlying structure is compelling in its beauty. A long history of attempts to construct such a unified theory exists.[1] But it was not until the concept of supersymmetry was developed that any reasonable hope of including matter fields presented itself.[2] In this Introduction we will sketch the development of the local (algebraic) approach to unification presented in this series of papers.

Since our intention is to include gravity, we have looked for a formulation which allows one to understand all the fields in a pictorial or geometrical fashion. We have chosen a technique which is not as rigorous as possible but which can be translated into a strict formalism through use of the theory of fiber bundles,[3,4] and yet employs notation which is only slightly different from the usual. See Secs. II and III of paper III (Ref. 4) for a sketch of how the rigor can be improved. It is interesting to discover that being more rigorous involves thinking about gauge freedom, a problem examined by Faddeev and Popov.[5] Though similar in many respects to the constructions of Utiyama and Kibble,[6] the history of this idea dates back at least to Cartan.[7]

Simply put, the idea is to extend a description of the base manifold in terms of independent, *basis* vector fields to include extra, internal, *basis* fields. A vector field is then written by giving its components in this basis. In the general case both components and bases can change under a displacement. To visualize this concept imagine a sphere being described locally by two independent vector fields. The normal vector field provides a picture of the extra, internal vector fields. As one moves about on the surface of the sphere, the local independent vectors tangent to the sphere will transform in some well-defined fashion. The normal vector field(s) can shrink, expand, or rotate in some other, quite general, fashion. The coefficient of transformation of the internal field which gives the difference in magnitude of the normal basis vector(s) between nearby (infinitesimally separated) points can be identified as the internal gauge field.[7] Similar differences between the tangent local basis vectors are identified as the (space-time) connection. Thus we will refer to the set of coefficients as either the gauge field or the connection.

By transporting this triad of vertical and horizontal vectors along two different small displacements and by finding the difference in values as the displacements shrink to zero (or by taking a small closed line integral on a singly sheeted surface) one can obtain the Riemann curvature of the manifold as the coefficient of the tangent (or horizontal) bases and the gauge field tensor as the coefficient of the internal (normal or vertical) bases. Details are found in Sec. II of this paper.

The extension of this idea to higher-dimensional base manifolds or more internal symmetries is conceptually similar and is only slightly more technically complicated. This use of independent vector fields for a description of curvature is familiarly known as the vierbein or tetrad formalism[8,9] when it is applied to the usual space-time manifold.

This local geometric formulation of curvature and gauge fields allows one to study topological aspects of these theories; see Sec. II of paper III.[4] Indeed an increasing number of papers exploiting topological concepts for an analysis of gauge theories has been appearing.[10] It might even be hoped that the increase in clarity of description afforded by this differential geometric language may lead to further insight into the physical nature of gauge theories. Such an apparent miracle of language has certainly happened before. Think of the Lagrangian formulation of classical mechanics. In fact, part of what we will be constructing is a theory of Lagrangians.[4] We will view them as a kind of Hilbert product[11] of curvatures (generalized, as above, to include internal-symmetry fields).

The Hilbert product of two functions is viewed as the integral of a function times an n-form (volume form times a function; see Sec. II of paper III).[4,8,11] The generalization to line elements or area elements proceeds by viewing them as one- or two-forms, respectively, etc. They are then multiplied by $(n-1)$-forms or $(n-2)$-forms, etc. and integrated over the n-manifold. If complex matrices of forms are considered, an appropriate adjoining operation (e.g., Hermitian conjugation) is included on the $(n-1)$-forms or $(n-2)$-forms, etc. For those unfamiliar with forms, descriptions and discussions can be found in Sec. II of this paper and Sec. II of paper III.[4]

But what about the inclusion of matter? In Sec. III we see that there are a number of approaches to this question. Since in this viewpoint gauge fields B_μ turn out to be coupled to the infinitesimal space-time displacement δx^μ through the label μ, $B=B_\mu dx^\mu$ and since matter fields ϕ do not carry this space-time vector index, one has a variety of options to explore. One can associate a matter field ϕ with the differential of an extra (fifth) degree of freedom, $d\theta$, which could be Bose matter, in which case it must effectively disappear from all relevant quantities. This can be effected in a manner similar to that suggested by Kaluza and Klein.[12] That is, it must be a small, compact (or circular) degree of freedom attached to each point of space-time, and fields are independent of it (the cylindricity assumption). If, however, the extra coordinate is a Grassmann (anticommuting) variable getting rid of it is quite easy and cylindricity is almost automatic.

Another approach is to extend the concept of connection. From the four independent displacements dx^μ, one can construct six independent arc elements, four independent three-volume elements, and one four-volume element; see Table I of paper III.[4] However, we have omitted the basis element I associated with points. Just as dx^μ is a basis for $B=B_\mu dx^\mu$, the gauge fields, it is possible to view the identity I as a basis for functions $\Phi=\phi I$.[4] We extend the concept of connection from just things proportional to displacements to include things proportional to the identity. We call such an extended connection an augmented connection. No cylindricity assumption is required at all. Further, when one computes the curvature of an augmented connection (the augmented curvature), one piece is the gauge field tensor and the other is the gauge-covariant derivative of matter, unless the base manifold (space-time) is curved, in which case there are curvature contributions also. Without space-time curvature one simply takes the appropriately defined Hilbert square[4,11] of the augmented curvature to find the action of the gauge fields coupled to matter. With gravity, minor complications arise and a special Hilbert product yields the action for the full system. The curvature is multiplied by an object which ensures that its indices get contracted to form the curvature scalar. The gauge and matter fields are squared. This yields Einstein gravity coupled to the gauge fields and matter fields.[13] That there is not a single principle for both sectors is a curious and interesting feature which we consider in some detail. Ultimately we do suggest a single principle for both internal and space-time sectors. However, the fact that such a splitting into horizontal and vertical exists is certainly not incompatible with two principles: e.g., a linear action for the horizontal and quadratic action for the vertical.

Such a geometric picture is pleasing as far as it goes, but a way of including spinors must be available (Sec. IV). One might try viewing some of the gauge fields as currents, but that does not yield a framework which gives a natural structure leading to the usual Dirac action.[4,14] By going back to the early works on spinors,[15] a reinterpretation of the local basis vectors as local basis matrices constructed as currents of spinor and conjugate spinor bases is the easiest framework in which to formulate a theory of spinors. Thus, initially, we adjoin an extra set of internal bases: four spinor bases and four conjugate spinor bases. As these bases are transported from point to infinitesimally separated point their coefficient of infinitesimal transformation (connection) is taken to be the spin-$\frac{1}{2}$ representation of the transformation. That is, if a spin-$\frac{1}{2}$ representation of a local Lorentz transformation gives the relation between nearby bases, the vector bases, formed as a Dirac current from the spinor bases, will transform as a spin-1 representation of this local Lorentz transformation. All possible vector basis transformations can be generalized in this fashion (using the vierbein formalism, in which one transforms coordinates to local Lorentz bases). Thus it is sufficient to consider transformations of the spinor bases.

We introduce the notation of fiber bundle theory and briefly review fiber bundle connections. More details are found in Secs. II and III of paper III.[4] The idea is presented that a rational action in a fiber bundle can account for the effects of gravitation.

Because of the applications of the OSp graded Lie algebras as structure groups for curved real superspaces, we consider the possibility that the real physical manifold could be imbedded in a complex manifold so that a similar application of the $SU(n|m)$ supergroups could be made. This requires viewing the coordinates as being constructed from the complex coordinates. There seems to be no way other than the rational action fiber bundle approach to do this.

By augmenting the spinor basis connection (or by using one of the other techniques described above), one obtains spinor fields coupled to gravity via the connection. The augmented curvature has as one sector the Riemann tensor and as the other sector the gravity-coupled gauge-covariant spinor derivative. If the horizontal action is taken to be linear, one finds Dirac fields coupled to Einstein gravity a beautiful and simple unification. However, we find that the spinors couple to the connection via an axial-vector current.[16] This could lead to anomalies.[17]

Having laid the foundations of the theory of moving frames, we develop the physical applications to full interacting theories in the second paper. By extending the spinor bases to include an extra lable (spin and color symmetry) and by having extra bases we can form a colored, curved, and flavored Weinberg-Salam[18] type model, having a geometric interpretation. We try to emphasize the closeness of this model to a particular supergroup (Sec. II of paper II). The action is related to the flat-space version by minor but intuitively plausible modifications: The metric becomes curved, the determinant factor must be included, spinors are coupled minimally to all fields including gravity (in the tetrad basis), and the curvature scalar gives the kinetic term for the metric. The contributions from the horizontal sector are linear in the gauge-covariant derivatives of the fields (spinor and connection) and the contributions from the vertical sector are quadratic in the gauge-covariant derivatives of the fields (scalar and gauge).

In order to have a truly unified theory, it is desirable to have a single prescription for both sectors. To this end we study the relationship between linear and quadratic actions. Motivated by the work of 't Hooft and Veltman[19] and Deser and others[20] in which it was shown that in order to one loop renormalize gravity, one must include terms quadratic in the curvature, we try a formalism which is entirely quadratic. We argue qualitatively that through the use of local Lorentz frames and gauge theory one can infer renormalizability of the full theory. The general transformation of a four-component spinor allowed by the Dirac Lie algebra, $U(2,2)$, leads one to a curvature tensor consisting of the usual terms plus extra pieces arising from the mixing of momentum and special conformal generators to give angular momentum generators. The Hilbert square[4,11] of this term consists of three pieces: a cross term which is the usual Einstein curvature scalar, a term quadratic in the Riemann tensor (this term can be related to a specific sum of the kinds of terms appearing in the renormalization program), and a cosmological constant term.[21]

By examining the action for pure electromagnetism we see that (see Sec. III of paper II)[4] electromagnetism can be described by an action linear in the first derivatives of the fields (Dirac formulation) or quadratic in the first derivatives of the potentials (potential or Klein-Gordon formulation). In analogy we suggest that the action, quadratic in the first derivatives of spinors, is actually an action for *spinor potentials* and that the usual Dirac spinors are related to these potentials in a fashion analogous to the relation between field and potential in electromagnetism. The idea of using spinors obeying quadratic equations of motion first appeared in a slightly different guise in the $V-A$ paper of Feynman and Gell-Mann[22]; indeed it was these potentials which suggested $V-A$. Feynman also points out that there are advantages to using quadratic actions in considering path integrals.[22]

Thus a single prescription for the action can be used. It is the Hilbert square of the curvature (or augmented curvature) for both horizontal and vertical sectors.

Now we approach the question from a different direction in order to join up with concepts in supersymmetry. For those unfamiliar with superalgebras, Kats[23] has elucidated the simple graded Lie algebras (and thus the others). Those of current physical interest can be described by block $m+n$ by $m+n$ matrices. The on-diagonal $m\times m$ and $n\times n$ sectors represent the L_0, or Bose sector, and the off-diagonal $n\times m$ and $m\times n$ sectors represent the L_1, or Fermi sectors. These algebras have a graded Lie product such that $[a,b]=-(-1)^{ij}[b,a]$ and $[a[b,c]]=[[a,b],c]+(-1)^{ij}[b,[a,c]]$ for $a\in L_i$ and $b\in L_j$ and c in either sector. The two principal series are the orthosymplectic, $\mathrm{OSp}(2m|n)$, and the special linear, $\mathrm{SL}(m|n)$ supergroups. A vertical line distinguishes the special linear supergroup from the special linear Lie algebra on an indefinite-metric space. As in the Cartan classification of the orthogonal groups, e.g., there are different series depending on whether n is odd or even (but more than two or equal to two). The Bose sectors of the $\mathrm{OSp}(2m|n)$ graded algebra are the $2m\times 2m$ symplectic matrices and the $n\times n$ orthogonal matrices; the Fermi sectors are a real $2m\times n$ matrix and its symplectic adjoint which is $n\times 2m$. The symplectic adjoint is minus the transpose followed by multiplication by the symplectic metric.

The special linear graded Lie algebra consists of complex special linear $m\times m$ and $n\times n$ algebras with one extra generator (along the diagonal). The Fermi sectors are not related to each other. We consider a subalgebra of $\mathrm{SL}(4|n)$ which we denote by by $D(4|n)$. It consists of all those elements in $\mathrm{SL}(4|n)$'s algebra which are skew-Dirac conjugate. Dirac conjugation for these $(4+n)\times(4+n)$ matrices is defined by introducing an extension of the γ_0 matrix which has γ_0 in its upper 4×4 corner and minus the identity in the lower $n\times n$ corner. Thus we take the Hermitian conjugate sandwiched between two of these extended γ_0 matrices and require it to be minus the original matrix. This procedure yields i times the Dirac algebra in the 4×4 sector, and i times the $\mathrm{SU}(n)$ algebra in the $n\times n$ sector with a piece proportional to the $n\times n$ identity matrix adjusted to give zero trace in combination with the 4×4 identity contribution. The $4\times n$ matrix is a fermion in the fundamental representation of the Dirac and $\mathrm{SU}(n)$ algebras. The $n\times 4$ matrix is its Dirac adjoint.

Further internal symmetries can be included by taking the upper diagonal to be a higher-dimensional representation of the Dirac matrices ($4p$ by $4p$). By allowing only the subalgebra consisting of four-by-four Dirac matrices and p-by-p $\mathrm{SU}(p)$ matrices, we can obtain an additional invariance group. We will take this symmetry group to be that of color. Of course color can be unified with flavor instead. But since color and spin have historically been related via parastatistics, we have explored this possibility.[24] We denote these groups by $D(4p|n)$.

We consider a connection on a manifold having four Bose coordinate differentials. Matter fields are included by using the basis for functions, I, the identity. This is an augmented connection, as described previously. The other techniques (inclusion of extra coordinate differentials) can also be used. One can either extend the base by including extra Bose or Fermi coordinate differentials. For definiteness consider a manifold having four Bose and one Fermi coordinate differentials. Consider a connection with components in a graded Lie algebra [such as $D(4|n)$]. The Bose (L_0) components of the connection B_μ are associated with the Bose coordinate differentials dx^μ, forming $B_\mu dx^\mu$. The Fermi (L_1) components of the connection Ψ are associated with the Fermi coordinate differentials $d\theta$ forming $\Psi d\theta$.

When one computes the curvature of this connection, one obtains the Yang-Mills-curvature tensors on the diagonal and the gauge-covariant derivative of the fermions on the minor diagonal. The action is given by the unique prescription that it is the Hilbert square of the curvature.

This action can be precisely the same as that which we obtain by considering the transformation of spinor bases. The reason is that the spinor basis transformation law can be taken to be given by a connection with coefficients in a superalgebra, $d\bar{s}_a=i\bar{s}_b\Omega^b_{ia}dy^i$. Here dy^i denotes the set $(dx^\mu, d\theta)$ or (dx^μ, I), for example. The matrices Ω^b_{ia} take values in a graded Lie algebra as described above. Thus the superalgebra matrices act as coefficients of "rotation" for the spinor frames providing a picture of the effect of gauging a superalgebra.

Requiring that the connection close to form the entire superalgebra $D(4|n)$, e.g., thus provides a unified description of gravitation, matter, and internal-symmetry fields. If we place no arbitrary restriction on the L_0 subsector consisting of the $\mathrm{SU}(2,2)$ generators identified with the Dirac Lie algebra (the identity is included also) we are led to the covering group of the conformal group $\mathrm{SO}(4,2)$. In the vierbein view of gravity there are ten fields: four vierbein fields Y^i_μ, and six connections, $\omega^i_{\mu j}$. The connections $\omega^i_{\mu j}$ are related to the usual connections $\Gamma^\mu_{\lambda\rho}$ by $\Gamma^\mu_{\lambda\rho}=Y^\mu_i(Y^i_{\rho|\lambda}+\omega^i_{\mu j}Y^j_\lambda)$. These can be related to the ten generators of $\mathrm{ISO}(3,1)$ or of its covering group. With appropriate care $\mathrm{ISO}(3,1)$ can be expanded to $\mathrm{SO}(4,1)$ [or $\mathrm{SO}(3,2)$, possibly] or their covering groups. But one still has the extra generators in the conformal group. For definiteness let Y^i_μ be associated with $P_i=\frac{1}{2}(I+\gamma_5)\gamma_i$ and $\omega^i_{\mu j}$ be associated with $M^j_i=\sigma^j_i$, the momentum and angular momentum generators. What are the additional fields implied by the superalgebra $D(4|n)\supset\mathrm{SU}(2,2)$ which go with $K_i=\frac{1}{2}(I-\gamma_5)\gamma_i$ and with $D=i\gamma_5$, the special conformal and dilation generators? Since any linear combination of the four P_i and K_i and the six M^i_j form a closed subalgebra perhaps that is all there is, physically. One might also speculate that in this region of space only ten are important and that elsewhere the others can be significant (depending on coupling strengths). If we make the assumption that the P_i and K_i fields and the I and D fields are proportional we are led to the idea that there might be axial contributions to electromagnetism. This possibility has been considered by Wolfenstein and by Herczeg.[25] The coefficient of proportionality can be quite large with the axial current term down from the vector current term only by 10^{-3} and not be in conflict with existing data on lepton-hadron interactions, if the axial current is conserved (no scalar terms, e.g.). In that view there is an interaction $eA^\lambda(j_\lambda^{(V)}+fj_\lambda^{(A)})$; f is 10^{-3}, j^V and j^A denote the vector electromagnetic current and the hadronic axial-vector current.[25] They examined this point in the linear Dirac formalism.

Motivated by Faddeev's use of "cohesions in a vector stratification," i.e., "connections in a fiber bundle," to solve the problem of gauge degeneracies in quantum fields,[5] we show how the theory of moving frames is related to the (more precise) theory of fiber bundles in the third paper.[3,4] Essentially a fiber bundle is a manifold which has a given space, such as the group SO(3), "attached" to each point in the base manifold. In our examples the base manifold is space-time. The bundle has a dimension equal to the sum of the base and the fiber dimensions [in the SO(3) example, see Sec. III of paper III,[4] it is $4+3=7$ dimensional]. When a connection (gauge field) is given in the bundle, it has extra components which are associated with the extra degrees of freedom. These precisely remove the gauge degeneracies. This shows that quantization in fiber bundles is a worthwhile problem to examine.

In the first two papers we have omitted reference to the extra degrees of freedom arising in a fiber bundle, since they add technical complexity to an already detailed subject. Once one understands how to handle them in a particular instance, the generalization to arbitrary bundles is easy. That is, we have ignored them for the same reason everyone else does.

To establish the language which is commonly used in the theory of fiber bundles (and in the theory of moving frames) we have reviewed the concept of differential forms and their associated operations (exterior differentiation, Hodge duality, and the Hilbert product of p-forms) in four dimensions. Detailed tables are provided (see Sec. II of paper III).[4]

The purpose of this third paper is twofold. One, it is hoped that providing further detail on some of the mathematical structures will flesh out those abstract constructions. Two, we hope to suggest possible ways in which this theory may be developed.

In this spirit we have suggested the fiber-bundle quantization problem. In the fourth section of paper III[4] we show that this formalism allows a uniform description of both Lie groups and general manifolds (through the use of local quantities). From there we show that many other structures can be easily included. For example, the postulates of general relativity about the local structure of space-time can be expressed as statements about the local choice of coordinates on the manifold. It is usually assumed that at any given point an orthonormal vector basis with vanishing (torsion-free) connection can be chosen. This corresponds to choosing the zeroth, first, and second terms in a Taylor expansion of the coordinate system (with respect to local R^n coordinates).[11] But what about manifolds which are locally flatter (such as a point on the bottom of a bowl)? The curvature information would be carried in higher terms of the Taylor series. We show how a manifold with third-order structure can be constructed. We also indicate some of the things that can happen if the local Lie group of rotations becomes a local manifold of rotations, a locally anisotropic situation. The Lie algebra of a bundle can be altered in a number of ways: by becoming graded or by having the Jacobi relation fail (as it does for the octonions or for other Mal'cev algebras).[26] These possibilities are of interest because they provide other mechanisms for breaking symmetries than that of Higgs and Kibble. We describe a model in which the Jacobi identity is broken but revives in the zero limit of a particular parameter. At that value the algebra becomes $u(1)+su(2)$.

Since the graded Lie algebra used in the second paper was designed to include the effects of gravitation, it was quite complicated. Simpler models are described in Sec. V of paper III. The first of these is based on $SL(2|1)$.[23] Since the group $SL(2,\mathbb{C})$ only covers the Lorentz and not the Pioncaré group the structure is greatly simplified. The spinors are two-component spinors. Nonetheless some of the features of the model in Sec. III of paper II will be present and the smaller version may give some insight into the structure of the theory.

Finally (in Sec. VI of paper III) we show how the standard spin-2–spin-$\frac{3}{2}$ supergravity action can easily be derived using moving frames. We point out that a simple supergroup is not being used [the nearest simple supergroup is $OSp(1,4)$][23] and write out a general model arising from a nonsimple supergroup. It has a supergravity, colored Weinberg-Salam-type action.

Clearly the infinitesimal viewpoint we have been describing unifies a great variety of beautiful and useful structures. But more importantly it gives us a tool with which to study these fascinating new objects.

II. MOVING FRAMES, CURVATURE, TORSION, AND ACTION FORMS

We will ultimately be extending the local frame of tangents to a point from 4 to $4+n$. But to begin, the usual basis of tangents to a point P is constructed by moving that point in each of the four coordinate directions an infinitesimal amount to neighboring points $P'_{(\mu)}$ where (μ) labels in which of the four directions the new point lies. Thus infinitesimally,

$$\delta P_{(1)} = P'_{(1)} - P = \frac{\partial P}{\partial x^1}\, dx^1, \tag{1}$$

when P is varied in the first direction. If P' is in arbitrary relation to P (but only infinitesimally separated) one finds[7]

$$dP = P' - P = \frac{\partial P}{\partial x^\mu}\, dx^\mu \equiv e_\mu dx^\mu. \tag{2}$$

Please see Fig. 1.

The point is simply that an infinitesimal displacement dP can be described in terms of a basis of tangent vectors e_μ and infinitesimal displacement dx^μ so that $dP = e_\mu dx^\mu$. This should be no more confusing than writing $\vec{v} = e_i v^i$ for a vector. What may be slightly more abstract is the idea that as one moves from point to point one's choice of basis, e_μ, can change. Thus it is important to consider the change in e_μ from P to P'. As with the well-known introduction to the connection, this is a primitive concept. It can be motivated by the concept of globally integrable orthogonal rotations [which includes the indefinite-metric Lorentz group SO(3,1)]. For this case a frame at P is given by flatly translating to 0 and expressing the components in terms of $e_\nu(0)$,

$$e_\mu(P) = e_\nu(0)\{\exp[\omega^{\alpha\beta}(P)\Sigma_{\alpha\beta}]\}^\nu_{\ \mu}, \tag{3}$$

where $\omega^{\alpha\beta}(P)$ is a single-valued function of position P (and the initial position 0) and $\Sigma_{\alpha\beta}$ is the vector representation of the given orthogonal group generators. In this case

$$de_\mu = e'_\mu(P') - e_\mu(P) = e_\nu \frac{\partial \omega^{\alpha\beta}}{\partial x^\lambda}\, dx^\lambda (\Sigma_{\alpha\beta})^\nu_{\ \mu} \tag{4}$$

or

$$de_\mu = e_\nu \omega^\nu_{\ \lambda\mu} dx^\lambda, \tag{5}$$

where

$$\omega^\nu_{\ \lambda\mu} \equiv \frac{\partial \omega^\nu_{\ \mu}}{\partial x^\lambda} \tag{6}$$

and we express Σ in terms of the metric

$$(\Sigma_{\alpha\beta})^{\mu\nu} = \tfrac{1}{2}(g^\mu_\alpha g^\nu_\beta - g^\nu_\alpha g^\mu_\beta) \tag{7}$$

for SO(p,q). To generalize to a curved manifold one can begin with the statement $de_\mu = e_\nu \omega^\nu_{\ \lambda\mu} dx^\lambda$. Then, only if the change in frames is independent of route, can one integrate the connection.[7] But, as we shall see shortly, if such route independence exists, the manifold is said to be flat. When we consider internal symmetries such globally integrable connections will correspond to gauge fields which consist only of pure gauge transformations of the vacuum such as $A_\mu = \partial_\mu \lambda$. Here the analogy is $A_\mu \leftrightarrow \omega_\mu{}^\alpha{}_\beta$ and $\partial_\mu\lambda \leftrightarrow \partial_\mu \omega^\alpha_\beta$. The introduction of connections which are not pure terms is in analogy to the introduction in the action of kinetic pieces for the gauge fields which were initially only required for gauge invariance of matter fields.

Now if the tangent basis is arbitrarily extended from the four legs (vierbein) e_μ to a $4+n$ basis, $\bar{e}_I = (e_\mu, \bar{e}_a)$, where there are n $\bar{e}_a$'s, while maintaining the idea that there are still only four displacements dx^μ, one arrives at an extended frame which is capable of including internal-symmetry structures. We can refer to the e_μ as horizontal and the $\bar{e}_a$ as vertical. To make this pictorial think of the usual two-sphere, S^2.

One can construct not only the two tangents $e_\mu = (e_1, e_2)$ but also a normal e_3. See Fig. 2. Here there are manifestly only the two infinitesimal displacements associated with moving about on the surface of the sphere. And yet it is possible to consider still further basis elements such as the normal e_3.

Thus, in general, for a four-dimensional base manifold consider $e_I = (e_\mu, \bar{e}_a)$ and $e'_I = (e'_\mu, \bar{e}'_a)$ at P and P', respectively. See Fig. 3. Then if e'_I and e_I are infinitesimally separated we consider the following changes in the frames:

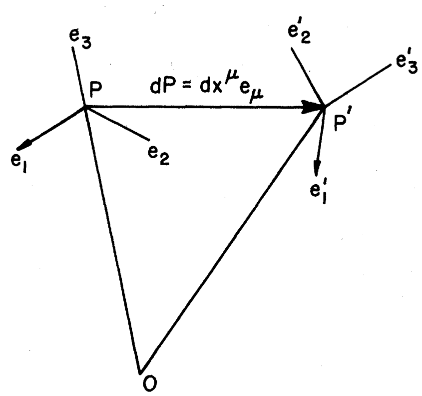

FIG. 1. Frames at neighboring points.

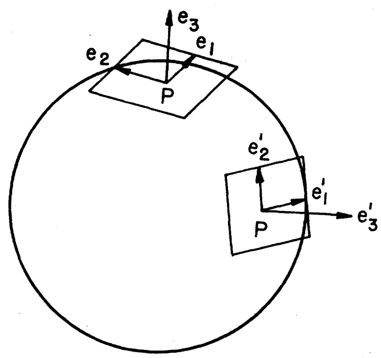

FIG. 2. Tangent and normal bundle for the two-sphere.

$$\text{Horizontal: } de_\mu = e_\nu \omega^\nu_{\lambda\mu} dx^\lambda \text{ and } dP = e_\mu dx^\mu, \qquad (8)$$

$$\text{Vertical: } d\bar{e}_a = \bar{e}_b B^b_{\lambda a} dx^\lambda. \qquad (9)$$

These will be identified with the connection, basis, and gauge fields, respectively.

No terms mixing horizontal and vertical bases have been included. Vertical fields are not mingled with horizontal or else "matter" or "isospin" could become "position."

Now let us consider the route dependence of these coefficients by examining the difference between two second-order paths. Thus let dx^λ and δx^ρ denote infinitesimal displacements. We will consider the effect of making first one displacement then the other and then doing these displacements in the other order. See Fig. 4. When two displacements are made, δde_I, there can be contributions not only from the change in basis but also from the change in the connection coefficients, $\omega^\nu_{\lambda\mu}$ and $B^b_{\lambda a}$, which, to leading order, is just their first derivatives. Thus

$$\text{Horizontal: } \delta dP = \delta(e_\lambda dx^\lambda) = \bar{e}_\beta \omega^\beta_{\rho\lambda} \delta x^\rho dx^\lambda, \qquad (10)$$

$$\delta d\bar{e}_\mu = \delta(\bar{e}_\nu \omega^\nu_{\lambda\mu} dx^\lambda) = \bar{e}_\nu(\omega^\nu_{\lambda\mu|\rho} + \omega^\nu_{\rho\alpha}\omega^\alpha_{\lambda\mu})\delta x^\rho dx^\lambda, \qquad (11)$$

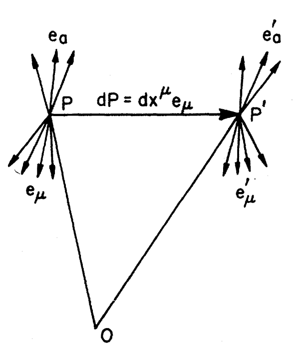

FIG. 3. Moving frame for space-time with internal symmetry.

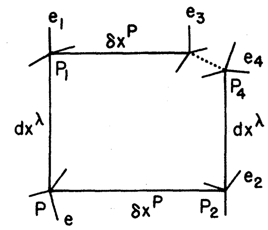

FIG. 4. Second-order displacement for points and frame; $P_3 \neq P_4$, on a curved manifold.

$$\text{Vertical: } \delta d\bar{e}_a = \delta(\bar{e}_b B^b_{\lambda a} dx^\lambda) = \bar{e}_b(B^b_{\lambda a|\rho} + B^b_{\rho c} B^c_{\lambda a})\delta x^\rho dx^\lambda. \qquad (12)$$

By doing these displacements in reverse order and then comparing the difference one obtains the following relations (note that quantities with barred indices are to be antisymmetrized on those indices; e.g., $A_{\rho\underline{\lambda}\underline{\mu}} \equiv A_{\rho\lambda\mu} - A_{\mu\lambda\rho}$):

$$(d\delta - \delta d)P = e_\beta[\tfrac{1}{2}\omega^\beta_{\underline{\rho}\underline{\lambda}}(\delta x^\rho dx^\lambda - dx^\rho \delta x^\lambda)], \qquad (13)$$

$$(d\delta - \delta d)e_\mu = e_\nu[\tfrac{1}{2}(\omega^\nu_{\underline{\lambda}\mu|\underline{\rho}} + \omega^\nu_{\underline{\rho}\alpha}\omega^\alpha_{\underline{\lambda}\mu}) \times (\delta x^\rho dx^\lambda - dx^\rho \delta x^\lambda)], \qquad (14)$$

$$(d\delta - \delta d)\bar{e}_a = \bar{e}_b[\tfrac{1}{2}(B^b_{\underline{\lambda}a|\underline{\rho}} + B^b_{\underline{\rho}c}B^c_{\underline{\lambda}a}) \times (\delta x^\rho dx^\lambda - dx^\rho \delta x^\lambda)]. \qquad (15)$$

The combination $\frac{1}{2}(\delta x^\rho dx^\lambda - dx^\rho \delta x^\lambda)$ is the antisymmetric tensor product of infinitesimals. It is written as $dx^\rho \wedge dx^\lambda$. See Sec. II of paper III.[4] The connection symbols $\omega^\nu_{\lambda\mu}$ are usually written as $\Gamma^\nu_{\mu\lambda}$. The nature of the internal-symmetry group is determined by placing conditions on the matrices $B^a_{\lambda b}$. For example, for SU(n), $\delta^a_b B^b_{\lambda a} = 0$, the trace vanishes, and $\bar{B}^b_{\lambda a} = -B^a_{\lambda b}$, the matrices are skew-Hermitian. We can write these relationships in a more familiar way by setting $B_\lambda \equiv B^a_{\lambda b}\bar{e}_a \otimes e^b$. Then $\text{tr}B_\lambda = 0$ and $B^\dagger_\lambda = -B_\lambda$. Or still more familiarly write $B^A_\lambda(i\Lambda^b_{Aa})$ for $B^b_{\lambda a}$ and note that $\text{tr}\Lambda_A = 0$ and $\Lambda^\dagger_A = \Lambda_A$. In particular, if there are three $\bar{e}_a$'s, we could take the matrices Λ_A to be the well-known Gell-Mann matrices[27] for SU(3). Another related case which could be considered is one in which there are eight $\bar{e}_a$'s, then we could take $\Lambda^b_{Aa} = -\epsilon^b_{Aa}$, the structure constants for SU(3). These correspond to the fundamental and adjoint representations, respectively.

We remark that in the rest of this paper we will write an overbar as a generic symbol for adjoint Thus $\bar{B}_\lambda \equiv B^\dagger_\lambda$ but on fermions, $\bar{\psi}$, will mean $\psi^\dagger\gamma_0$. e_μ is written without a bar because of its reality, $e_\mu \equiv \bar{e}_\mu$.

Now we are prepared to write the relations given above in a more familiar fashion (recall that $\Gamma^\beta_{\underline{\lambda}\underline{\rho}} = \Gamma^\beta_{\lambda\rho} - \Gamma^\beta_{\rho\lambda}$):

$$\tfrac{1}{2}(d\delta-\delta d)P=e_\beta(\tfrac{1}{2}\Gamma^{\beta}_{\underline{\lambda}\,\underline{\rho}}dx^\rho\wedge dx^\lambda)$$
$$=e_\beta(\tfrac{1}{2}T^\beta_{\rho\lambda}dx^\rho\wedge dx^\lambda)=e_\beta T^\beta, \quad (16)$$

$$\tfrac{1}{2}(d\delta-\delta d)e_\mu=e_\nu[\tfrac{1}{2}(\Gamma^\nu_{\mu\underline{\lambda}|\underline{\rho}}+\Gamma^\nu_{\alpha\underline{\rho}}\Gamma^\alpha_{\mu\underline{\lambda}})dx^\rho\wedge dx^\lambda]$$
$$=e_\nu(\tfrac{1}{2}R^\nu_{\mu\underline{\rho}\,\underline{\lambda}}dx^\rho\wedge dx^\lambda)=e_\nu R^\nu_\mu, \quad (17)$$

$$\tfrac{1}{2}(d\delta-\delta d)\bar{e}_a=\bar{e}_b[\tfrac{1}{2}(B^A_{\underline{\lambda}|\underline{\rho}}+i\epsilon^A_{BC}B^B_\rho B^C_\lambda)(i\Lambda_A)^b_a dx^\rho\wedge dx^\lambda]$$
$$=\bar{e}_b[\tfrac{1}{2}F_{\rho\lambda}(i\Lambda_A)^b_a dx^\rho\wedge dx^\lambda]=\bar{e}_b F^A(i\Lambda_A)^b_a. \quad (18)$$

T^β is the torsion, R^μ_ν is the curvature two-form (not the Ricci tensor, which we do not use without explicit mention). F^A is the Yang-Mills field tensor. We remark that the torsion is the antisymmetic part of the connection in a coordinate basis.

Equivalently, one can consider a closed line integral. By taking an infinitesimal square $[PQRSP]$, where P, Q, R, S are the four vertices in sequence, we can write the line integral of δe_μ as

$$\oint \delta e_\mu=\int_{[PQRSP]}\delta e_\mu=\int_{[PQR]}\delta e_\mu-\int_{[PSR]}\delta e_\mu \quad (19)$$

and obtain $R^\nu_\mu e_\mu$ as the result. Similarly

$$\oint \delta\bar{e}_a=\int_{[PQRSP]}\delta\bar{e}_a=\int_{[PQR]}\delta\bar{e}_a-\int_{[PSR]}\delta\bar{e}_a$$
$$=(F^A i\Lambda^b_{Aa})\bar{e}_b\equiv F^b_a\bar{e}_b. \quad (20)$$

Using the basis (e^μ, e^a) dual to $(e_\mu, \bar{e}_a)$ (where $\langle e^\mu|e_\nu\rangle=\delta^\mu_\nu$, etc.), set $\theta\equiv dP$, $\omega\equiv de_\mu\otimes e^\mu$, and $B=d\bar{e}_a\otimes e_a$. Now we can write basis-independent formulas for the torsion, curvature, and field tensor[3,4,11]

$$T=d\theta+\omega\wedge\theta=T^\beta e_\beta, \quad (21)$$

$$R=d\omega+\omega\wedge\omega\equiv R^\mu_\nu e_\mu\otimes e^\nu, \quad (22)$$

$$F=dB+B\wedge B\equiv F^b_a\bar{e}_b\otimes e^a. \quad (23)$$

Here the exterior derivative d, whose action on a one-form $A_\lambda dx^\lambda$ is given as $d(A_\lambda dx^\lambda)=\frac{1}{2}A_{\lambda|\rho}dx^\rho\wedge dx^\lambda$, has been used. A short computation will then show that these are the objects described above. See Sec. II of paper III[4] for sample calculations. It is worth noting that this formalism is well adapted to treated Lie groups and manifolds on similar footing. See Sec. III of paper III.[4]

It is convenient to introduce the notation $\theta=e_\mu\theta^\mu$, and then $T^\mu=d\theta^\mu+\omega^\mu_\nu\wedge\theta^\nu$, where $\theta^\mu=dx^\mu$ in a coordinate basis. It is not necessary to restrict a local frame to being associated with a coordinate patch; any system of four independent vectors would work, Indeed a natural choice is one which is orthonormal everywhere. Thus introduce $\theta^i=Y^i_\mu dx^\mu=Y^i_\mu\theta^\mu$ and $e_i=Y^\mu_i e_\mu$ with $Y^\mu_i Y^j_\mu=\delta^j_i$. Y^i_μ is the vierbein or tetrad field. Then define the symmetric bilinear map called the metric:

$$g(\theta^i,\theta^j)\equiv\eta^{ij}=\theta^i\cdot\theta^j=Y^i_\mu Y^j_\nu dx^\mu\cdot dx^\nu$$
$$=Y^i_\mu Y^j_\nu g^{\mu\nu}\equiv Y^i_\mu Y^j_\nu g(dx^\mu, dx^\nu) \quad (24)$$

or

$$g(e_i,e_j)\equiv\eta_{ij}=e_i\cdot e_j=Y^\mu_i Y^\nu_j e_\mu\cdot e_\nu$$
$$=Y^\mu_i Y^\nu_j g_{\mu\nu}\equiv Y^\mu_i Y^\nu_j g(e_\mu, e_\nu). \quad (25)$$

Equivalently $g_{\mu\nu}=Y^i_\mu Y^j_\nu\eta_{ij}$. The use of the vierbein field reveals the underlying Lorentz group structure. In the new basis $T^i=d\theta^i+\omega^i_j\wedge\theta^j$ with $\omega^i_j=\omega^i_{kj}\theta^k$ the connection, T^i the torsion. Now

$$d(\eta_{ij})=d(e_i\cdot e_j)=de_i\cdot e_j+e_i\cdot de_i\equiv e_l\omega^l_i\cdot e_j+e_i\cdot e_l\omega^l_j$$
$$=\omega_{ij}+\omega_{ji}, \quad (26)$$

since $\omega^k_i\cdot e_k\cdot e_j=\omega_{ij}$. We can choose the basis e_i so that $d(\eta_{ij})=0$, and we find ω_{ij} is skew-symmetric (generates the orthogonal group). For η_{ij} the global Minkowski metric, ω_{ij} generates the Lorentz group. If e_i is actually a coordinate basis (such that $\theta^i=dy^i$), the metric is globally Minkowski; the space is flat. Gauging only the Lorentz group is not enough to yield curvature; the vierbein, Y^i_μ (translations) must be gauged[6] (see Sec. III of paper on extended gauge transformations[4]).

When spinors are included, use of the vierbein field allows us to view gravity as coupling through a gauge-covariant derivative. Thus under a Lorentz transformation $\psi'=e^{-i\omega^{ij}\sigma_{ij}}\psi$ with $\sigma_{ij}=\frac{1}{8}i[\gamma_i,\gamma_j]$. So a gauge-covariant derivative is $\nabla_i=\bar{\partial}_i-i\omega^{jk}_i\sigma_{jk}$ with $\bar{\partial}_i=Y^\mu_i\partial_\mu$ in analogy with $\psi'=e^{-i\alpha}\psi$ producing a gauge-covariant derivative $\nabla_\mu=\partial_\mu-iA_\mu$. The fact that $\sigma_{ij}=-\sigma_{ji}$ is compatible with a skew-symmetric connection $\omega^{ij}=-\omega^{ji}$.

We note that using forms $\theta^i=Y^i_\mu dx^\mu$ or directional derivatives $\theta_i=Y^\mu_i\partial_\mu$ emphasizes the similarity between Lie groups and general manifolds. See Sec. IV of paper III.[4] A single description can be used for both.[7]

It is possible to express the relationship between the connections in coordinate and noncoordinate bases as follows:

Basis: $$\theta=e_i\theta^i=(e_\mu Y^\mu_i)(Y^i_\nu dx^\nu), \quad (27)$$

Connection: $$\omega=\omega^j_{ik}e_j\otimes e^k\theta^i$$
$$=[Y^\rho_k Y^\mu_i(-Y^j_{\rho|\mu}+\omega^\lambda_{\rho\mu}Y^j_\lambda)]e_j\otimes e^k\theta^i, \quad (28)$$

Yang-Mills: $$B=B^I_i\theta^i(i\Lambda_I)=(B^I_\mu Y^\mu_i)\theta^i(i\Lambda_I). \quad (29)$$

Here $\omega^\lambda_{\rho\mu}=\Gamma^\lambda_{\mu\rho}$ are the connection coefficients on a coordinate basis. They bear the usual relation to the metric $g_{\mu\rho}$ when there is no torsion. Indeed the general relationship between connection and metric can be obtained from $dg_{\mu\nu}=\omega_{\mu\nu}+\omega_{\nu\mu}=g_{\mu\nu|\alpha}dx^{\alpha}$. Using this three times, one finds

$$\Gamma^{\lambda}_{\mu\rho}=\tfrac{1}{2}g^{\lambda\alpha}(g_{\mu\alpha|\rho}+g_{\rho\alpha|\mu}-g_{\mu\rho|\alpha}) \tag{30}$$

for a coordinate basis. In a noncoordinate basis one finds, in general,[28]

$$\omega_{ij}=[\tfrac{1}{2}(C_{ijk}+C_{ikj}-C_{jki})+\tfrac{1}{2}(g_{ij|k}+g_{ik|j}-g_{jk|i})]\,\theta^{k}. \tag{31}$$

Here differentiation is performed with respect to the forms θ^k (with respect to $\bar{\partial}_i=Y^{\mu}_{i}\partial_{\mu}$). Thus $dY^{i}_{\mu}=Y^{i}_{\mu|\lambda}Y^{\lambda}_{k}\theta^{k}$, for example. The coefficients $C_{ijk}=C^{l}_{ij}g_{lk}$ can be found from $d\theta^{i}+\tfrac{1}{2}C^{i}_{jk}\theta^{j}\wedge\theta^{k}=T^{i}=0$. ω_{ij} can be expressed by starting from $g_{\mu\nu}=Y^{i}_{\mu}\eta_{ij}Y^{j}_{\nu}$ and using this formula. First we find C_{ijk},

$$\begin{aligned}C_{ijk}=C^{l}_{ij}\eta_{lk}&=(Y^{k}_{\lambda|\mu}Y^{\lambda}_{i}Y^{\mu}_{j})\eta_{lk}\\&=Y^{k}_{\lambda|\mu}Y^{\lambda}_{i}Y^{\mu}_{j}\eta_{lk}-Y^{k}_{\lambda|\mu}Y^{\lambda}_{j}Y^{\mu}_{i}\eta_{lk},\end{aligned} \tag{32}$$

and then

$$\omega_{ij}=\tfrac{1}{2}(C_{ijk}+C_{ikj}-C_{jki})\theta^{k}, \tag{33}$$

since

$$g_{ij|k}=\eta_{ij|k}=0. \tag{34}$$

See Sec. IV of paper III for the relationship between Lie groups and manifolds.

We remark that the metric $g_{\mu\nu}$ can be viewed as a composite of the quartet of vector fields Y^{i}_{μ}. Here one views the index i as an internal index and the index μ as the space-time vector index. In fact some authors refer to the μ index as the space-time index and i as a Lorentz index. The Y^{i}_{μ} are both Lorentz and space-time vectors in this language, whereas $g_{\mu\nu}$ is a space-time tensor and a Lorentz scalar.

The connections ω_{ij} can be written out in full detail as

$$\begin{aligned}\omega_{ij}&=\tfrac{1}{2}(Y^{\mu}_{i}Y^{\nu}_{j}Y^{m}_{\mu|\nu}\eta_{mk}+Y^{\mu}_{i}Y^{\nu}_{k}Y^{m}_{\mu|\nu}\eta_{mj}-Y^{\mu}_{j}Y^{\nu}_{k}Y^{m}_{\mu|\nu}\eta_{mi})\theta^{k}\\&=\omega_{kij}\theta^{k},\end{aligned} \tag{35}$$

where

$$\theta^{k}=Y^{k}_{\mu}dx^{\mu}. \tag{36}$$

It is this object contracted with the tensor $\sigma^{ij}=\tfrac{1}{8}i[\gamma^{i},\gamma^{j}]$ which appears in the spinor gauge-covariant derivative $\nabla_{k}=Y^{\mu}_{k}\partial_{\mu}+\omega_{kij}\sigma^{ij}$. We remark that $\tfrac{1}{2}\{\gamma^{i},\gamma^{j}\}=\eta^{ij}$ and $\gamma^{\mu}=Y^{\mu}_{i}\gamma^{i}$ satisfies $\tfrac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\}=g^{\mu\nu}$.

So far we have not made much use of the Grassmann algebra notation except incidentally.[4,8] We will note that when one writes the coefficients ω_{kij} contracted with θ^{k} as $\omega_{kij}\theta^{k}=\omega_{ij}$, the object ω_{ij} is called a connection one-form, since $\theta^{k}=Y^{k}_{\mu}dx^{\mu}$ is a one-form (has one dx^{μ} in it). The object $B=B^{A}_{\mu}(i\Lambda_{A})dx^{\mu}$ is also a connection one-form for the internal-symmetry sector. We could call it the Yang-Mills one-form or gauge-field one-form. The objects T, R, and F are all two-forms since they can be expressed as $T=T_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$, $R=R_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$, and $F=F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$. p-forms involve the p-fold product of the dx^{μ}'s, of course, as described in Sec. II of paper III. In that section we also tabulate the effect of the operator denoted by an asterisk, the Hodge dual operator, which takes p-forms on an n-manifold (in our case $n=4$) into $n-p$ ($=4-p$)-forms on that manifold. We can use this fact to introduce a natural Hilbert product on p-forms for a given manifold, generally with nontrivial topology.[4] As we shall see shortly, this will allow us to define the action as the natural Hilbert product of curvature forms.

The Hilbert product for functions, zero-forms, is given as[4,11]

$$\langle f|h\rangle=\int f*\wedge h=\int (fI)*\wedge hI=\int fh\sqrt{-g}\,d^{4}x. \tag{37}$$

If f and h are complex

$$\langle f|h\rangle=\int \bar{f}*\wedge h=\int (\bar{f}I)*\wedge hI=\int \bar{f}h\sqrt{-g}\,d^{4}x. \tag{38}$$

For one-forms (raising is done by $g_{\mu\nu}$)

$$\begin{aligned}\langle A|B\rangle=\int A*\wedge B&=\int (A_{\mu}dx^{\mu})*\wedge(B_{\nu}dx^{\nu})\\&=\int A_{\mu}B^{\mu}\sqrt{-g}\,d^{4}x.\end{aligned} \tag{39}$$

For two-forms

$$\begin{aligned}\langle C|R\rangle&=\tfrac{1}{2}\int C*\wedge R\\&=\tfrac{1}{2}\int (C_{\mu\nu}dx^{\mu}\wedge dx^{\nu})*\wedge(R_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta})\\&=-\int C_{\mu\nu}R^{\mu\nu}\sqrt{-g}\,d^{4}x.\end{aligned} \tag{40}$$

For p-forms

$$\langle E|F\rangle\equiv\frac{1}{p!}\int \bar{E}*\wedge F. \tag{41}$$

The factor $1/p!$ is an optional normalization. For matrices of forms such as $B=B^{A}_{\mu}dx^{\mu}(i\Lambda_{A})$ or $F=F^{A}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}(i\Lambda_{A})$ a scalar on the A index is formed by taking $\Lambda_{A}\circ\Lambda_{B}=-\delta_{AB}$. The centered circle will denote an inner product. In case the Λ_{A} matrices are normalized suitably one can set $i\Lambda_{A}\circ i\Lambda_{B}=-\mathrm{tr}\,i\Lambda_{A}\circ i\Lambda_{B}=+\delta_{AB}$. So, for example, we can consider the Hilbert product of matrices of two-forms such as the Yang-Mills force tensor and find the action on a manifold having a topology consistent with the fields[4]

$$\tfrac{1}{2}\langle F|F\rangle = \tfrac{1}{2}\int \bar{F} * \mathring{\wedge} F$$

$$= \tfrac{1}{4}\int \tfrac{1}{2} F^A_{\mu\nu}(dx^\mu \wedge dx^\nu)^* \wedge F^B_{\alpha\beta}(dx^\alpha \wedge dx^\beta) \times (\bar{i}\bar{\Lambda}_A \circ i\Lambda_B)$$

$$= -\tfrac{1}{4}\int F^A_{\mu\nu} F^B_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} \delta_{AB}\sqrt{-g}\, d^4x. \qquad (42)$$

Here $F = \frac{1}{2}F^A_{\mu\nu}(i\Lambda_A)dx^\mu \wedge dx^\nu$ and $(\bar{i}\bar{\Lambda}_A) = -\bar{i}\bar{\Lambda}_A = -i\Lambda_A$. This choice of metric on the matrices may be somewhat clearer in the skew-Hermitian basis $E_A = i\Lambda_A$. $\bar{E}_A = -E_A$ and $E_A \circ E_B = -\delta_{AB}$, therefore $\bar{E}_A \circ E_B = \delta_{AB}$. We recall the theorem that a unitary matrix is the exponential of a skew-Hermitian matrix. The overbar of course is Hermitian conjugation in this context.

Indeed, with these conventions we can recognize the Hilbert product $\langle F|F\rangle$ as being the usual action for a Yang-Mills theory written in a basis-independent fashion. Here the analogy is $\vec{V}\cdot\vec{V} = V^iV^je_i\cdot e_j = V^iV^jg_{ij}$. On the one hand, we have a basis-independent notation, on the other, the explicit component notation. Each has its own advantages.

It is possible to express the Einstein gravitational action in this notation also. However, an interesting distinction between the internal-symmetry action and the Einstein action arises. For internal symmetry we consider the inner product of the Yang-Mills two-form with itself but for gravity the inner product with a constant curvature tensor (two-form) is taken. This is what introduces the gravitational constant $\kappa = c^3/G\hbar \sim 2.61\times 10^{-66}$ cm^{-2} and allows one to recover the Newtonian limit as we shall see. Thus define the "constant" metric two-form (constructed from the metric on bivectors, the coefficients of two-forms)

$$C = \kappa(g^j_k g^i_l - g^i_k g^j_l)\theta^k \wedge \theta^l \Sigma_{ij}. \qquad (43)$$

Here $\kappa = c^3/G\hbar$; $\Sigma_{ij} = \frac{1}{2}(E_{ij} - E_{ji})$, with E_{ij} the matrix whose components are $(E_{ij})_{kl} = g_{ik}g_{jl}$. The coefficient $\kappa(g^i_k g^j_l - g^j_k g^i_l) = C^{ij}_{kl}$ is the Riemann tensor when the curvature is constant; $\theta^k = Y^k_\mu dx^\mu$ as usual. We can write the curvature as follows: $R = \frac{1}{2}R^{ij}_{kl}\theta^k \wedge \theta^l \Sigma_{ij}$; then note that $\mathrm{tr}(E_{ij}\cdot E_{kl}) = \frac{1}{2}(g_{il}g_{jk} - g_{jl}g_{ik})$. Now we can write the Einstein action as

$$\langle C|R\rangle = \int \tfrac{1}{2}\mathrm{tr}(\bar{C} * \wedge R)$$

$$= \int d^4x(\kappa R^{ij}_{ij})\sqrt{-g} = \kappa\int R_S\sqrt{-g}\, d^4x, \qquad (44)$$

where R_S is the curvature scalar of space-time.

If we had introduced $\langle R|R\rangle$ as the action, which equals $-\int d^4x(R^{ij}_{kl}R^{kl}_{ij})\sqrt{-g}$, by analogy with gauge theories ($\langle F|F\rangle$) we would not recover the usual matter coupling, since we would not have introduced the gravitational constant, κ or $1/G$. However, one can construct a theory which has the same set of metrics, $g_{\mu\nu} = Y^i_\mu \eta_{ij} Y^j_\nu$, but is derived from a quadratic action.

Note that $d\theta^i + \omega^i_j \wedge \theta^j = T^i$ allowed us to find the connection coefficients by using the fact that the coefficients C^i_{jk} could be computed from

$$d\theta^i + (\tfrac{1}{2}C^i_{kj}\theta^k) \wedge \theta^j = d\theta^i + \omega^i_j\theta^j = 0 = T^i, \qquad (45)$$

since

$$d\theta^i = \tfrac{1}{2}(Y^i_{\mu|\nu} Y^\nu_{\underline{k}} Y^\mu_{\underline{j}})\theta^k \wedge \theta^j. \qquad (46)$$

Then

$$C^i_{jk} = Y^\mu_{\underline{j}} Y^\nu_{\underline{k}} Y^i_{\mu|\nu}. \qquad (47)$$

This formula was used to obtain

$$\omega_{ij} = \tfrac{1}{2}(C_{ijk} + C_{ikj} - C_{jki})\theta^k, \qquad (48)$$

which can in turn be used to find

$$\tfrac{1}{2}R_{(ij)(kl)}\theta^k \wedge \theta^l = d\omega_{ij} + \omega_{ik} \wedge \omega^k_j. \qquad (49)$$

Indeed

$$\omega_{ij} = \omega_{kij}\theta^k \Rightarrow \omega_{kij} = \tfrac{1}{2}(C_{ijk} + C_{ikj} - C_{jik}) = \Gamma_{ijk}. \qquad (50)$$

Thus

$$\tfrac{1}{2}(R_{ijlk})\theta^l \wedge \theta^k = \tfrac{1}{2}(\omega_{\underline{l}i\underline{j}|k} + \omega_{mij}\omega^m_{\underline{k}\,\underline{l}} + \omega^m_{\underline{k}i}\omega_{\underline{l}mj})\theta^k \wedge \theta^l. \qquad (51)$$

To include torsion add $(-\omega_{mij}T^m_{kl})$.

From this formula the symmetries $R_{(ij)(kl)} = -R_{(ji)(kl)} = -R_{(ij)(lk)}$ can be read off. The curvature scalar R_S can be calculated

$$R_S = \eta^{ik}\eta^{jl}R_{ijkl} = (\omega_{\underline{l}}{}^{kl}{}_{|\underline{k}} + \omega^{kl}_m\omega^m_{\underline{k}\,\underline{l}} + \omega^{km}_{\underline{k}}\omega^l_{\underline{l}m}). \qquad (52)$$

Using the more familiar Γ_{ijk} symbols we can write

$$R^i{}_{jkl} = (\Gamma^i{}_{\underline{jl}|\underline{k}} + \Gamma^i_{m\underline{k}}\Gamma^m_{\underline{j}l} + \Gamma^i_{jm}\Gamma^m_{\underline{k}\,\underline{l}}), \quad \Gamma^m_{jk} = \eta^{mi}\Gamma_{ijk}. \qquad (53)$$

Thus

$$R_S = \eta^{jk}R^l_{jlk} = \eta^{jk}(\Gamma^l{}_{\underline{jk}|l} + \Gamma^l_{\underline{m}l}\Gamma^m_{\underline{j}k} + \Gamma^l_{jm}\Gamma^m_{\underline{l}\,\underline{k}}). \qquad (54)$$

Now

$$\Gamma^l_{jk} = \tfrac{1}{2}\eta^{lm}(C_{mjk} + C_{mkj} - C_{jkm})$$

and (55)

$$\Gamma^m_{\underline{k}\,\underline{l}} = -C^m_{kl},$$

and

$$R_S = [-2C_{jl}{}^{l|j} + \tfrac{1}{4}C^{lmk}(3C_{lkm} + C_{mkl} + C_{lmk})], \quad (56)$$

$$C^l_{jl} = \eta^{lm}C_{jlm}, \quad C^{lmk} = \eta^{li}\eta^{mu}\eta^{kj}C_{ikj}; \quad (57)$$

each is raised in place. Observe that for a fully antisymmetric constant C_{ijk} which occurs for the rotation group $C_{ijk} = \epsilon_{ijk}$ where ijk run over only three values and then $R_S = -\frac{1}{4}\epsilon^{lmk}\epsilon_{lmk}$. See Sec. IV of paper III.[4] Since the derivatives of the vierbein field Y^i_μ satisfy the equation

$$\begin{aligned} d\theta^i &= -\tfrac{1}{2}C^i_{jk}\theta^j \wedge \theta^k \\ &= -\omega^i_{jk}\theta^j \wedge \theta^k + T^i_{jk}\theta^j \wedge \theta^k, \end{aligned} \quad (58)$$

we can reinterpret the Lagrangian $R_S\sqrt{-g}$, using Eq. (56), as a theory of pure torsion ($\omega^i_{jk} \equiv 0$, but $T^i_{jk} \equiv -\frac{1}{2}C^i_{jk}$). The higher derivatives of Y^i_μ in $C_{jl}{}^{l|j}$, can be removed using by-parts integration. Of course, in the gauge $C^l_{jl} = 0$ these terms are also missing. However, when one integrates by parts, the symmetric part of the Y^i_μ derivative also appears. Thus define

$$D^i_{jk} \equiv (Y^\mu_j Y^\nu_k + Y^\mu_k Y^\nu_j)(Y^i_{\mu|\nu}) \quad (59)$$

analogous to C^i_{jk}. Now find the quantity Δ^l_{ik} defined implicitly by

$$\begin{aligned} \omega^\alpha_{\nu\mu} &= Y^\alpha_l Y^i_\mu Y^k_\nu\{\tfrac{1}{2}[\eta^{lj}(C_{jik} + C_{jki}) + D^l_{ik}]\} \\ &\equiv Y^\alpha_l Y^i_\mu Y^k_\nu \Delta^l_{ik} \\ &= \Gamma^\alpha_{\mu\nu}. \end{aligned} \quad (60)$$

$\Delta^l_{ik} = \Delta^l_{ki}$ implies that $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$.

The by-parts integrated action is simply

$$A = \kappa \int \eta^{ik}\Delta^l_{\underline{km}}\Delta^m_{l\underline{i}}(Y d^4x). \quad (61)$$

Note that this is not the naive action $T_{jkl}T^{jkl}$ or $C_{mkl}C^{jkl}$ for the frame to which one would naturally be led by analogy with gauge theories.

In fact in a gaugelike theory for both θ^i and ω^i_j the obvious choice would be $\kappa\langle T|T\rangle + \langle R|R\rangle$ as the action. Here κ has dimensions of length to the minus second power. Introducing[8] the Planck length L satisfying $L^2 = 1/\kappa$ and noting that dim$[R]$ is zero and so is dim$[(1/L)T]$, we can write the action using $\hat{T} = (1/L)T$ as

$$\begin{aligned} \langle\hat{T}|\hat{T}\rangle + \langle R|R\rangle &= \left(\frac{1}{L}\right)^2 \langle T|T\rangle + \langle R|R\rangle \\ &= \kappa\langle T|T\rangle + \langle R|R\rangle, \end{aligned} \quad (62)$$

or,

$$A = -\int (\kappa T^{klm}T_{klm} + R_{ijkl}R^{ijkl})\sqrt{-g}\,d^4x. \quad (63)$$

Note that when functional integrals are evaluated carefully, they are defined in Euclidean space. The metric η^{ij} is replaced by δ^{ij}. The internal structure becomes ISO(4) and ghost fields are removed. Here the connection coefficients and the vierbein fields are to be varied separately. We will not examine this possibility further here; even though this action is the one, one would be led to based on analogy with gauge theories' quadratic actions. See Secs. III of paper II and IV of paper III.[4] Here the gauge group is the Poincaré group, ISO(3, 1), where ω^i_j is associated with SO(3, 1) and θ^i with R^4, translations.

$$\begin{pmatrix} \omega^i_j & \theta^i \\ 0 & 0 \end{pmatrix}$$

is the matrix[3] of connections. This matrix can be obtained by contracting the group SO(4, 1) or SO(3, 2) connections.

It is important to note that the action A differs from the gauging of a purely internal ISO(3, 1) because of various factors arising from derivatives and differentials. More specifically we can examine the purely R^4 piece of the action and find

$$\begin{aligned} \tfrac{1}{4}\eta_{in}T^i_{jk}T^n_{lm}(\tfrac{1}{2}\epsilon^{jk}{}_{st}\theta^s \wedge \theta^t) \wedge \theta^l \wedge \theta^m &= (\tfrac{1}{4}\eta_{in}Y^i_{\sigma|\lambda}Y^n_{\alpha|\beta})[\tfrac{1}{2}\epsilon^{jk}{}_{st}\epsilon^{\mu\nu\alpha\beta}(Y^\sigma_j Y^\lambda_k)(Y^s_\mu Y^t_\nu)]\,d^4x \\ &= -(\tfrac{1}{4}\eta_{in}Y^i_{\sigma|\lambda}Y^n_{\alpha|\beta})\{(\tfrac{1}{2}Y^\sigma_j Y^\lambda_k Y^\alpha_l Y^\beta_m)[\det(Y^i_\mu)](\eta^{jl}\eta^{km} - \eta^{jm}\eta^{kl})\}\,d^4x. \end{aligned} \quad (64)$$

A purely internal R^4 would yield only $\eta_{in}Y^i_{\sigma|\lambda}Y^n_{\alpha|\beta}\frac{1}{2}g^{\sigma\bar{\alpha}}g^{\lambda\bar{\beta}}d^4x$ with $g^{\alpha\beta} = \eta^{\alpha\beta}$. If it were not for the factor $Y^\sigma_j Y^\lambda_k Y^\alpha_l Y^\beta_m \det Y^i_\mu$, the internal and space-time actions would be the same. This would then permit quantization along the lines indicated by 't Hooft.[19]

From a gauge theory viewpoint we should quantize Y^i_μ, not $g_{\mu\nu}$; then $g_{\mu\nu} = Y^i_\mu\eta_{ij}Y^j_\nu$ is a composite of the Y^i_μ's. Even if the torsion squared (or translation squared) action is renormalizable (possibly under some conditions on the coefficient $Y^\sigma_j Y^\lambda_k Y^\alpha_l Y^\beta_m \det Y^i_\mu$, appropriately symmetrized), it is far from clear that the action of Eq. (61) or

$$[-2T_{jl}{}^{l|j} + \tfrac{1}{4}T^{lmk}(3T_{lkm} + T_{mkl} + T_{lmk})]Y \quad (65)$$

would then be renormalizable. We remark that gauging only translations is structurally simpler than Einstein's theory. For further remarks see Secs. III of paper II and IV of paper III.[4]

What happens in the case of Yang-Mills fields on a curved manifold? Here we take

$$K = \begin{pmatrix} C & 0 \\ 0 & F \end{pmatrix}, \quad P = \begin{pmatrix} R & 0 \\ 0 & F \end{pmatrix}, \quad (66)$$

$$R = d\omega + \omega \wedge \omega, \quad F = dB + B \wedge B, \quad C = C^{ij}_{kl}\theta^k \wedge \theta^l \Sigma_{ij},$$

and define the action (Y^i_μ and B^A_μ are the dynamical fields)

$$\mathcal{A} = \langle K | P \rangle = \int \tfrac{1}{2}\overline{K} * \overset{\circ}{\wedge} P$$

$$= \int (\kappa R_S - \tfrac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A)\sqrt{-g}\, d^4x. \tag{67}$$

In the rest of this paper we will suppress reference to the choice of topology on the base manifold. We will assume it to be quite general and consistent with the field dynamics. In Ref. 4 it is pointed out that sources and holes in the manifold are intimately related. Manifolds with holes admit regular connections which can behave as if there were distribution valued sources. E.g., let $\chi_D(x)$ be the characteristic function of D, a subset of space-time M, and let $\overline{D}$ be its complement. Then if $\overline{D}$ is excised from M the action is $\int_D (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) d^4x$. The equations of motion are $\chi_D(x)(\partial^2 A_\mu - \partial_\mu \partial \cdot A) = 0$. $(\partial^2 A_\mu - \partial_\mu \partial \cdot A)$ can be arbitrary on the interior of $\overline{D}$; only the surface is relevant. By extending the action using conservation of energy, effective equations of motion for $\overline{D}$ arise. In the limit as $\overline{D}$ becomes a set point moving in time, one finds that they obey the classical equations of electrodynamics.[4]

III. BOSE MATTER FIELDS

Now we will consider a formalism leading to actions having matter as well as gauge and vierbein fields. We will wind up with actions having a conventional expression in terms of their component fields, but we will start from a viewpoint which may ultimately lead to a deeper understanding of the role of matter fields in the geometry and topology associated with gauge theories.

Since matter fields ϕ interacting with gauge bosons B_μ carry no space-time index μ, they are scalars, that is, zero-forms or functions. We will deal with spinors and other objects carrying Lorentz indices shortly. Thus it is natural to consider an object which has sectors containing gauge bosons (one-forms) *and* matter fields (zero-forms). Another technique[4,12] for including matter fields is to go to a $(4+1)$-dimensional space-time, but this approach is somewhat artificial since the fields depend only on x, the cylindricity assumption, if the extra direction is Bose. The matter fields are associated with the fifth direction which is to be integrated over. We will give details at the end of this section.

To make this idea more concrete let us consider a specific example. Let SU(n) be the gauge group and let C^n (complex n-space) be the domain of the matter fields. When transporting a frame $\overline{e}_a$ along a line $x(\lambda)$, we found

$$\delta \overline{e}_a = \lim_{\lambda_e \to 0} \int_0^{\lambda_e} \overline{e}_b \left[B^b_{\mu a}(x(\lambda)) \frac{dx^\mu(\lambda)}{d\lambda} \right] d\lambda$$

$$\equiv \lim_{\lambda_e \to 0} \int_0^{\lambda_e} \overline{e}_b [B^b_a(\lambda)]\, d\lambda. \tag{68}$$

Here $x(\lambda)$ is a curve in space parametrized by $\lambda \in [0, \lambda_e]$ with $x(0)$ the initial point and $x(\lambda_e)$ the end point of the arc. $B^b_a(\lambda) \equiv B^b_{\mu a}(x(\lambda)) dx^\mu(\lambda)/d\lambda$ is an SU(n)-valued function of λ. To introduce matter fields we must extend the number of basis elements by at least one: $\overline{e}_a \to (\overline{e}_a, \overline{e})$. Now in the limit that $\lambda_e \to 0$ we found $\delta \overline{e}_a = \overline{e}_b [B^b_{\mu a}(x) \delta x^\mu] = \overline{e}_b B^b_a(x)$, which depended on the initial direction of the arc through δx^μ. Since matter fields are not dependent on the direction of the arc (they carry no index), we expect the coefficient to be a function. This can be obtained through the artifice of a distribution valued connection as follows:

$$[\overline{e}_a, \overline{e}] = \lim_{\lambda_e \to 0} \int_0^{\lambda_e} [\overline{e}_b, \overline{e}] \begin{bmatrix} B^b_a(\lambda) & \phi^b(\lambda) \\ 0 & 0 \end{bmatrix} d\lambda, \tag{69}$$

where

$$B^b_a(\lambda) = B^b_{\mu a}(x(\lambda)) \frac{dx^\mu(\lambda)}{d\lambda}, \tag{70}$$

$$\Phi^b(\lambda) = \phi^b(x(\lambda)) \delta_D(\lambda). \tag{71}$$

$\delta_D(\lambda)$ is the Dirac δ function and represents I for line integrals. Just as the first term B^b_a can be associated with one-forms (infinitesimal displacement) the second can be associated with zero-forms [functions at $x(0)$]. Indeed we find the limits above to be given as

$$\delta[\overline{e}_a, \overline{e}] = [\overline{e}_b, \overline{e}] \begin{bmatrix} B^b_{\mu a} \delta x^\mu & \phi^b I \\ 0 & 0 \end{bmatrix}, \tag{72}$$

$\delta x^\mu \in \Lambda^1(M)$, forms, $I \in \Lambda^0(M)$ functions. We will suppress the I as understood. We notice that the relation

$$\delta[\overline{e}_a, \overline{e}] = [\overline{e}_b, \overline{e}] \begin{bmatrix} B^b_{\mu a} \delta x^\mu & \phi^b I \\ 0 & 0 \end{bmatrix} = [\overline{e}_b, \overline{e}][\Omega] \tag{73}$$

defines the augmented connection Ω. In the $(4+1)$-dimensional approach[4,12] the I is replaced by $d\theta$, an infinitesimal for the circle or a Grassmann manifold. B and ϕ are independent of θ and $B^A_\theta = 0$.

By making first one displacement, then another, we find

$$\tfrac{1}{2}(d\delta-\delta d)[\bar{e}_a,\bar{e}]=d[\bar{e}_b B^b_{\mu a}\delta x^\mu,\bar{e}_b\phi^b]-\delta[\bar{e}_b B^b_{\nu a}dx^\nu,\bar{e}_b\phi^b]$$

$$=\{\bar{e}_b[\tfrac{1}{2}(B^b_{\underline{\mu}a|\underline{\nu}}+B^b_{\underline{\nu}c}B^c_{\underline{\mu}a})](dx^\nu\delta x^\mu-\delta x^\nu dx^\mu),\bar{e}_b(\bar{\phi}^b_{|\nu}+B^b_{\nu a}\phi^a)(dx^\nu-\delta x^\nu)\}$$

$$=[\bar{e}_b,\bar{e}]\begin{bmatrix}\tfrac{1}{2}F^b_{a\mu\nu}dx^\nu\wedge dx^\mu & \phi^b_{|\nu}+B^b_{\nu a}\phi^a\\ 0 & 0\end{bmatrix}$$

$$\equiv[\bar{e}_b,\bar{e}][P],\text{ with } P \text{ the augmented curvature,} \tag{74}$$

$$F_{\nu\mu}=B_{\underline{\mu}|\underline{\nu}}+[B_\nu,B_\mu]. \tag{75}$$

We note that $P\equiv d\Omega+\Omega\wedge\Omega$ can be defined and calculated using the standard rules of exterior differentiation. But we have introduced these ideas in a pictorial fashion using "real" displacements. The forms and the displacements can be distinguished: $\hat{d}x^\nu\wedge\hat{d}x^\mu\equiv\frac{1}{2}(dx^\nu\delta x^\mu-\delta x^\nu dx^\mu)$ and $\hat{d}x^\nu=\frac{1}{2}(dx^\nu-\delta x^\nu)$. However, the carets have been dropped. The product $\hat{d}x^\nu\wedge\hat{d}x^\mu$ can be associated with an area element and $\hat{d}x^\nu$ with a line element as in Fig. 5. Note that

$$(\hat{d}x^\nu)^*\wedge\hat{d}x^\mu=\eta^{\nu\mu}d^4x. \tag{76}$$

The action can now be obtained from P, the augmented curvature. By taking its Hilbert square, the open circle denotes another product such as trace

$$\mathfrak{a}=\langle P|P\rangle=\tfrac{1}{2}\int\begin{bmatrix}\bar{F}^* & 0\\ \bar{D}^* & 0\end{bmatrix}\overset{\circ}{\wedge}\begin{bmatrix}F & D\\ 0 & 0\end{bmatrix}=\tfrac{1}{2}\int\frac{1}{n}\mathrm{tr}\begin{bmatrix}\bar{F}^*\wedge F & \bar{F}^*\wedge D\\ \bar{D}^*\wedge F & \bar{D}^*\wedge D\end{bmatrix}$$

$$=\int[-\tfrac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A+\tfrac{1}{2}(\bar{\phi}_{|\nu}+\bar{\phi}\bar{B}_\nu)(\phi^{|\nu}+B^\nu\phi)]\,d^4x, \tag{77}$$

$$D\equiv(\phi_{|\nu}+B_\nu\phi)dx^\nu,\quad F=(F_{\mu\nu})dx^\nu\wedge dx^\mu,\quad \Lambda_A\circ\Lambda_B\equiv\frac{1}{n}\mathrm{tr}(\Lambda_A\Lambda_B)=\delta_{AB}, \tag{78}$$

which is recognizable as the action for C^n scalar fields interacting with gauge fields in SU(n).

In the five-dimensional approach the connection is assumed to have the form

$$d[\bar{e}_a,\bar{e}]=[\bar{e}_b,\bar{e}]\begin{bmatrix}B^A_\mu(x)i\Lambda^b_{Aa}dx^\mu & \phi^b(x)d\theta\\ 0 & 0\end{bmatrix}, \tag{79}$$

where

$$B^b_{\mu a}=B^A_\mu i\Lambda^b_{Aa}\text{ defines }B^A_\mu. \tag{80}$$

The curvature is

$$[\bar{e}_a,\bar{e}]P=\tfrac{1}{2}[d,\delta][\bar{e}_a,\bar{e}]$$

$$=[\bar{e}_b,\bar{e}]\begin{bmatrix}\tfrac{1}{2}(B^b_{\underline{\mu}a|\underline{\nu}}+B^b_{\underline{\nu}c}B^c_{\underline{\mu}a})dx^\nu\wedge dx^\mu & (\phi^b_{|\nu}+B^b_{\nu a}\phi^a)dx^\nu\wedge d\theta\\ 0 & 0\end{bmatrix}. \tag{81}$$

With an appropriate definition of the product $\circ$ we find that the action is

$$\langle P|P\rangle=\int[-\tfrac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A+(\bar{\phi}_{|\nu}-i\bar{\phi}\Lambda_AB^A_\nu)(\phi^{|\nu}+iB^{\nu A}\Lambda_A\phi)]\,d^4x\,d\theta. \tag{82}$$

Since the integrand is independent of $d\theta$ we can integrate over the fifth variable (normalized to one) and find the usual action

$$\langle P|P\rangle=\int[-\tfrac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A+(\bar{\phi}_{|\nu}-i\bar{\phi}\Lambda_AB^A_\nu)(\phi^{|\nu}+iB^{\nu A}\Lambda_A\phi)]\,d^4x. \tag{83}$$

This choice of connection makes the mathematics of the "augmented" connection obvious. The point is that a connection with values as described above, $B_\mu(x)dx^\mu$ and $\phi(x)d\theta$, can always be associated with an augmented connection $B_\mu(x)dx^\mu$ and $\phi(x)I$ in this fashion. The physics is the same. The main difficulty with the (4 + 1)-dimensional version is a philosophical one. Why no dependence on this fifth variable?[12] It acts only as a crutch to get the usual action.

Indeed this procedure can be extended to larger groups. One can form a unified theory on a space of 4 + 1 = 5 dimensions, and then break the unified group by a special choice of connection. That is, if G_n is the initial group [having an action on an n-dimensional space as SU(n) does], a breaking to $G_p\times G_q$ with

$p+q=n$ can be performed. Let $dy^M=(dx^\mu, \not d\theta)$; then a G_n connection on five-space is broken by restricting it from the most general

$$\begin{bmatrix} B^a_{Mb}(y)dy^M & \phi^a_{Mj}(y)dy^M \\ \phi^i_{Mb}(y)dy^M & B^i_{Mj}(y)dy^M \end{bmatrix} \text{ to } \begin{bmatrix} B^a_{\mu\, b}(x)dx^\mu & \phi^a_j(x)\not d\theta \\ \phi^j_b(x)\not d\theta & B^i_{\mu\, j}(x)dx^\mu \end{bmatrix}. \tag{84}$$

Define $\not d\theta^i$ to be $\theta^i d\theta^i$ (or $d\theta^i$) if $d\theta^i \wedge d\theta^j$ is symmetric (or antisymmetric). There would therefore be no diaton gauge fields. The Yang-Mills two-form is given as follows ($d\not d\theta$ terms are omitted since their dual vanishes):

$$\begin{bmatrix} (B^a_{\mu|\nu b}+B^a_{\nu c}B^c_{\mu\, b})dx^\nu \wedge dx^\mu & (\phi^a_{j|\nu}+B^a_{\nu b}\phi^b_j-\phi^a_k B^k_{\nu j})dx^\nu \wedge \not d\theta \\ (\phi^j_{b|\nu}-\phi^j_c B^c_{\nu b}+B^j_{\nu k}\phi^k_b)dx^\nu \wedge \not d\theta & (B^i_{\mu|\nu j}+B^i_{\nu k}B^k_{\mu\, j})dx^\nu \wedge dx^\mu \end{bmatrix}. \tag{85}$$

The action is found from $\langle P|P\rangle=\mathfrak{G}$. After $\not d\theta$ integration it is

$$\int \left(-\tfrac{1}{4}\bar{F}^{ab}_{\nu\mu}F^{\nu\mu}_{ab}-\tfrac{1}{4}\bar{F}^{ij}_{\nu\mu}F^{\nu\mu}_{ij}+\bar{D}^{aj}_\nu D^\nu_{aj}+\bar{D}^{ja}_\nu D^\nu_{ja}\right)d^4x, \tag{86}$$

where

$$F^{ab}_{\nu\mu}=B^{ab}_{\underline{\mu}|\underline{\nu}}+B^a_{\underline{\nu}c}B^{cb}_{\underline{\mu}}, \quad F^{ij}_{\nu\mu}=B^{ij}_{\underline{\mu}|\underline{\nu}}+B^i_{\underline{\nu}k}B^{kj}_{\underline{\mu}}, \tag{87}$$

and

$$\begin{aligned} D^{aj}_\nu &= \phi^{aj}_{|\nu}+B^a_{\nu b}\phi^{bj}-\phi^a_k B^{kj}_\nu, \\ D^{jb}_\nu &= \phi^{jb}_{|\nu}-\phi^j_c B^{cb}_\nu+B^j_{\nu k}\phi^{kb}. \end{aligned} \tag{88}$$

Appropriate definition of the product $\circ$ has been used. The effective action is invariant under $G_p \times G_q$. When unitary groups are decomposed in this fashion, an extra u(1) invariance arises, since

$$\mathrm{su}(n) \to \mathrm{su}(p)+\mathrm{su}(q)+\mathrm{u}(1)+(C^p)^q.$$

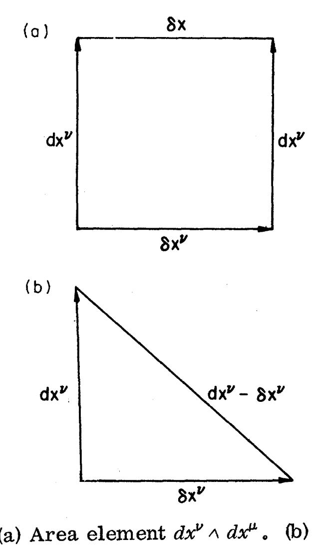

FIG. 5. (a) Area element $dx^\nu \wedge dx^\mu$. (b) Diagonal line element.

This symmetry breaking by use of an extra dimension can be represented diagrammatically. See Fig. 6. The coset[4,31] generators $(C^p)^q$ are associated with $d\theta$, the generator for S^1. It is precisely this sort of identification of cosets with $d\theta$ which allowed us to use augmented connections. That is, instead of considering $B_\mu dx^\mu$ with $B_\mu \in \mathrm{su}(p)+\mathrm{su}(q)+\mathrm{u}(1)$ and $\phi d\theta$ with $\phi \in (C^p)^q$, the augmented connection uses the same $B_\mu dx^\mu$ but replaces $\phi d\theta$ by ϕI. The pair (I, dx^μ) replaces $(d\theta, dx^\mu)$. After integration on θ the effective actions are identical. Distinctions between SU(n) and SU(p,q) with $p+q=n$ are being ignored here. Please note that this technique is reminiscent of Kaluza and Klein's.[12]

The concept of Fermi coordinates permits another interpretation of augmented connections. Knowing that, if θ_i is a Fermi coordinate, $\int d\theta_i = 0$ and $\int \theta_i d\theta_i = 1$ for each i, and remembering that duality can be viewed as the operation which yields the appropriate volume element, e.g.,

$$g_{\nu\lambda}[(dx^\nu)^* \wedge dx^\lambda] = 4\sqrt{-g}\,d^4x, \tag{89}$$

as noted in Sec. II of paper III, we define a generalized duality operating on supersymmetric coordinates which has a similar effect. Thus on a two-dimensional Fermi manifold

$$\int (\not d\theta_1)^* \wedge \not d\theta_1 \equiv \int (\theta_1 d\theta_1) \wedge (\theta_2 d\theta_2) = 1. \tag{90}$$

Or for a (4, 2) (Bose, Fermi) supermanifold we define the operation * on the various kinds of two-forms in the following way:

$\underline{FF}$: $$(\not d\theta_1 \wedge \not d\theta_2)^* \wedge (\not d\theta_1 \wedge \not d\theta_2) \equiv -(\theta_1 d\theta_1)\wedge(\theta_2 d\theta_2)(\sqrt{-g}\,d^4x), \tag{91}$$

$\underline{FB}$: $$(\not d\theta_1 \wedge dx^\nu)^* \wedge (\not d\theta_1 \wedge dx^\lambda) \equiv (\theta_1 d\theta_1)\wedge(\theta_2 d\theta_2)[\sqrt{-g}(-g^{\nu\lambda})d^4x], \tag{92}$$

$\underline{BB}$: $$(dx^\rho \wedge dx^\nu)^* \wedge (dx^\sigma \wedge dx^\lambda) \equiv -(\theta_1 d\theta_1)\wedge(\theta_2 d\theta_2)[\sqrt{-g}\,(g^{\rho\bar\sigma}g^{\nu\bar\lambda})d^4x]. \tag{93}$$

Algebra of Fields	su(n)	su(p) + su(q) + u(1) + $(\mathbb{C}^p)^q$; p + q = n
Components of Algebra	$\begin{bmatrix} iA_a dy^a & i\phi_a dy^a \\ i\bar{\phi}_a dy^a & iB_a dy^a \end{bmatrix}$	$\begin{bmatrix} iA_\mu dx^\mu & i\phi d\theta \\ i\bar{\phi} d\theta & iB_\mu dx^\mu \end{bmatrix}$
Differentials in Base Manifold	$dy^a = \{dx^\mu, d\theta\}$	dx^μ; $d\theta$

FIG. 6. This is a diagram of the breaking of su(u) to su(p) + su(q) + u(1) by associating a coset decomposition with either a fifth infinitesimal (or one can replace $d\theta$ by I for the augmented version); $d\theta$ can be a differential of a Bose or Grassmann variable. The coefficients are functions of x^μ only.

For the case of a (4, 1) (Bose, Fermi) manifold we find

$$FB:\ (d\theta \wedge dx^\nu)^* \wedge (d\theta \wedge dx^\lambda) \equiv (\theta d\theta) \wedge (\sqrt{-g}\, g^{\nu\lambda} d^4x)\,, \quad (94)$$

$$BB:\ (dx^\rho \wedge dx^\nu)^* \wedge (dx^\sigma \wedge dx^\lambda) \equiv -(\theta d\theta) \wedge (\sqrt{-g}\, g_{\rho\bar{\sigma}} g_{\nu\bar{\lambda}}\, d^4x)\,, \quad (95)$$

and similarly for (4, n) (Bose, Fermi) manifolds and for p-forms with $p \neq 2$. Note that this definition of the asterisk implies that only forms with Bose coefficients have a nonvanishing Hilbert square. It forces the cylindricity condition.[12]

Thus consider the following connection on a (4, 1) (Bose, Fermi) manifold. We omit any θ dependence since its Hilbert square would vanish:

$$\Omega = \begin{bmatrix} B^a_{\nu b}(x) dx^\nu & \phi^a(x) d\theta \\ 0 & 0 \end{bmatrix}. \quad (96)$$

It has the following curvature:

$$P = d\Omega + \Omega \wedge \Omega = \begin{bmatrix} \tfrac{1}{2}(B^a_{\underline{\nu} b}|_{\underline{\mu}} + B^a_{\underline{\nu} c} B^c_{\underline{\mu} b}) dx^\nu \wedge dx^\mu & (\phi|^a_\nu + B^a_{\nu b}\phi^b) dx^\nu \wedge d\theta \\ 0 & 0 \end{bmatrix}. \quad (97)$$

Taking the Hilbert square yields, on a flat base manifold,

$$\langle P | P \rangle = \int (-\tfrac{1}{4}\bar{F}^{ab}_{\mu\nu} F^{\mu\nu}_{ab} + \bar{D}^a_\nu D^\nu_a)(\theta d\theta) d^4x = \int (-\tfrac{1}{4}\bar{F}^{ab}_{\mu\nu} F^{\mu\nu}_{ab} + \bar{D}^a_\nu D^\nu_a) d^4x \quad (98)$$

with

$$F^{ab}_{\nu\mu} = B^{ab}_{\underline{\mu}}|_{\underline{\nu}} + B^a_{\underline{\nu} c} B^{cb}_{\underline{\mu}} \text{ and } D^a_\nu = \phi|^a_\nu + B^a_{\nu b}\phi^b\,. \quad (99)$$

Generalization to other coset decompositions of other internal-symmetry groups is obvious.

IV. FERMI MATTER FIELDS

Now we will show how to include fermions. To do this we follow the lead of Pauli, Cartan, Dirac, Weyl, and others.[15] Simply stated, the idea is to replace the basis e_μ by matrices γ_μ with respect to a spinor basis $\bar{s}_\alpha$ (we will employ Greek letters near the beginning of the alphabet for spinors and the others for vectors). Thus $\gamma_\mu = \gamma_\mu{}^\alpha{}_\beta \bar{s}_\alpha \otimes s^\beta$; these $\bar{s}_\alpha$ will be the four spinor basis vectors for the spin-$\tfrac{1}{2}$ representation and γ_μ will be the usual Dirac γ matrices. Thus $e_\mu \cdot e_\nu = g_{\mu\nu}$ becomes $\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu} I$, where I is the matrix, $I = I^\alpha_\beta \bar{s}_\alpha \otimes s^\beta$, identity in spinor space. In general the γ_μ matrices are $2^n \times 2^n$ if μ takes on $2n$ or $2n+1$ values. We can think of spinors (Dirac spinors) as being the 2^n component column vectors on which these matrices act, though, more accurately, the spinors are the elements of the full Clifford algebra (all 2^n elements) on a space of dimension n.[15] Thus, in four dimensions, a spinor is an element of the two-dimensional Clifford algebra ($\chi = \chi^0 I + \chi^1\sigma_1 + \chi^2\sigma_2 + \chi^{12}\sigma_1\sigma_2$; here the σ matrices are Pauli's). There is a linear map relating the χ's to the ψ's. For details Cartan's *Theory of Spinors* is as clear a reference as one is likely to find.[15] Now, as described above, we use the vierbein (tetrad) formalism, since this is compatible with an antisymmetric connection and a constant metric. Thus we replace e_i by γ_i, where $e_i \cdot e_j = \eta_{ij}$ and $\frac{1}{2}\{\gamma_i, \gamma_j\} = \eta_{ij} I$. The spin-$\frac{1}{2}$ representation of the Lorentz generators is $\sigma_{ij} = \frac{1}{8} i[\gamma_i, \gamma_j]$. Here

$$[\sigma_{ij}, \sigma_{kl}] = i\epsilon_{(ij)(kl)}^{(mn)} \sigma_{mn} \tag{100}$$

with

$$\begin{aligned}\epsilon_{(kj)(kl)}^{(mn)} &= \eta_{\overline{i}\underline{k}} \eta_{\underline{l}}^{m} \eta_{\overline{j}}^{n} \\ &= \eta_{ik}\eta_l^m \eta_j^n + \eta_{jl}\eta_k^m \eta_i^n - \eta_{il}\eta_k^m \eta_j^n - \eta_{jk}\eta_l^n \eta_i^n .\end{aligned} \tag{101}$$

Underlined indices are antisymmetrized together as are those with overbars. The matrices γ_i and σ_{ij} can be written as

$$\gamma_i = \gamma_i{}^{\alpha}_{\beta} \bar{s}_{\alpha} \otimes s^{\beta} \text{ and } \sigma_{ij} = \sigma_{ij}{}^{\alpha}_{\beta} \bar{s}_{\alpha} \otimes s^{\beta} . \tag{102}$$

We will suppress the symbol $\otimes$ and write, e.g., $\gamma_i = \bar{s}_\alpha \gamma_i{}^{\alpha}_{\beta} s^{\beta}$. Now the frame we will be interested in is $\bar{s}_\alpha$ and $\bar{s}$ (in analogy with $\bar{e}_\alpha$ and $\bar{e}$). The infinitesimal change is given as

$$\delta\bar{s}_\alpha = \bar{s}_\beta(-i\Gamma_i{}^{\beta}_{\alpha} Y^i_\mu \delta x^\mu) = \bar{s}_\beta(-i\Gamma_i{}^{\beta}_{\alpha}\theta^i) , \tag{103}$$

$$\delta\bar{s} = \bar{s}_\alpha \psi^\alpha . \tag{104}$$

Here $\Gamma_i{}^{\beta}_{\alpha} \equiv \Gamma^j_{ik}[\sigma^\kappa_j]^\beta_\alpha$. If instead of the spin-$\frac{1}{2}$ representation we had used the spin 1, $\bar{s}_\alpha$ is replaced by e_j and $[\sigma^k_j]^\beta_\alpha$ becomes replaced by the spin-1 representation of the Lorentz generators $[\frac{1}{2}(\eta^{km}\eta_{jl} - \eta^m_j \eta^k_l)]$. Indeed one could generalize this procedure to other representations of the Lorentz group.

With this choice of connection for $\Gamma_i{}^{\beta}_{\alpha}$ one can show that

$$\begin{aligned}\delta\gamma_i = \delta(\gamma_i{}^{\alpha}_{\beta}\bar{s}_\alpha \otimes s^\beta) &= \Gamma^k_{ji}\gamma_k{}^{\alpha}_{\beta}(\bar{s}_\alpha \otimes s^\beta) Y^j_\mu \delta x^\mu \\ &= \Gamma^k_{ji}\gamma_k \theta^i .\end{aligned} \tag{105}$$

Since γ_i corresponds to e_i, we find that the change of basis γ_i is the same as the familiar change of vector basis e_i. Thus we can proceed to calculate the curvature associated with changes in $\bar{s}_\alpha$ and relate it to the curvature of a space-time manifold. Let us see this in more detail.

Consider the transport of the frame $[\bar{s}_\alpha, \bar{s}]$ around a loop

$$\tfrac{1}{2}(d\delta - \delta d)\bar{s}_\alpha = \bar{s}_\beta\{R^i_{jkl}\theta^k \wedge \theta^l \tfrac{1}{2}[\sigma^j_i]^\beta_\alpha\}, \tag{106}$$

$$\tfrac{1}{2}(d\delta - \delta d)\bar{s} = \bar{s}_\alpha(\psi|^\alpha_i - i\Gamma_i{}^{\alpha}_{\beta}\psi^\beta)\theta^i \tag{107}$$

$$= \bar{s}_\alpha[\psi|^\alpha_i - i\Gamma^j_{ik}[\sigma^k_j]^\alpha_\beta \psi^\beta]\theta^i . \tag{108}$$

Here[8]

$$\begin{aligned}R^i_{jkl} &= \Gamma^i_{j\underline{l}}|_{\underline{k}} + \Gamma^i_{m\underline{k}}\Gamma^m_{j\underline{l}} + \Gamma^i_{jm}(\Gamma^m_{\underline{kl}} - T^m_{\underline{kl}}), \\ \theta^i &= Y^i_\mu dx^\mu .\end{aligned} \tag{109}$$

In order to obtain the action we can first extract the coefficients of the augmented curvature

$$\tfrac{1}{2}(d\delta - \delta d)[\bar{s}_\alpha, \bar{s}] = [\bar{s}_\beta, \bar{s}]\begin{bmatrix} \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l[\sigma^j_i]^\beta_\alpha & [\psi^\alpha_i - i\Gamma^j_{ik}(\sigma^k_j)^\alpha_\beta\psi^\beta]\theta^i \\ 0 & 0 \end{bmatrix} \equiv [\bar{s}_\alpha, \bar{s}][P] . \tag{110}$$

To obtain Einstein's action for the curvature of space-time it was necessary to introduce the constant curvature tensor C^{ij}_{kl} and an associated two-form which we denoted as

$$C = \tfrac{1}{2}\kappa(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l)\theta^k \wedge \theta^l \Sigma_{ji} , \tag{111}$$

where $(\Sigma_{ij})^k_l = \frac{1}{2}(E_{ij} - E_{ji})^k_l$ was the vector representation of the Lorentz generators. Here we replace $(\Sigma_{ij})^k_l$ by $(\sigma_{ij})^\alpha_\beta$, the spin-$\frac{1}{2}$ representation of those generators, and

$$C = \tfrac{1}{2}\kappa(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l)\theta^k \wedge \theta^l \sigma_{ji} = \kappa\sigma_{lk}\theta^k \wedge \theta^l . \tag{112}$$

Just as the introduction of the constant curvature form leads to an action linear in the gauge-covariant derivative of the Γ^j_{ik}'s we find that by introducing an appropriate one-form we can have Dirac spinors on a curved manifold. It seems that there may be a deep relationship between Einstein's action and Dirac's action. Thus we introduce the augmented version of the constant curvature two-form. C is no longer constant and instead it is given as follows. We will refer to this as the K-structure:

$$K \equiv \begin{bmatrix} -\tfrac{1}{2}\kappa(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l)\theta^k \wedge \theta^l(\sigma_{ij})^\alpha_\beta & -i\gamma^\alpha_{i\beta}\theta^i\psi^\beta \\ 0 & 0 \end{bmatrix} \quad \alpha, \beta \text{ indices suppressed on next lines,}$$

$$= \begin{bmatrix} -\kappa\sigma_{kl}\theta^k \wedge \theta^l & -i\gamma_i\theta^i\psi \\ 0 & 0 \end{bmatrix} = i\begin{bmatrix} \tfrac{1}{4}\kappa \not\theta \wedge \not\theta & -\not\theta\psi \\ 0 & 0 \end{bmatrix}, \quad \not\theta = \gamma_i\theta^i. \tag{113}$$

We can construct the action. Here the open circle will be proportional to the trace, and, for typographical reasons, we will use parentheses with bars, $\overline{(\,,\,)}$, to denote that the enclosed matrix is to be (Dirac) conjugated. Thus $\overline{(K)} \equiv \bar{K}$.

$$\mathfrak{a} = \langle K | P\rangle = \int \frac{1}{2}\overline{\left(\begin{bmatrix} -\tfrac{1}{4}\kappa i \not\theta \wedge \not\theta & -i\not\theta\psi \\ 0 & 0 \end{bmatrix}^*\right)} \overset{\circ}{\wedge} \begin{bmatrix} \tfrac{1}{2}R^i_{jkl}\,\theta^k \wedge \theta^l\sigma^j_i & (\psi_i - i\Gamma^j_{ik}\sigma^k_j\psi)\theta^i \\ 0 & 0 \end{bmatrix}$$

$$= \int \tfrac{1}{2}[\kappa R_S + i\bar{\psi}\gamma^i(\vec{\partial}_i - i\Gamma^j_{ik}\sigma^k_j)\psi]\sqrt{-g}\; d^4x; \quad \sqrt{-g} = \det(Y^i_\mu). \tag{114}$$

The bar denotes Dirac adjoint. To recover the usual fermion action we need the contribution from the adjoint basis, s^β and s. We use a formalism which admits mixing between s^α and $\bar{s}_\alpha$. The connection is

$$\Omega = \begin{bmatrix} i\Gamma^j_{ik}[\sigma^k_j]^\beta_\alpha\theta^i & \psi^\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{(i\Gamma^j_{ik}[\sigma^k_j]^\beta_\alpha\theta^i)} & \bar{\psi}_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{115}$$

$$\delta\bar{s}_\alpha = \bar{s}_\beta i\Gamma^j_{ik}[\sigma^k_j]^\beta_\alpha\theta^i,$$

$$\delta\bar{s} = \bar{s}_\beta\psi^\beta,$$

and (116)

$$\delta s^\alpha = [\bar{i}\Gamma^j_{ik}\bar{\sigma}^k_j]^\alpha_\beta s^\beta,$$

$$\delta s = \bar{\psi}_\alpha s^\alpha,$$

since $\delta[\bar{s}_\alpha, \bar{s}, s^\alpha, s] = [\bar{s}_\beta, \bar{s}, s^\beta, s]\Omega$ in this peculiar notation.

$$P = \begin{bmatrix} \tfrac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l\sigma^j_i & (\psi_{|i} - i\Gamma^j_{ik}\sigma^k_j\psi)\theta^i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{(\tfrac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l\sigma^j_i)} & \overline{([\psi_{|i} - i\Gamma^j_{ik}\sigma^k_j\psi]\theta^i)} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{117}$$

$$K=\begin{bmatrix} -\frac{1}{4}\kappa i \not{p}\wedge\not{p} & -i\not{p}\psi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}\kappa\overline{(i\not{p}\wedge\not{p})} & -\overline{(i\,\theta\overline{\psi})} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{118}$$

$$\begin{aligned}\mathcal{A}=\langle K|P\rangle &= \int [\kappa R_s+\overline{\psi}[\gamma^i(\tfrac{1}{2}i\cdot\overleftrightarrow{\partial}_i)+\tfrac{1}{2}\Gamma^j_{ik}\{\gamma^i,\sigma^k_j\}]\psi]\sqrt{-g}\,d^4x \\ &= \int [\kappa R_s+\overline{\psi}[\gamma^i(\tfrac{1}{2}i\overleftrightarrow{\partial}_i)+\tfrac{1}{4}\Gamma^j_{ik}\epsilon^{ik}_{jm}\gamma_5\gamma^m]\psi]\sqrt{-g}\,d^4x \\ &= \int [\kappa R_s+\overline{\psi}[\tfrac{1}{2}\overleftrightarrow{\not{P}}]\psi+\tfrac{1}{4}\Gamma^*_m\overline{\psi}\gamma_5\gamma^m\psi]\sqrt{-g}\,d^4x \end{aligned} \tag{119}$$

with

$$\overleftrightarrow{\not{p}}\equiv\gamma^i i\overleftrightarrow{\partial}_i=\gamma^i_i(\overrightarrow{\partial}_i-\overleftarrow{\partial}_i)\text{ and }\Gamma^*_m\equiv\Gamma^j_{ik}\,\epsilon^{ik}_{jm}\,. \tag{120}$$

Note that

$$\gamma_5\equiv i\,\frac{\epsilon^{ijkl}}{4!}\,\gamma_i\gamma_j\gamma_k\gamma_l$$

changes in another coordinate basis to $\sqrt{-g}\,\gamma_5$.

Let us consider the term $\frac{1}{4}\Gamma^*_m\overline{\psi}\gamma_5\gamma^m\psi$. By returning to an examination of the idea of gauging the Lorentz group we see that since $\psi'=e^{-i\omega\cdot\sigma}\psi$ under SO(3, 1), the spinor gauge-covariant derivative is

$$\nabla_i=\partial_i-i\,\omega_i\cdot\sigma=\partial_i-i\,\Gamma^j_{ik}\sigma^k_j\,. \tag{121}$$

The symmetrized Fermi kinetic term is

$$\tfrac{1}{2}i\overline{\psi}(\gamma^j\overrightarrow{\nabla}_j-\overleftarrow{\nabla}_j\gamma^j)\psi=\overline{\psi}\tfrac{1}{2}i\overleftrightarrow{\not{\partial}}\psi+\tfrac{1}{4}\Gamma^*_m\overline{\psi}\gamma_5\gamma^m\psi\,, \tag{122}$$

$$\Gamma^*_m=\tfrac{1}{2}\epsilon^{jk}_{mi}\,Y^i_{\mu\,|\nu}\,Y^\mu_j\,Y^\nu_k\,. \tag{123}$$

Not only is there axial coupling, there is derivative coupling to the vierbein field, unless the Y^i_μ and ω^{ij}_μ are varied independently as in a first-order formalism. When flavor or color is included there will be vector coupling. This could lead to anomalies.[17]

While this term may present problems for the quantum version of the theory, it is there because it was required by local Lorentz invariance for the spinor fields. Several authors[13,16] have pointed out that this term leads to an action which is no longer invariant under index exchange for the connection ($\Gamma^\lambda_{\mu\nu}\leftrightarrow\Gamma^\lambda_{\nu\mu}$). What one can write is that if $\nabla_\lambda g_{\mu\nu}=0$, then

$$\Gamma^\lambda_{\mu\nu}=\{{}^{\,\lambda}_{\mu\nu}\}-K^{\,\lambda}_{\mu\nu} \tag{124}$$

with

$$K^{\,\lambda}_{\mu\nu}=g^{\lambda\rho}(T_{\mu\nu\rho}-T_{\mu\rho\nu}+T_{\rho\mu\nu}) \tag{125}$$

the contortion and

$$T^{\,\lambda}_{\mu\nu}=g^{\lambda\rho}T_{\mu\nu\rho} \tag{126}$$

the torsion. $\{{}^{\,\lambda}_{\mu\nu}\}$ is the usual Christoffel symbol. Since $\Gamma^\lambda_{\mu\nu}$ is no longer symmetric, there must be torsion when there are spinors. But because of spin averaging most macroscopic consequences are minor.

In the theory whose action is given by Eq. (119), called the Einstein-Cartan-Sciama-Kibble[29] theory, one can show that

$$K_{\mu\nu\lambda}=K^\rho_{\mu\nu}g_{\rho\lambda}=\tfrac{1}{4}\overline{\psi}\gamma_{\underline{\mu}}\gamma_{\underline{\nu}}\gamma_\lambda\psi\,. \tag{127}$$

Note that all three indices $\mu\nu\lambda$ are antisymmetrized. Where there are spinors there must be torsion.

Clearly one can form other basis currents besides $\gamma_k{}^\alpha_\beta\,\overline{s}_\alpha\otimes s^\beta$ and $\sigma_{kl}{}^\alpha_\beta\,\overline{s}_\alpha\otimes s^\beta$. In addition, there are

$$I^\alpha_\beta\,\overline{s}_\alpha\otimes s^\beta,\quad(\gamma_k\gamma_5)^\alpha_\beta\,\overline{s}_\alpha\otimes s^\beta,\text{ and }(i\gamma_5)^\alpha_\beta\,\overline{s}_\alpha\otimes\overline{s}^\beta. \tag{128}$$

Denote by γ_K the set of 16 Dirac matrices satisfying $\gamma_K=\gamma_0\gamma_K^\dagger\gamma_0$. We will show in Sec. III of paper II that these transform as the $\overline{4}\otimes 4$ representation of U(2, 2) [or SU(2, 2)] the covering group of the conformal group, SO(4, 2). Further, the fermions ψ themselves can be included to form a closed graded Lie algebra which we denote by $D(4\,|\,1)$. Sometimes this graded Lie algebra is called SU(2, 2 | 1). But since Dirac conjugation is used to form the adjoint spinors $\overline{\psi}\equiv\psi^\dagger\gamma_0$ instead of just $\psi^\dagger$, we will symbolize this graded Lie algebra by $D(4\,|\,n)$; see Sec. I.

Let us ignore the spinor fields ψ temporarily.

The Bose piece of the structure we have been considering has the following set of basis transformations:

$$d\bar{s}_\alpha = i\,\bar{s}_\beta[\omega_{\mu k}^{j}(\sigma_j^k)_\alpha^\beta]dx^\mu\,,$$
$$de_k = e_j[\Gamma_{\mu k}^j]dx^\mu\,, \qquad (129)$$
$$dP = e_j[Y_\mu^j]dx^\mu\,.$$

We have implemented the idea that e_j transform like $\gamma_{j\alpha}^{\beta}\bar{s}_\beta \otimes s^\alpha$ by requiring $\Gamma_{\mu k}^j \equiv \omega_{\mu k}^j$. Thus we have tied together a piece of the internal-symmetry gauge fields with the connection for the space-time manifold. We will refer to this operation as pinning.

For the $D(4)$ or SU(2, 2) structure we can introduce the connections

$$d\bar{s}_\alpha = i\,\bar{s}_\beta[\omega_\mu I_\alpha^\beta + \omega_\mu^k \gamma_{k\alpha}^{\ \beta} + \omega_{\mu k}^j(\sigma_j^k)_\alpha^\beta + \omega_\mu^{*k}(\gamma_5\gamma_k)_\alpha^\beta + \omega_\mu^*(i\gamma_5)_\alpha^\beta]dx^\mu\,,$$
$$de_k = e_j[\Gamma_{\mu k}^j]dx^\mu\,, \qquad (130)$$
$$dP = e_k[Y_\mu^k]dx^\mu\,.$$

We now require that $\Gamma_{\mu k}^j \equiv \omega_{\mu k}^j$ and $m\,Y_\mu^k = \omega_\mu^k$. m has the dimensions of mass; see Sec. III of paper II.[4]

For those familiar with fiber bundles we note that this is an SU(2, 2) bundle over a curved base manifold whose structures have been "pinned" together. For those not familiar with fiber bundles a brief review of our notation and some of the concepts involved is in Secs. II and III of paper III.[4]

I will briefly describe some of the essential features of connections on fiber bundles. Intuitively, one likes to think of a bundle B as the union of local copies G_p of a given gauge group G where the union is taken over all points p in a manifold M called the base manifold:

$$B = \bigcup_{p\in M} G_p = \bigcup_{p\in M}\{p\}\times G\,. \qquad (131)$$

But in fact it is more accurate to say that a bundle B is a manifold having a right action by a symmetry group G called the structure group, which allows one to define equivalence classes. The set of equivalence classes $M = B/G$ is called the base manifold. The map $\pi: B\to M = B/G$ is called canonical projection. The set $\pi^{-1}(p) = G_p$ is called the fiber at p. Every fiber is isomorphic to the fixed group G, $\pi^{-1}(p)\approx G$.

A connection on this bundle must respect the right action by G. This forces the connection to have a special dependence on the group parameters. Indeed, in a local coordinate system (x^μ, ϕ^a) for an open set $W = U\times G$ of the bundle ($U\subset M$; G is the group), we can show that the connection can be expressed as

$$\omega = \Omega^{-1}A_\mu(x)dx^\mu\Omega + \Omega^{-1}\partial_\mu\Omega dx^\mu + \Omega^{-1}\partial_a\Omega d\phi^a\,, \qquad (132)$$

where, for each x, $\Omega(x,\phi) = \exp[\tau^D(x,\phi)\Lambda_D]$ is a diffeomorphism of $\pi^{-1}(x)$ onto G. $\{\Lambda_D\}$ is a realization of the generators for G. Note that if $\{\Lambda_D\}$ are differential operators, B is a principal bundle. If $\{\Lambda_D\}$ are matrices, B is an associated bundle. We note that by changing the frame ω reduces to A,

$$d\bar{e}'_A = \bar{e}'_B\omega_A^B(x,\phi)\,,$$
$$d\bar{e}_A = \bar{e}_B A_A^B(x)\,, \qquad (133)$$
$$\bar{e}_A = \bar{e}'_B\Omega_A^B\,.$$

The action is independent of frame. As long as we are not concerned with the details of gauge choice, we can get by with the bases in which the connection takes its more restricted form. The consequences for gauge choices are considered in Sec. III of paper III.[4]

The curvature two-form is horizontal, that is, it involves only $dx^\mu\wedge dx^\nu$ terms:

$$\begin{aligned}P &= d\omega + \omega\\ &= \Omega^{-1}[dA + A\wedge A]\Omega\\ &= \Omega^{-1}[F]\Omega\\ &= \Omega^{-1}[\tfrac{1}{2}(\partial_\mu A_\nu + [A_\mu, A_\nu])dx^\mu\wedge dx^\nu]\Omega\,.\end{aligned} \qquad (134)$$

Introducing $*dx^\mu\wedge dx^\nu = \sqrt{-g}\,(\tfrac{1}{2}\epsilon^{\mu\nu}{}_{\lambda\rho})dx^\lambda\wedge dx^\rho$ we see that the action is independent of ϕ^a:

$$\begin{aligned}A &= \tfrac{1}{4}\int \mathrm{tr}[P^*\wedge P]\\ &= \tfrac{1}{4}\int \mathrm{tr}[F^*\wedge F]\\ &= -\tfrac{1}{4}\int F_{\mu\nu}^A F_A^{\mu\nu}\sqrt{-g}\,d^4x\,.\end{aligned} \qquad (135)$$

In the case of the SU(2, 2) bundle described above, there are nontrivial connections both in the horizontal (base) and vertical (fiber) structures. We have taken the action to be only that of the fiber while requiring an exact equality of mY_μ^i with ω_μ^i and $\Gamma_{\mu j}^i$ with $\omega_{\mu j}^i$. If contributions from the horizontal structure were included, further ambiguities in the relative magntidues (coupling constants) of the terms in the action would arise, since one has no *a priori* way of assigning relative weights of horizontal and vertical pieces. We can of course weight them equally in an *ad hoc* choice.

Another way out is possible. In this approach we consider a *flat* base manifold and a fiber with the Poincaré (or possibly de Sitter) group as structure group. But we give the vertical gauge fields an action which looks exactly like the action for a curved base manifold. It will be a specific rational (ratio of polynomials) action in these gauge fields and their derivatives. When the solutions are examined, we will find exactly the same solutions as for a cruved manifold, since the equations of motion will be the same. We will not deal with questions about patching together solutions, asymptotic flatness, and singularities in the equations here. Only the interpretation will have been changed. Indeed this is the sort of viewpoint that many particle theorists tend to favor. The gauge fields may have a complicated set of interactions but they are on a flat background. In fact, in terms of computing the relevant diagrams one is effectively led to this viewpoint.

Thus instead of considering a curved manifold we consider a special fiber bundle whose structure group is related to that of the symmetries of space-time but whose base is flat. The infinitesimal transformations are given by

$$\begin{aligned} dP &= e'_\mu dx^\mu , \\ de'_\mu &= 0 , \\ de &= e_j m Y^j_\mu dx^\mu , \\ de_j &= e_k \Gamma^k_{\mu j} dx^\mu . \end{aligned} \tag{136}$$

m is a massive coupling constant. We will consider both the Poincaré group and its extension, the de Sitter group, as structure group. Introduce

$$\begin{aligned} T^i_{\mu\nu} &= \partial_{\underline{\mu}} Y^i_{\underline{\nu}} + \Gamma^i_{\underline{\mu} j} Y^j_{\underline{\nu}} , \\ R^{ij}_{\mu\nu} &= \partial_{\underline{\mu}} \Gamma^{ij}_{\underline{\nu}} + \Gamma^i_{\underline{\mu} k} \Gamma^{kj}_{\underline{\nu}} , \\ C^{ij}_{\mu\nu} &= Y^i_{\underline{\mu}} Y^j_{\underline{\nu}} , \end{aligned} \tag{137}$$

and λ, a parameter which will determine if the group is de Sitter or Poincaré. Set

$$T^{ij}_{\mu\nu} = R^{ij}_{\mu\nu} + \lambda m^2 C^{ij}_{\mu\nu} . \tag{138}$$

The limit $\lambda = 0$ is the Poincaré group; $\lambda \neq 0$ is the de Sitter group. The metric is

$$\begin{aligned} g(e_i, e_j) &= \eta_{ij} , \\ g(e, e) &= -1 . \end{aligned} \tag{139}$$

Given a form $dx^\mu \wedge dx^\nu$ we will define the map $\star$ (which will not actually be the Hodge dual, it will just look like it) using the gauge fields Y^i_μ or $\theta^i = Y^i_\mu dx^\mu$,

$$\star(\theta^i \wedge \theta^j) \equiv \tfrac{1}{2} \epsilon^{ij}{}_{kl} \theta^k \wedge \theta^l . \tag{140}$$

Now

$$\begin{aligned} \star(\theta^i \wedge \theta^j) \wedge (\theta^m \wedge \theta^n) &\equiv -\eta^{i\bar{m}} \eta^{j\bar{n}} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \\ &\equiv -\eta^{i\bar{m}} \eta^{j\bar{n}} \det(Y^k_\tau) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 . \end{aligned} \tag{141}$$

Therefore

$$\star(dx^\mu \wedge dx^\nu) \wedge (dx^\lambda \wedge dx^\rho) = -[Y^\mu_i Y^\nu_j Y^\lambda_m Y^\rho_n] \eta^{i\bar{m}} \eta^{j\bar{n}} [\det Y^k_\tau] d^4x \tag{142}$$

To do this we need Y^i_μ to be a nonsingular 4×4 matrix. This requires that the vertical structure have a four-dimensional coset. We can therefore construct the action for this fiber bundle as

$$\begin{aligned} \mathfrak{A} &= \tfrac{1}{4} \int \operatorname{tr}(\star \overline{T} \wedge T) \\ &= -\tfrac{1}{4} \int (m^2 T^i_{\mu\nu} \eta_{ij} T^j_{\lambda\rho} + \tfrac{1}{4} T^{ij}_{\mu\nu} \eta_{ik} \eta_{jl} T^k_{\lambda\rho}) \\ &\quad \times \left(\frac{Y^\mu_m Y^\lambda_n Y^\nu_q Y^\rho_p \eta^{mn} \eta^{q\rho}}{Y^\sigma_r Y^\xi_s Y^\tau_t Y^\omega_\mu \epsilon_{\sigma\xi\tau\omega} \epsilon^{rstu}} \right) d^4x . \end{aligned} \tag{143}$$

If $\lambda \to 0$, the de Sitter group reduces to the Poincaré group. If $m \to 0$, the Poincaré group reduces to the Lorentz group.

It we take the structure group to be the full set of symmetries of the flat manifold, we are led to SO(4, 2), the conformal group. We remark that the infinitesimal action on a coordinate is given by

$$\delta x_\mu = \delta a_\mu + \delta\omega^\nu_\mu x_\nu + 2\delta b \cdot x x_\mu - \delta b_\mu x^2 + \delta\rho x_\mu , \tag{144}$$

where δa_μ, $\delta\omega^\nu_\mu$, δb_μ, and $\delta\rho$ are 15 small parameters.

We pick the structure group to be (at least locally) isomorphic to the group of symmetries of the base manifold. Of course, they do not act on the tangent space in this viewpoint; indeed the connections are given by

$$\begin{aligned} dP &= e_\mu dx^\mu , \\ de_\mu &= 0 , \\ de_I &= e_K \epsilon^K_{IM} \omega^M dx^\mu . \end{aligned} \tag{145}$$

The e_μ are tangent bases. The e_I are an adjoint basis for SO(4, 2). ϵ^K_{IM} are the SO(4, 2) structure constant. In fact we choose to work with the covering group of SO(4, 2) called SU(2, 2) so that we can introduce spinors. Thus we work in the fundamental associated bundle for SU(2, 2). By including the identity we can add on one extra U(1) to form U(2, 2). The connections are therefore

$$\begin{aligned} dP &= e_\mu dx^\mu , \\ de_\mu &= 0 , \\ d\bar{s}_\alpha &= -i \bar{s}_\beta \gamma^\beta_{I\alpha} \omega^I_\mu dx^\mu . \end{aligned} \tag{146}$$

The reader can show that the objects $\gamma^\alpha_{K\beta} \bar{s}_\alpha \otimes s_\beta$ transform (under displacement) just like the e_K's.

By extracting the vector piece of the connection, $\omega^i_\mu dx^\mu \gamma_i$, we can define a map $\star$ as above which yields an effective gravitational Lagrangian:

$$\star(dx^\mu \wedge dx^\nu) \wedge (dx^\lambda \wedge dx^\rho)$$

$$\equiv [\omega_i^\mu \omega_j^\nu \omega_m^\lambda \omega_n^\rho]\eta^{i\bar{m}}\eta^{j\bar{n}}[\det \omega_\tau^k]d^4x . \quad (147)$$

Note that the coefficient of d^4x has dimension zero. Now define

$$P^I\gamma_I = d\omega^I\gamma_I - i\omega^K\gamma_K \wedge \omega^L\gamma_L \quad (148)$$

and take the action to be

$$\mathfrak{a} = \frac{1}{N}\int \mathrm{tr}[\star(\bar{P}^I\bar{\gamma}_I) \wedge (P^J\gamma_J)] . \quad (149)$$

N is a normalizing factor.

Of course, given the richness of the *polynomial* SU(2, 2) structure (without the ratio of ω_μ^i's), it may be that the solutions to its equations of motion already fall within the envelope of current gravitational experiments. Such a polynomial theory would be conventionally renormalizable. But then we would lose the natural relationship between ω_μ^i and the vierbein field.

Just as SO(3, 1) is the structure group for a real four-manifold with local metric (1, −1, −1, −1), SU(2, 2) is the structure group for a complex four-manifold with local metric (1, 1, −1, −1). The orthogonal groups can be extended to the orthosymplectic group as a structure group for a real superspace (having Bose and Fermi coordinates). The SU(n) groups can be extended to the SU($n|m$) groups as structure groups for a complex superspace. But space-time is a real four-manifold (unless one admits Kaluza-Klein[12] type modifications). Thus it seems unreasonable to view SU(2, 2$|m$) as the physical manifold's structure group unless there is some natural way to select a real, vector four-manifold from the complex four-manifold.[30]

As far as I know there is no natural way currently available to embed a real four-manifold in the complex superspaces whose structure is given by SU(2, 2$|m$). But I will describe what is currently known.

Penrose and MacCallum[31] have related the column matrices on which SU(2, 2) acts to null lines in space-time (equipped with spin information). They call these SU(2, 2) column matrices twistors. By intersecting two Minkowski null lines, points in space-time can be selected. But, because the null-cone structure is tied to the curvature, it does not seem to be possible to replace coordinates by global twistors for use in the field theory. Thus we are led back to the flat base with a curved bundle.

Of course, there we can make such a relationship; however, other difficulties persist. Each twistor has eight real parameters. Two of them (required for intersection) have 16. But a position contains only four parameters. It is not clear how the connections should depend on the rest of these parameters. One might think of forming vector coordinates out of the bilinears, but even then there are problems. There are 16 bilinears:

$$\begin{aligned} S &= \bar{\chi}\chi , \\ P^\mu &= \bar{\chi}\tfrac{1}{2}(I+\Gamma^5)\Gamma^\mu\chi , \\ M^{\mu\nu} &= \bar{\chi}\tfrac{1}{8}i[\Gamma^\mu, \Gamma^\nu]\chi \equiv \bar{\chi}\Sigma^{\mu\nu}\chi , \\ K^\nu &= \bar{\chi}\tfrac{1}{2}(I-\Gamma^5)\Gamma^\nu\chi , \\ D &= \bar{\chi}i\Gamma^5\chi . \end{aligned} \quad (150)$$

They are formed from the eight real parameters of χ using the Γ matrices for the base manifold $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}I$. But, of course, they do not depend on the overall phase of χ, leaving only seven parameters. Thus there must be nine constraint equations among the bilinears. Some of these are the following:

$$\begin{aligned} P^2 &= 0 , \\ K^2 &= 0 , \\ SD &= \vec{L}^2 - \vec{B}^2 , \\ S^2 - D^2 &= \vec{L}\cdot\vec{B} , \end{aligned} \quad (151)$$

where $L^i = \frac{1}{2}M^{jk}\epsilon^{\ i}_{jk}$ and $B^i = M^{0i}$.

We note that $V^2 \equiv (P+K)^2 > 0$ and $A^2 \equiv (P-K)^2 < 0$. The sum is in the forward light cone. The difference is in the spacelike region. Neither combination spans all of space-time. A combination that does is $S(P^\mu + K^\mu) + D(P^\mu - K^\mu) \equiv SV^\mu + DA^\mu$. But it covers space more than once. $V^\mu = (P^\mu + K^\mu)$ would be admissible only if one accepted the idea that space-time is in the forward light cone of a single point.

The transformation properties of the bilinears under an infinitesimal SU(2, 2) transformation are as follows:

$$\begin{aligned} e^{-i\omega^K\Gamma_K} &\approx I - i\omega^K\Gamma_K \\ &\approx I - i[\rho^\lambda\tfrac{1}{2}(I+\Gamma^5)\Gamma_\lambda + \omega^{\lambda\mu}\Sigma_{\lambda\mu} \\ &\qquad + \kappa^\mu\tfrac{1}{2}(I-\Gamma^5)\Gamma_\mu + \epsilon i\Gamma^5] , \end{aligned} \quad (152)$$

$$\delta\chi \approx -i\omega^K\Gamma_K\chi , \quad (153)$$

$$\begin{aligned} \delta S &= 0 , \\ \delta P_\mu &= 8\rho^\lambda M_{\lambda\mu} - 2\kappa_\mu D + \omega_\mu^\nu P_\nu - 2\epsilon P_\mu , \\ \delta M_{\mu\nu} &= -\tfrac{1}{4}\rho_{\underline{\mu}}P_{\underline{\lambda}} + \tfrac{1}{4}\kappa_{\underline{\mu}}K_{\underline{\lambda}} + \omega_\mu^\nu M_{\nu\lambda} , \\ \delta K_\nu &= -8\kappa^\lambda M_{\lambda\nu} - 2\rho_\nu D + \omega_\nu^\mu K_\mu + 2\epsilon K_\nu , \\ \delta D &= \rho^\mu P_\mu + \kappa^\mu K_\mu . \end{aligned} \quad (154)$$

But under the action of an infinitesimal SO(4, 2) transformation on coordinates x^μ with parameters $\{a^\mu, \omega^{\mu\nu}, b_\nu, c\}$ we find

$$\delta x_\mu = a_\mu + \omega^\nu_\mu x_\nu + 2b\cdot x x_\mu - b_\mu x^2 + c x_\mu . \qquad (155)$$

The special conformal action (parameters b_μ) is quadratic; the bilinears transform linearly. Of course, one could simply use some vector subset of the bilinears which spanned all of space-time such as $SV^\mu + DA^\mu$ as the parameters, x^μ. Then equivalent sets of parameters can be reached by using either SO(4, 2) or SU(2, 2) reparametrizations. But, of course, the SU(2, 2) reparametrizations lead us out of the initial four-parameter set.[32] We are obliged to examine the dependence of the connections on the rest of the seven bilinear parameters. Whatever the dependence may be, the result is the same. Since there is no evidence for further timelike or spacelike degrees of freedom, the connections can depend at most on some sort of cyclic fashion. The result is an action depending only on four-vector parameters.

Let us suggest one of the possible constructions. Take two vectors $x^\mu = [(SV^\mu + DA^\mu)/(|SD|)^{1/2}$, e.g.] and $y^\mu = [(SV^\mu - DA^\mu)/(|SD|)^{1/2}$, e.g.] that satisfy $x^2 = y^2$. They have seven free parameters from which all the bilinears may be found. Introduce the invertible map $M^{\mu\nu}\Sigma_{\mu\nu} \xrightarrow{f} M^{ij}\sigma_{ij}$ which sends components into components in the obvious way. Now take the connections to be

$$\omega^K\gamma_K = \Omega^{-1}\omega^K_\nu(x)dx^\nu\gamma_K\Omega + \Omega^{-1}d\Omega ,\qquad (156)$$
$$\Omega = \exp[-if(x^\mu y^\nu \Sigma_{\mu\nu})] .$$

The dependence on the extra parameters can be gauged away as in a fiber bundle.

This entire approach is fraught with ambiguities and apparently arbitrary decisions. The hope that a curved vector four-manifold can be "naturally" embedded in the curved spinor four-manifold seems unrealistic. The theory ends up (at best) looking like the SU(2, 2) bundle over a flat vector base. There appears to be no advantage in adopting any other viewpoint than that of moving frames (fiber bundles).

V. SUMMARY

In this paper we motivated the concept of local frames as bases for the infinitesimal displacements in the manifold.[7] In the case of four dimensions they form a system of four independent vectors tangent to the manifold. We extended this basis to include extra "legs" which did not correspond to space-time displacements (they can be related to internal displacements; see Sec. III of paper III). A visual, simple example is that of the normal bundle to the two-sphere. The extra "leg" is a basis for the line attached to each point. Displacements are only allowed on the sphere's surface.

Under an infinitesimal displacement the bases can be rotated (transformed) into each other. We assumed that the space-time and internal pieces do not mix. The coefficients of these transformations are the connection coefficients when they act on the space-time bases and the Yang-Mills (gauge field) coefficients when they act on the internal bases. By computing a second-order displacement both ways and by comparing the results (take the difference) we discover that the curvature and the Yang-Mills field tensors arise as coefficients of the bases. If the curvature and Yang-Mills tensor vanish, then the bases are integrable. When these tensors are contracted with the antisymmetric tensor product of the infinitesimal displacements, they are called the curvature and Yang-Mills two-forms.

We introduced the Hodge dual (generalized Maxwell dual) and formed the action as a Hilbert product of the curvature (or Yang-Mills tensor) with its dual or some other suitably constructed two-form.

We then discussed three ways to include Bose matter fields (Sec. III). First, one introduces an extra basis element (the others have been used up). Now there are a number of options. Since matter fields carry no space-time index μ they can be associated with a fifth basis element,[4,12] $d\!\!\!/\theta$, which may be either Bose or Fermi, or they can be associated with the element I which is a basis for the functions on the manifold (this can also be related to distribution valued connections). Since the effects are ultimately the same, it is a matter of taste which technique one prefers.

Having introduced matter fields as part of the internal-symmetry structure (this is done with an eye to superunified algebras; see Secs. II and III of paper II),[4] we compute the action for Bose matter fields coupled to gauge fields as the Hilbert square of the curvature[4,11] of the full internal structure (augmented or not). This yields the usual minimally coupled action for scalar fields coupled to gauge fields. So far no potential terms are included (see Sec. II of paper II).

In the fourth section we introduce spinor fields coupled to gravity. After going to a local Lorentz basis, one can construct spinor frames by the condition that they transform as the spin-$\frac{1}{2}$ representation of the Lorentz action on the vector bases. The role of the gauge fields is played by the space-time connection for a Lorentz frame. In terms of spinor basis transformations the matrix format of the connection is similar to the Bose construction given earlier (Sec. III of paper II). However, the action is constructed as the

Hilbert product of the curvature P with another two-form K constructed by index saturation; this yields an action linear in both the connection and Dirac spinor. The question of whether there is a deeper way to understand this auxiliary form is examined in Sec. III of paper II. The product $\langle K|P\rangle$, suitably symmetrized, is the minimally coupled action for spinors on a curved manifold.[13,16] It is interesting to note that the connection couples as an axial vector to the spinors. In a theory with vector couplings the problem of anomalies arises.[17]

Having introduced the spinor bases, we consider the question of whether they may be more fundamental than vector bases and whether the coordinates x^μ can be similarly replaced by spinor coordinates.

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