Path-integral representation for the S matrix

C. L. Hammer and J. E. Shrauner*
Ames Laboratory—USDOE, Iowa State University, Ames, Iowa 50011

B. De Facio
Physics Department, University of Missouri-Columbia, Columbia, Missouri 65211
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We present a formulation of quantum field theory as a path-integral representation for elements of the U or S matrix in the coherent-state basis. These matrix elements are shown to serve as generating functionals for all the usual S-matrix elements between states of definite particle number. The coherent-state formalism for general bilinear quantum field theories is described and then incorporated into the construction of a path integral such that the formulation is independent of any canonical formalism. We discuss the relationship of this formulation to the usual path-integral formulation and show in what sense they are equivalent for canonical theories. We also discuss how this formulation is more general than the usual one in that it is well defined even for theories having no canonical form and for which a Lagrangian action and the usual path-integral formulation may not apply. Applications of this formulation to specific calculations are exhibited for the cases of a quantized field interacting with a given external source and for the renormalization of the simple quantum field theory model of a scalar meson field interacting with a nonrelativistic nucleon field.

I. INTRODUCTION

We consider the formulation of a quantum field theory that combines the path-integral representation with the coherent-state representation for elements of the U or S matrices. This formulation is useful both for proving fundamental properties of quantum field theories and for performing specific calculations.

The path-integral formulation of quantum field theory has been very prominent recently in elementary particle physics, and there exist many fine reviews of the subject in the literature.[1] Some of the earliest work in this subject was by Feynman, who developed a path-integral representation for quantum amplitudes and applied them to integrating out (the degrees of freedom of) the electromagnetic field in quantum electrodynamics.[2]

Coherent-state techniques have also been rising to prominence in general quantum field theory and elementary particle physics as well as in quantum optics. The application of coherent-state techniques to problems involving large quantum numbers in quantum optics is by now rather well known.[3] Recently coherent states have been used as the foundation for statistical theories of correlations of many-particle production processes at high energies.[4] These statistical theories and other phenomenological theories correspond to detailed microscopic field theories at a stage at which some subset of the interacting field variables have been integrated out as mentioned above. Also Bolsterli[5] has used coherent-state methods, without path integrals or lattice limits, to study covariant theories of mesons with static sources. He has shown these methods to be powerful.

The power of combining the path-integral formulation and the coherent-state representation has been shown in the recent solution by Dente[6] of the long-standing problem of demonstrating that quantum electrodynamics has a classical limit. In this work it was shown by integrating out the electromagnetic field in transition matrix elements between nonvacuum coherent states that QED in its usual form has the expected classical limit without the necessity of imposing any artificial alteration of the theory as was done in the "absorber theory" of Feynman and Wheeler.[7]

The coherent-state formalism for general bilinear quantum field theories as described by DeFacio and Hammer[8] is incorporated in an essential way into our path-integral formulation. The formalism of bilinear field theories, including their quantization, derives totally from their free-field equations of motion and associated currents and does not depend on a canonical quantization procedure.[8,9,10,11] This generalization makes our path-integral formulation in the basis of coherent states of general bilinear field theories applicable even for such theories for which no canonical form exists.

Earlier work[12] uniting the coherent-state formalism and path-integral formalism has been done by Klauder and Schweber, who adapted the lattice-integration methods of Feynman, much the same as we do here. More recently fundamental work in this area has been done by Faddeev, Berezin, Klauder, and others.[13] Thus, the basic mathematical details in this area have been, with the exceptions of a few possible lacunae, rather thoroughly developed. In the present work we assemble the formulation with modest innovations as described above, emphasizing the extended generality and the potential for modeling and treating important phenomenological problems. We start on a program of explicit calculations using this formalism.

A path-integral representation for U- and S-matrix elements in the coherent-state basis is thus constructed. Because of the properties of the coherent-state matrix elements of field operators, the quantum field theory problem is reduced to a quantum mechanics problem. The path-integral formulation then reduces the quantum mechanics problem to a consideration of an effective classical action problem. Thus a convenient and natural formulation is obtained for the treatment of transitions between initial and final states, i.e., end points of the functional path integral that are nontrivial, nonvacuum states such as classical condensates, solitons,[14] vexictons,[15] vortices,[16] etc. This, combined with the fact that this formulation is in terms of the U- and S-matrix elements, means that we have a powerful method for approaching spontaneous symmetry breaking in quantum field theory in a way not previously used for this purpose.

In this paper we describe in Sec. II the general formalism of the coherent-state basis for a general bilinear quantum field theory. In Sec. III we show how the S-matrix element in the coherent-state basis of a given field may serve as a generating functional for all the usual S-matrix elements between states of the occupation-number basis for that field. We also discuss there the time dependence of the operators of a general bilinear quantum field theory upon which is based not only the asymptotic condition but also the extension to the coherent-state formalism of a method of quantization that is independent of any postulates of canonical quantization or locality conditions.[8,9,10,11] In Sec. IV we construct a path-integral representation of U- and S-matrix elements in the coherent-state basis for general bilinear quantum field theories. In Sec. V we demonstrate the application of this formulation to the calculation of a specific problem with the prototype of a quantized field interacting with a given external current source. In Sec. VI we extend application to the renormalization of the simple scalar meson model of interacting fields. In Sec. VII we discuss the relation to our formulation of quantum field theory to the usual path-integral representation. Specifically, we show that our path integral is equivalent to the usual one for the case of canonical field theories. However, our formulation is more general in that it remains well defined even for theories that have no canonical form and for which the Lagrangian and its action, and therefore, possibly, the usual path-integral formulation, does not exist.

II. THE COHERENT-STATE BASIS

For a field theory of the usual class[8] in which there exists, as a consequence of the field equations of motion, a conserved current $J_\mu(\psi_1, \psi_2)$ that is bilinear in ψ_1 and ψ_2, which are any pair of solutions of the field equations, the timelike current component $J_0(\psi_1, \psi_2)$ defines a bilinear structure that can be used to construct invariant inner products as

$$(\phi(t), \chi(t)) \equiv \int d^3x\, J_0(\phi(\vec{x}, t), \chi(\vec{x}, t)). \tag{2.1}$$

This inner product can be formed with any pair of suitably well-behaved functions. It will be independent of time t whenever ϕ and χ each satisfy the field equations from which the conserved current J_μ is defined.

Instead of the usual canonical commutation relations (CCR's) we adopt the more general form[8,17] required by the symmetries of the equations of motion:

$$[\psi, S] = s\psi, \tag{2.2}$$

where S is a symmetry operator

$$S = \int d^3x\, J_0(\psi, s\psi).$$

This expression, coupled with the locality assumption for independent fields

$$[\psi_i(\vec{x}, t), \psi_j(\vec{y}, t)]_\pm = 0, \tag{2.3}$$

can be used to obtain the remaining equal-time commutators. They are not necessarily CCR's.[8,17]

Of particular interest for our use is the operator

$$\Sigma_\phi(t) \equiv z^{*-1/2}(\psi(t), \phi(t))\eta_A - z^{-1/2}\bar{\eta}_A(\phi(t), \psi(t)), \tag{2.4}$$

where $\phi(\vec{x}, t)$ is any suitably well-behaved c-number function; $\psi(\vec{x}, t)$ is an operator-valued solution of the field equations, and z is a renormalization constant. For bosons $\eta_A = \bar{\eta}_A = \eta_B = 1$, and for fermions $\eta_A = \eta_F$ and $\bar{\eta}_A = \bar{\eta}_F$ are a pair of anticommuting c-number quantities for which

$$\{\eta_F, \bar{\eta}_F\} = \{\eta_F, \eta_F\} = \{\bar{\eta}_F, \bar{\eta}_F\} = \{\eta_F, \psi\} = \{\bar{\eta}_F, \psi\} = 0.$$

The commutator of the operator $\Sigma_\phi(t)$ with the field operator follows from the form of Eq. (2.2) as

$$[\psi(\vec{x}, t), \Sigma_\phi(t)] = z^{*-1/2}\eta_A \phi(\vec{x}, t). \tag{2.5}$$

This commutation relation implies that the operator Σ_ϕ generates transformations of the field operator ψ of the form

$$e^{-\Sigma_\phi(t)}\psi(\vec{x},t)e^{\Sigma_\phi(t)} = \psi(\vec{x},t) + z^{*-1/2}\eta_A\phi(\vec{x},t)\,. \tag{2.6}$$

This means that $\exp[\Sigma_\phi(t)]$ generates coherent states of the field $\psi(\vec{x},t)$ as

$$\begin{aligned} e^{\Sigma_\phi(t)}|\Omega\rangle &\equiv |\phi(t)\rangle\,,\\ \langle\Omega|e^{-\Sigma_\phi(t)} &\equiv \langle\phi(t)|\,, \end{aligned} \tag{2.7}$$

where $|\Omega\rangle$ is the Fock-space vacuum state. These coherent states are eigenstates of the field operator $\psi(\vec{x},t)$ as

$$\langle\phi(t)|\psi(\vec{x},t)|\phi(t)\rangle = z^{*-1/2}\eta_A\phi(\vec{x},t)\,, \tag{2.8}$$

or if $\psi_+(\vec{x},t)$ is the pure annihilation-operator part of $\psi(\vec{x},t)$,

$$\psi_+(\vec{x},t)|\phi(t)\rangle = z^{*-1/2}\eta_A\phi(\vec{x},t)|\phi(t)\rangle. \tag{2.9}$$

The function $\phi(\vec{x},t)$ is arbitrary, with the condition that it is well behaved with respect to the inner products that it enters for the specific theory under consideration in a given case. This *coherent-state wave function* $\phi(\vec{x},t)$ can also be used as a variational parameter in our formulation of quantum field theory in the coherent-state basis.

Using the Baker-Campbell-Hausdorf theorem we obtain the important relations

$$\begin{aligned} e^{\Sigma_\phi(t)} = {}& \exp[z^{*-1/2}(\psi(t),\phi(t))\eta_A]\\ &\times\exp[-z^{-1/2}\bar{\eta}_A(\phi(t),\psi(t))]\\ &\times\exp[-\tfrac{1}{2}|z|^{-1/2}(\phi(t),\phi(t))]\,, \end{aligned} \tag{2.10}$$

$$\begin{aligned} e^{\Sigma_\phi(t)}e^{\Sigma_\chi(t)} = {}& e^{\Sigma_{\phi+\chi}(t)}\\ &\times\exp\{\tfrac{1}{2}|z|^{-1}[(\chi(t),\phi(t)) - (\phi(t),\chi(t))]\}\,, \end{aligned} \tag{2.11}$$

and therefore

$$\begin{aligned} \langle\phi(t)|\chi(t)\rangle = \exp\{-\tfrac{1}{2}|z|^{-1}[&(\chi(t),\chi(t)) + (\phi(t),\phi(t))\\ &-2(\phi(t),\chi(t))]\}\,, \end{aligned} \tag{2.12}$$

where $\phi(\vec{x},t)$ and $\chi(\vec{x},t)$ are arbitrary coherent-state wave functions and $\psi(\vec{x},t)$ is the quantized field operator. The coherent states $|\phi(t)\rangle$ satisfy the completeness relation

$$\int\left[\frac{D^2\phi}{\pi|z|}\right]|\phi(t)\rangle\langle\phi(t)| = 1\,. \tag{2.13}$$

The differential element for our functional integrations on the coherent-state wave-function space is written in the forms

$$\begin{aligned} \left[\frac{D^2\phi}{\pi|z|}\right] &= \left[\frac{D\,\mathrm{Re}\phi}{(\pi|z|)^{1/2}}\right]\left[\frac{D\,\mathrm{Im}\phi}{(\pi|z|)^{1/2}}\right]\\ &= \left[\frac{D\phi}{(2\pi z)^{1/2}}\right]\left[\frac{D\phi^*}{(2\pi z^*)^{1/2}}\right]. \end{aligned} \tag{2.14}$$

Here we have adopted the notation for any constant b,

$$[Db\phi] = \prod_k (bd\beta_k) = (bd\beta_1)(bd\beta_2)\cdots\,,$$

where β_k are the expansion coefficients of the field ϕ expanded in normal modes.

All the relations for coherent states mentioned above apply for any operator-valued field ψ satisfying canonical equal-time commutation or anti-commutation relations and their associated conserved currents.[9,10] However, in what follows we will always be using complete sets of states that are coherent states of free fields and so the inner products involved in the generators and matrix elements of such states will always be based on the conserved currents of the free fields; i.e., the usual type of inner products. In this case the commutation relation Eq. (2.5) on which our coherent-state formalism is based is well defined and determined even if no canonical commutation relation or locality (or quasilocality) condition exists.[8,9,10] Quite generally, all operators of a quantized field theory may be expressed in bilinear form such as we use for the Σ_ϕ operator.[8,11]

III. S-MATRIX ELEMENT IN COHERENT-STATE BASIS AS GENERATING FUNCTIONAL FOR USUAL S-MATRIX ELEMENTS

The S-matrix element in the coherent-state basis of a given field in a given theory may serve as a generating functional for all the usual S-matrix elements between states of definite numbers of quanta of that field. The mechanism of this generation is the variation with respect to the projection of the coherent-state wave function ϕ onto the single-particle modes of the in-states of this field. The reason that all the usual particle S-matrix elements may be obtained from a coherent-state S-matrix element is because a coherent state is comprised of a definite combination of all the n-particle Fock states of all modes of the field. This may be shown by projecting the in-field operator ψ_{in} and the coherent-state wave function ϕ onto the positive-energy free-particle wave function of the kth mode $\hat{\phi}_k(x)$ as

$$\psi_{k,\mathrm{in}} \equiv (\hat{\phi}_k(t),\psi_{\mathrm{in}}(t))\,,\quad \text{independent of } t$$

and

$$\beta_k(t) \equiv (\hat{\phi}_k(t),\phi(t))\,,$$

where the $\hat{\phi}_k(x,t)$'s are assumed to form a complete orthonormal set with $(\hat{\phi}_k, \hat{\phi}_q) = \delta^3_{k,q}$. So then the generator of the coherent in state $|\phi(t)\rangle_{\text{in}}$ is expanded as

$$\Sigma_{\phi\,\text{in}}(t) = (\psi_{\text{in}}(t), \phi(t))\eta_A - \bar{\eta}_A(\phi(t), \psi_{\text{in}}(t))$$
$$= \sum_k [\psi^\dagger_{k,\text{in}}\eta_A\beta_k(t) - \bar{\eta}_A\psi_{k,\text{in}}\beta^*_k(t)], \qquad (3.1)$$

from which follows

$$|\phi(t)\rangle_{\text{in}} = e^{\Sigma_{\phi\,\text{in}}(t)}|\Omega\rangle_{\text{in}}$$
$$= \prod_k e^{\psi^\dagger_{k,\text{in}}\eta_A\beta_k(t)}e^{-(1/2)|\beta_k(t)|^2}|\Omega\rangle_{\text{in}}$$
$$= \prod_k \left[\sum_{n_k}\frac{[\eta_A\beta_k(t)]^{n_k}}{(n_k!)^{1/2}}|n_k\rangle_{\text{in}}\right]e^{-(1/2)|\beta_k(t)|^2}. \qquad (3.2)$$

Thus the general particle in-state of the Fock basis is

$$|l_1 l_2\cdots l_k\cdots\rangle_{\text{in}} = \prod_k (\partial/\partial\beta_k)^{l_k}(l_k!)^{-1/2} \times e^{(1/2)|\beta_k|^2}|\phi\rangle_{\text{in}}\Big|_{\phi=0=\beta_k}, \qquad (3.3)$$

where the t dependence has been suppressed, as it goes away after the $\phi\to 0, \beta_k\to 0$ limit. This implies that the usual S-matrix element between such states of definite particle number is

$$_{\text{out}}\langle l_1\cdots l_k\cdots\,|n_1\cdots n_q\cdots\rangle_{\text{in}} = \prod_{k,q}(\partial/\partial\beta'^*_k)^{l_k}(\partial/\partial\beta_q)^{n_q}(l_k!n_q!)^{-1/2} \times e^{1/2(|\beta'_k|^2+|\beta_q|^2)}{}_{\text{out}}\langle\phi'|\phi\rangle_{\text{in}}\Big|_{\substack{\phi=0\\ \phi'=0}}, \qquad (3.4)$$

with $\beta'_k(t) \equiv (\hat{\phi}_k(t), \phi'(t))$. These relations are direct consequences of the definitions of the coherent states.

The asymptotic in/out limits of our coherent states require some discussion. The usual asymptotic condition is given as the weak operator limit relating (matrix elements of) interacting unrenormalized Heisenberg fields $\psi(\vec{x}, t)$ with the free asymptotic in- or out-fields $\psi_{\text{in, out}}(\vec{x}, t)$, each in terms of their particle-mode projections as

$$\underset{t\to-\infty(+\infty)}{\text{wk-lim}}[(\hat{\phi}_k(t), \psi(t)) - z^{1/2}(\hat{\phi}_k(t), \psi_{\text{in, out}}(t))] = 0, \qquad (3.5)$$

where z is a wave-function renormalization constant. The term $(\hat{\phi}_k(t), \psi_{\text{in, out}}(t))$ of this relation is independent of the time t since both $\hat{\phi}_k(\vec{x}, t)$ and $\psi_{\text{in}}(\vec{x}, t)$ satisfy the free-field equation which determines the conserved current that is used to construct these inner products. The same inner product based on the free-field conserved current is used in both terms of this relation.

The generator of the coherent in/out states, $\Sigma_{\phi\,\text{in/out}}$, is given by Eq. (3.1) in terms of this free-field-based inner product and these same $\hat{\phi}_k$'s. If we now use this same free-field-based inner product in

$$\Sigma_\phi(t) = z^{*-1/2}(\psi(t), \phi(t))\eta_A - z^{-1/2}\bar{\eta}_A(\phi(t), \psi(t)), \qquad (3.6)$$

then it is clear that

$$\underset{t\to-\infty\,(+\infty)}{\text{wk-lim}}[\Sigma_\phi(t) - \Sigma_{\phi\,\text{in (out)}}(t)] = 0. \qquad (3.7)$$

Justification of the use of the free-field inner product in the $\Sigma_\phi(t)$ defined in Eq. (3.6) will be more apparent after a discussion of the t dependence of these operators, which also leads us into our U-matrix treatment in Sec. IV.

The time dependence of the Heisenberg field operators is described as

$$\psi(t) = e^{iHt}\psi(0)e^{-iHt}.$$

We can also define a set of fields $\psi_0(t)$ that propagate as free fields with the Hamiltonian $H_0 = H - V$ and coincide with the interacting fields of the Heisenberg and Dirac pictures all at time $t = 0$ as

$$\psi_0(t) = e^{iH_0t}\psi(0)e^{-iH_0t}$$
$$= U_c(t, 0)\psi(0),$$

where $U_c(t, 0)$ is the c-number time translation operator for the free field ψ_0. For example $U_c(t, 0) = \exp(it\nabla^2/2m)$ in the case of a Hamiltonian theory of nonrelativistic particles of mass m. Then

$$e^{-iH_0t}(\phi(0), \psi(0))e^{iH_0t} = (\psi(0), U_c^{-1}(t, 0)\psi(0))$$
$$= (U_c(t, 0)\phi(0), \psi(0)).$$

For the purposes of the rest of the present paper we shall restrict our considerations to coherent-state wave functions that satisfy the free-field equations, so that $U_c(t, 0)\phi(0) = \phi(t)$ [and incidentally $\beta_k(t)$ defined above becomes time independent], and so

$$e^{iHt}e^{-iH_0t}(\phi(0), \psi(0))e^{iH_0t}e^{-iHt} = (\phi(t), \psi(t))$$

and (3.8)

$$\Sigma_\phi(t) = e^{iHt}e^{-iH_0t}\Sigma_\phi(0)e^{iH_0t}e^{-iHt}.$$

Thus the coherent state evolves in time as

$$|\phi(t)\rangle = e^{iHt}e^{-iH_0t}|\phi(0)\rangle$$
$$= e^{\Sigma_\phi(t)}|\Omega(t)\rangle, \qquad (3.9)$$

where

$$|\Omega(t)\rangle = e^{iHt}e^{-iH_0t}|\Omega\rangle .$$

The asymptotic in/out states are given as

$$\underset{t\to-\infty\,(+\infty)}{\text{wk-lim}}\,[e^{iHt}e^{-iH_0t}|\phi(0)\rangle - |\phi(t)\rangle_{\text{in}}] = 0\,, \quad (3.10)$$

and the S-matrix element in the coherent-state basis is given by

$$\underset{\substack{t\to-\infty\\ t'\to+\infty}}{\text{wk-lim}}\,[\langle\phi'(0)|U(t',t)|\phi(0)\rangle - {}_{\text{out}}\langle\phi'(t')|\phi(t)\rangle_{\text{in}}] = 0\,, \quad (3.11)$$

where

$$U(t',t) = e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}$$

is the familiar time evolution operator.[18] The time $t=0$ is when our free and interacting Heisenberg fields coincide and when the Heisenberg and Dirac pictures coincide.

This justifies the use of the free-field inner products in constructing the generator $\Sigma_\phi(t)$ in Eq. (3.6), because all our coherent states can be viewed as being built at time $t=0$ when the free and interacting Heisenberg fields coincide and then they evolve in time to other values of t according to the above descriptions. Finally, in terms of the above limits and the Heisenberg field operators, the S-matrix element Eq. (3.4) is

$$\begin{aligned}{}_{\text{out}}\langle l_1\cdots l_k\cdots|n_1\cdots n_q\cdots\rangle_{\text{in}} &= \lim_{\substack{t\to-\infty\\ t'\to\infty}}\prod_{k,q}(\partial/\partial\beta_k'^*)^{l_k}(\partial/\partial\beta_q)^{n_q}(l_k!n_q!)^{-1/2}\\ &\quad\times e^{+(1/2)|z|^{-1}(|\beta_k'|^2+|\beta_q|^2)}\\ &\quad\times\langle\phi'(t')|\phi(t)\rangle|_{\substack{\phi=0\\ \phi'=0}}\,. \end{aligned} \quad (3.12)$$

IV. PATH-INTEGRAL REPRESENTATION OF U AND S MATRICES IN COHERENT-STATE BASIS

The interaction Hamiltonian in the Dirac picture is defined as

$$V(t) = e^{iH_0t}Ve^{-iH_0t}\,. \quad (4.1)$$

Then the U operator above in the limit that $t'-t=\epsilon$ becomes infinitesimally small is

$$U(t+\epsilon,t) = e^{-i\epsilon V(t)}\,. \quad (4.2)$$

This expression is valid to lowest order in ϵ even if V should depend explicitly on time.

The field operators of which H, H_0, and V are constructed are assumed to be separated into pure annihilation and creation parts as $\psi=\psi_+ + \psi_-$. A normal ordering is defined so that in matrix elements between coherent states the annihilation parts act to the right on their eigenstates as $\psi_+(\vec{x},t)|\phi(0)\rangle = \phi(\vec{x},t)|\phi(0)\rangle$, and the creation parts ψ_- act similarly to the left on their eigenstates. This definition of the normal ordering clearly is the same as the conventional normal ordering. A result is that in the infinitesimal-time-interval U-matrix element the exponential operator can be replaced by a c-number functional as

$$\begin{aligned}&\langle\phi'(0)|e^{-i\epsilon V(\psi_-(t),\psi_+(t))}|\phi(0)\rangle\\ &\quad=\langle\phi'(0)|\phi(0)\rangle e^{-i\epsilon V(z^{-1/2}\phi'(t')^*,\,z^{*-1/2}\phi(t))}\,. \end{aligned} \quad (4.3)$$

We consider $\eta_A=\bar\eta_A=1$ here and in the following. Because our matrix element is first order in the infinitesimal ϵ we have some liberty as to the convention for specifying its time arguments. For example, it might be convenient to label them as $V(\phi'(t+\epsilon)^*,\phi(t))$ in the exponent functional. We shall also make use of the choice $V(\phi'(t)^*,\phi(t))$.

To obtain the finite-time-interval matrix elements we start by inserting a complete set of intermediate coherent states as

$$\begin{aligned}\langle\phi'(0)|U(t',t)|\phi(0)\rangle &= \int\left[\frac{D^2\phi_1(0)}{\pi|z|}\right]\langle\phi'(0)|U(t',t_1)|\phi_1(0)\rangle\\ &\quad\times\langle\phi_1(0)|U(t_1,t)|\phi(0)\rangle\,, \end{aligned} \quad (4.4)$$

and we do this for each $t_l=t+l\epsilon$, $l=1,2,\ldots,N$, with $t'-t=(N+1)\epsilon$. After integrating over all the sets of coherent intermediate states we let $N\to\infty$ and $\epsilon\to 0$. This gives

$$\begin{aligned}\langle\phi'(0)|U(t',t)|\phi(0)\rangle = \lim_{\substack{N\to\infty\\ \epsilon\to 0}}\int &\langle\phi'(0)|\phi_N(0)\rangle\left[\frac{D^2\xi_N}{\pi}\right]e^{-i\epsilon V(z^{-1/2}\phi'^*(t'),\,\xi_N(t_N))}\\ &\times\langle\phi_N(0)|\phi_{N-1}(0)\rangle\left[\frac{D^2\xi_{N-1}}{\pi}\right]e^{-i\epsilon V(\xi_N^*(t_N),\,\xi_{N-1}(t_{N-1}))}\cdots\\ &\times\langle\phi_1(0)|\phi(0)\rangle e^{-i\epsilon V(\xi_1^*(t_1),\,\phi(t)z^{*-1/2})}\,, \end{aligned} \quad (4.5)$$

where the integration variables are defined as $\xi_l\equiv z^{*-1/2}\phi_l$. Altogether a path integral is built up in which each succeeding set of intermediate states may be labeled by the l associated with t_l. A representative integration is

$$\begin{aligned}\int\left[\frac{D^2\xi_l(0)}{\pi}\right]&\langle\phi_{l+1}(0)|\phi_l(0)\rangle e^{-i\epsilon V(\xi_{l+1}^*(t_{l+1}),\,\xi_l(t_l))}\\ &\times e^{-i\epsilon V(\xi_l^*(t_l),\,\xi_{l-1}(t_{l-1}))}\langle\phi_l(0)|\phi_{l-1}(0)\rangle\,. \end{aligned} \quad (4.6)$$

This results in the product of $\langle\phi_{l+1}(0)|\phi_{l-1}(0)\rangle$, which just describes the propagation of the kinematical weight factor, times a dynamical part from the V's. Clearly this feature will propagate through the whole lattice giving for the matrix element for the finite interval $t'-t$

$$\langle\phi'(0)\,|U(t',t)\,|\,\phi(0)\rangle=\langle\phi'(0)\,|\,\phi(0)\rangle \times\exp\left[-i\int_t^{t'}dt''\mathcal{V}(t'')\right], \quad (4.7)$$

where $\mathcal{V}(t)$ is an effective interaction potential functional of the fields ϕ' and ϕ that describes the results of the path integration.

V. INTEGRATION OF EXTERNAL CURRENT PROBLEM

The problem of a quantized field interacting with a given external source current is the prototype for calculations with interacting quantum field theories. Our treatment in subsequent articles of more complex interacting theories will refer to this simple prototype.

We consider a system of a quantized field $\psi(\vec{x},t)$ interacting with a given external source current $j(\vec{x},t)$ through the interaction Hamiltonian

$$V=\int d^3x\,\psi(\vec{x},t)j(\vec{x},t)+\text{H.c.} \quad (5.1)$$

The normal-ordered U-matrix element for an infinitesimal time interval in the coherent-state basis in the Dirac interaction picture is

$$\langle\phi'(0)\,|U(t+\epsilon,t)\,|\,\phi(0)\rangle =\exp\left\{-i\epsilon\int d^3x[z^{*-1/2}\phi'^*(\vec{x},t+\epsilon)j(\vec{x},t) +z^{-1/2}j^*(\vec{x},t+\epsilon)\phi(\vec{x},t)]\right\}\langle\phi'(0)\,|\,\phi(0)\rangle. \quad (5.2)$$

Note that the kinematic weight factor is evaluated at $t=0$. The functional integrations over complete sets of coherent states at each of the intermediate lattice times which comprise the path integral are in this case Gaussian. That this is true independently of the structure of the inner product in the above expression for $U(t+\epsilon,t)$ is readily seen by reexpressing it in terms of expansions on a complete set of eigenfunctions $\hat{\phi}_k(\vec{x},t)$ of the free Hamiltonian. Defining $\beta_k'(t)=(\hat{\phi}_k(t),\phi'(t))$ and $\beta_k(t)=(\hat{\phi}_k(t),\phi(t))$ as before and similarly

$$\sum_k\beta_k^*(t)j_k(t)\equiv\int d^3x\,\phi^*(\vec{x},t)j(\vec{x},t)\,,$$

we have

$$\langle\phi'(0)\,|U(t+\epsilon,t)\,|\,\phi(0)\rangle=\exp\left\{-i\epsilon\sum_k[z^{*-1/2}\beta_k'(t+\epsilon)^*j_k(t)+z^{-1/2}j_k(t+\epsilon)^*\beta_k(t)]\right\} \times\exp\left(-\tfrac{1}{2}|z|^{-1}\sum_k[|\beta_k'|^2+|\beta_k|^2-2\beta_k'^*\beta_k]\right) \quad (5.3)$$

and

$$\left[\frac{D^2\phi_l(0)}{\pi|z|}\right]=\prod_k\left(\frac{d\,\text{Re}\beta_{lk}}{(\pi z^*)^{1/2}}\right)\left(\frac{d\,\text{Im}\beta_{lk}}{(\pi z)^{1/2}}\right) \quad (5.4)$$

for the integration labeled by the time $t_l=t+l\epsilon$, with $l=0,1,2,\ldots,N$, $\epsilon(N+1)=(t'-t)$. In this decomposition it is thus clear that all our integrals are Gaussian. After integrating N intermediate stages the finite-lattice path integral is

$$\langle\phi'(0)\,|U(t',t)\,|\,\phi(0)\rangle=\prod_k\left\{\exp\left[\frac{-1}{2|z|}(|\beta_k'|^2+|\beta_k|^2-2\beta_k'^*\beta_k)\right] \times\exp\left[-i\sum_{l=1}^{N+1}[z^{*-1/2}\beta_k'^*(t_l)j_k(t_{l-1})+z^{-1/2}j_k^*(t_l)\beta_k(t_{l-1})]\right] \times\exp\left[-\epsilon^2\sum_{l=1}\sum_{r=1}^{l}j_k^*(t_{l+1})j_k(t_r)e^{-i\omega_k(t_l-t_r)}\right]\right\}. \quad (5.5)$$

The basic integral is seen by setting $N=1$ in this expression. In terms of the original inner products this may be written in the continuous limit $N\to\infty$, $\epsilon\to dt''$ as

$$\langle\phi'(0)\,|U(t',t)\,|\,\phi(0)\rangle=\langle\phi'(0)\,|\,\phi(0)\rangle\exp\left\{-i\int_t^{t'}d^3x\,dt''[z^{*-1/2}\phi'^*(t'')j(t'')+z^{-1/2}j^*(t'')\phi(t'')]\right\} \times\exp\left[-\int_t^{t'}dt''d^3x\int_t^{t''}dt'''d^3y\,j(\vec{x},t'')G(\vec{x},t'';\vec{y},t''')j(\vec{y},t''')\right], \quad (5.6)$$

where

$$G(\vec{x},t';\vec{y},t) \equiv \sum_k \hat{\phi}_k(\vec{x},t')\,\hat{\phi}_k^*(\vec{y},t)\,. \qquad (5.7)$$

As expected, the kinematic factor $\langle\phi'(0)\,|\,\phi(0)\rangle$ appears in the finite-interval U-matrix element in the same way as in the matrix element for the infinitesimal interval. The exponential factor containing the double time integral is common to all U-matrix elements of this theory, including all the particle-Fock-states matrix elements that can be generated from a matrix element between coherent states. In particular this factor by itself is clearly the Fock-space vacuum-vacuum U-matrix element. All these observations also apply equally well, after taking the asymptotic limits, to the corresponding S-matrix elements.

Taking the asymptotic limit and adapting our previous prescription Eq. (3.12) for deriving the usual particle S-matrix elements we obtain the well-known result for the vacuum-vacuum matrix element,

$$_{\text{out}}\langle\Omega\,|\,\Omega\rangle_{\text{in}} = {}_{\text{in}}\langle\Omega\,|S|\,\Omega\rangle_{\text{in}} = \lim_{\substack{t\to-\infty\\ t'\to\infty}} \langle\phi'(0)\,|\,U(t',t)\,|\,\phi(0)\rangle\,|_{\phi'=0=\phi}$$

$$= \lim_{\substack{t\to-\infty\\ t'\to\infty}} \exp\left[-\int_t^{t'} dt'' \int_t^{t''} dt''' \sum_k j_k^*(t'')j_k(t''')e^{-i\omega_k(t''-t''')}\right].$$

VI. SCALAR MESON MODEL

A simple quantum field theory that we use to illustrate our formulation and methods is the model of a neutral Hermitian scalar meson field interacting with very massive nonrelativistic nucleons.[19] This model is characterized by the Hamiltonian $H = H_0 + V$ with the free part

$$H_0 = \sum_q m\psi_q^\dagger\psi_q + \sum_k \omega_k a_k^\dagger a_k\,, \qquad (6.1)$$

in which the energies of the nucleons are taken to be independent of their momenta $\vec{q}$, the meson energies are $\omega_k = (k^2+\mu^2)^{1/2}$, and the nucleon and meson creation and annihilation operators, respectively, satisfy the equal-time commutation relations

$$\{\psi_{q'}, \psi_q^\dagger\} = \delta^3(\vec{q},\vec{q}')\,,$$
$$[a_{k'}, a_k^\dagger] = \delta^3(\vec{k},\vec{k}')\,. \qquad (6.2)$$

Here m and μ are the physical masses of the nucleons and the mesons.

The interaction is given by

$$V = \lambda \sum_{q,k} (\psi_{q+k}^\dagger\psi_q a_k g_k + \text{H.c.}) - \delta m \sum_q \psi_q^\dagger\psi_q\,, \qquad (6.3)$$

where λ is the coupling constant, $g_k = f(k^2)/(2\omega_k)^{1/2}$ with $f(k^2)$ a form factor, and δm is a mass-renormalization counterterm.

A. Renormalization

We shall first consider S-matrix elements for transitions between initial and final states that are coherent states of the meson field and that have a single nucleon in a state of definite momentum since it is this sector that contains all the essential renormalization aspects of the scalar meson model. Thus the S-matrix element of interest is

$$_{\text{out}}\langle p', \phi'\,|\,p, \phi\rangle_{\text{in}} = \lim_{\substack{t\to-\infty\\ t'\to\infty}} |z_p|^{-1}\langle\phi'(0)\,|\,\psi_{p'}e^{iH_0t'}e^{-iH(t'-t)}$$
$$\times e^{-iH_0t}\psi_p^\dagger\,|\,\phi(0)\rangle\,, \qquad (6.4)$$

where z_p is the nucleon wave-function renormalization constant.

Following the analysis of Sec. IV, we find the U-matrix element for the infinitesimal interval $t_{j+1} - t_j = \epsilon$ for the interaction potential of Eq. (6.3) to first order in ϵ is

$$M_{j+1,j} = \langle\phi_{j=1}(0)\,|\,\psi_{p_{j+1}}e^{-i\epsilon V(t_j)}\psi_{p_j}^\dagger\,|\,\phi_j(0)\rangle$$
$$= \langle\phi_{j+1}(0)\,|\,\phi_j(0)\rangle\left\{\delta^3(\vec{p}_{j+1},\vec{p}_j)(1+i\epsilon\delta m) - i\epsilon\lambda\sum_{q,k}\left[\delta^3(\vec{q},\vec{p}_j)\delta^3(\vec{q}-\vec{k},\vec{p}_{j+1})g_k\beta_{j,k}e^{-i\omega_k t_j}\right.\right.$$
$$\left.\left. + \delta^3(\vec{q},\vec{p}_{j+1})\delta^3(\vec{q}-\vec{k},\vec{p}_j)g_k^*\beta_{j,k}^*e^{i\omega_k t_j}\right]\right\}\,, \qquad (6.5)$$

with $\psi_{p_j}\,|\,\phi_l(0)\rangle = 0$ for all l and p_j, and where $\beta_{j,k} = (\hat{\phi}_k(0), \phi_j(0))$, ϕ_j is the coherent-state wave function in the jth lattice stage, and $\hat{\phi}_k$ is the kth normal-mode wave function as before. The reason that only one nucleon operator enters into the intermediate-state calculation is determined by the fermion-number superselection rule and the fact that our nonrelativistic model contains no antinucleons. The momentum δ functions, which arise from the anticommutation rules of the nucleon operators, Eq. (6.2), are now replaced by their familiar Fourier integral representation. Then the integration over the set of intermediate states at the first lattice point $t_1 = t+\epsilon$ is

$$M_{2,0}=\int\left[\frac{D^2\phi_1}{\pi}\right]d^3p_1\langle\phi_2(0)\,|\,\phi_1(0)\rangle\langle\phi_1(0)\,|\,\phi(0)\rangle$$
$$\times\int d^3x_1 e^{i\vec{x}_1\cdot(\vec{p}_1-\vec{p}_2)}\int d^3x\, e^{i\vec{x}\cdot(\vec{p}-\vec{p}_1)}e^{+2i\epsilon\delta m}\exp\Big\{-i\epsilon\lambda\sum_k\big[g_k(\beta_{1k}e^{i\vec{k}\cdot\vec{x}_1-i\omega t_1}+\beta_k e^{i\vec{k}\cdot\vec{x}-i\omega_k t})$$
$$+g_k^*(\beta_{2k}^* e^{-i\vec{k}\cdot\vec{x}_1+i\omega_k t_1}+\beta_{1k}^* e^{i\vec{k}\cdot\vec{x}+i\omega_k t})\big]\Big\}. \tag{6.6}$$

Summation over the momenta $\vec{p}_1$ of the intermediate nucleon states gives $\delta^3(\vec{x}-\vec{x}_1)$, which in turn saturates the d^3x_1 integration. Thus the first lattice integration gives

$$M_{2,0}=\langle\phi_2(0)\,|\,\phi(0)e^{2i\epsilon\delta m}\int d^3x\, e^{i\vec{x}\cdot(\vec{p}-\vec{p}_2)}\exp\Big\{-i\epsilon\lambda\sum\,\big[g_k^*\beta_{2k}^* e^{-i\cdot\vec{k}\cdot\vec{x}}(e^{-i\omega_k t_1}+e^{-i\omega_k t})$$
$$+g_k\beta_k e^{i\vec{k}\cdot\vec{x}}(e^{-i\omega_k t_1}+e^{-i\omega_k t})-i\epsilon\lambda\,|g_k|^2 e^{-i\omega_k(t_1-t)}\big]\Big\}. \tag{6.7}$$

After N such lattice integrations, and letting $\epsilon\to 0$ and $N\to\infty$ keeping $(N+1)\epsilon = t'-t$ fixed we get the continuum limit

$$\langle p',\phi'(0)\,|U(t',t)\,|p,\phi(0)\rangle=\langle\phi'(0)\,|\,\phi(0)\rangle\int d^3x\, e^{i\vec{x}\cdot(\vec{p}-\vec{p}')}\exp\Big(i\delta m\int_t^{t'}dt''\Big)$$
$$\times\exp\Big\{-i\lambda\int_t^{t'}dt''\sum_k\,[\beta_k'^* g_k^* e^{-i\vec{k}\cdot\vec{x}+i\omega_k t''}+\beta_k g_k e^{i\vec{k}\cdot\vec{x}-i\omega_k t''}]$$
$$-\lambda^2\int_t^{t'}dt''\int_t^{t''}dt'''\sum_k\,|g_k|^2 e^{-i\omega_k(t''-t''')}\Big\}. \tag{6.8}$$

The one-nucleon S-matrix element is obtained from this by setting $\phi=\phi'=0$. Thus

$${}_{\rm out}\langle p'\,|p\rangle_{\rm in}=\lim_{\substack{t\to-\infty\\ t'\to\infty}}\delta^3(\vec{p}-\vec{p}')\,|z_p|^{-1}\exp\Big[i\delta m\int_t^{t'}dt''-\lambda^2\int_t^{t'}dt''\int_t^{t''}dt'''\sum_k\,|g_k|^2 e^{-i\omega_k(t''-t''')}\Big]$$
$$\equiv\delta^3(\vec{p}'-\vec{p})\,. \tag{6.9}$$

Then equating logarithms of coefficients of $\delta^3(\vec{p}-\vec{p}')$ gives

$$\lim_{t'-t\to\infty}\,[\ln|z_p|-i\delta m(t'-t)]=\lim_{t'-t\to\infty}\,(-\lambda^2)\int_t^{t'}dt''\int_t^{t''}dt'''\sum_k\,|g_k|^2 e^{i\omega_k(t''-t''')}$$
$$=\lim_{t'-t\to\infty}\,(-\lambda^2)\sum_k\,|g_k|^2\Big[\frac{(t'-t)}{i\omega_k}+\frac{1}{\omega_k^2}(e^{-i\omega_k(t'-t)}-1)\Big]. \tag{6.10}$$

The Riemann-Lebesque lemma can be invoked to drop the oscillating terms. This gives the well-known[19] results

$$\delta m=-\lambda^2\sum_k\,|g_k|^2\omega_k^{-1}\,, \tag{6.11}$$

$$\ln|z_p|=\lambda^2\sum_k\,|g_k|^2\omega_k^{-2}\,. \tag{6.12}$$

B. Meson-nucleon and meson-meson scattering

The S-matrix elements for scattering in the one-nucleon sector, as obtained from Eq. (3.12), are

$${}_{\rm out}\langle q_1'q_2'\cdots q_{\rm in}';p'\,|q_1q_2\cdots q_l;p\rangle_{\rm in}=\delta_{m,l}\delta^3(\vec{p}'-\vec{p})\prod_{i=1}^{l}\delta^3(\vec{q}_i'-\vec{q}_i)\,, \tag{6.13}$$

where the renormalization conditions, Eqs. (6.1) and (6.12), have been applied, and where one notes that the $\delta(\omega_k)$ terms which arise are zero for the case of massive mesons. This expresses the well-known results that in this model there is no nontrivial meson-nucleon or meson-meson scattering.

C. N-N scattering and effective N-N potential

To consider nucleon-nucleon scattering with our formulation we must calculate S-matrix elements between initial and final states that are coherent states of the meson field and have two nucleons in states of definite momentum,

$$_{\text{out}}\langle p',q',\phi'|p,q,\phi\rangle_{\text{in}}=\lim_{\substack{t\to-\infty\\ t'\to\infty}}|z_p|^{-2}\langle\phi'(0)|\psi_{p'}\psi_{q'}e^{iH_0t'}e^{-iH(t'-t)}e^{iH_0t}\psi_p^\dagger\psi_q^\dagger|\phi(0)\rangle. \tag{6.14}$$

The elementary U-matrix element for the infinitesimal time interval, $t_j-t_i=\epsilon$, in which we are interested is

$$N_{j,i}=\langle\phi_j(0)|\psi_{p_j}\psi_{q_j}e^{-i\epsilon V(t_j)}\psi_{p_i}^\dagger\psi_{q_i}^\dagger|\phi_i(0)\rangle. \tag{6.15}$$

As before in the single-nucleon matrix element $M_{j,i}$ the way the nucleon operators enter $N_{j,i}$ is fixed by the fermion number superselection rule and the fact that there are no antinucleons in our nonrelativistic model. The anticommutation rules for the nucleon operators, Eq. (6.2), give rise to momentum δ functions in $N_{j,i}$ which we replace by their Fourier integral representations to give

$$N_{j,i}=\tfrac{1}{2}\langle\phi_j(0)|\phi_i(0)\rangle\int d^3x\,d^3y\,[e^{i\vec{x}\cdot(\vec{q}_i-\vec{p}_j)+i\vec{y}\cdot(\vec{q}_j-\vec{p}_i)}-e^{i\vec{x}\cdot(\vec{q}_i-\vec{q}_j)+i\vec{y}\cdot(\vec{p}_j-\vec{p}_i)}]$$
$$\times\left[1+2i\epsilon\,\delta m-i\epsilon\lambda\sum_k(e^{i\vec{k}\cdot\vec{x}}+e^{-i\vec{k}\cdot\vec{y}})g_k\beta_{ik}e^{-i\omega_kt_i}-i\epsilon\lambda\sum_k(e^{i\vec{k}\cdot\vec{x}}+e^{i\vec{k}\cdot\vec{y}})g_k^*\beta_{jk}^*e^{i\omega_kt_i}\right]. \tag{6.16}$$

The form of these expressions for $N_{j,i}$ with respect to its dependence on the variables of functional integration, ϕ_j or β_{jk} and ϕ_i or β_{ik}, is the same as in the earlier case for $M_{j,i}$. The differences occur only because of the more complicated double δ functions; $(e^{i\vec{k}\cdot\vec{x}}+e^{-i\vec{k}\cdot\vec{y}})$ replaces $e^{i\vec{k}\cdot\vec{x}}$, and $2\delta m$ replaces δm. Thus although the algebra is slightly more tedious, the same description propagates through at every succeeding lattice integration. The complete S-matrix element becomes

$$_{\text{out}}\langle p',q',\phi'|p,q,\phi\rangle_{\text{in}}=\lim_{t'-t\to\infty}\langle\phi'(0)|\phi(0)\rangle\int d^3x\,d^3y\,[e^{i\vec{x}\cdot(\vec{q}-\vec{p}')+i\vec{y}\cdot(\vec{q}'-\vec{p})}-e^{i\vec{x}\cdot(\vec{q}-\vec{q}')+iy\cdot(\vec{p}'-\vec{p})}]$$
$$\times\exp\left\{-i\lambda\int_t^{t'}dt''\sum_k[\beta_k'^*g_k^*(e^{i\vec{k}\cdot\vec{x}}+e^{-i\vec{k}\cdot\vec{y}})e^{i\omega_kt''}+\beta_kg_k(e^{-i\vec{k}\cdot\vec{x}}+e^{i\vec{k}\cdot\vec{y}})e^{-i\omega_kt''}]+2\lambda^2\sum_k|g_k|^2e^{i\vec{k}\cdot(\vec{x}+\vec{y})}\int_t^{t'}dt''\int_t^{t''}dt'''e^{-i\omega_k(t''-t''')}\right\}, \tag{6.17}$$

where we have already cancelled $2\ln|z_p|-2i\delta m(t'-t)$ out of the double time integral using the renormalization identities defined in the single-nucleon matrix element.

The exact nucleon-nucleon scattering matrix element is obtained by setting $\phi=0=\phi'$. To find the static potential between a pair of nucleons we expand to lowest order in the interaction, i.e., to order λ^2, giving

$$_{\text{out}}\langle p'q'|p,q\rangle_{\text{in}}\cong\tfrac{1}{2}\Big[\delta^3(\vec{q}'-\vec{p})\,\delta^3(\vec{p}'-\vec{q})-\delta^3(\vec{q}-\vec{q}')\,\delta^3(\vec{p}-\vec{p}')+\delta^3(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')2\lambda^2T\left(\frac{|g_{q-p'}|^2}{i\omega_{q-p'}}-\frac{|g_{q-q'}|^2}{i\omega_{q-q'}}\right)\Big], \tag{6.18}$$

where we retained only the leading asymptotic behavior

$$\lim_{\substack{t\to-\infty\\ t'\to\infty}} \int_t^{t'} dt'' \int_t^{t''} dt''' e^{-i\omega_k(t''-t''')} = \lim_{t'-t\to} \frac{t'-t}{i\omega_k} \equiv \frac{T}{i\omega_k}. \quad (6.19)$$

We can identify T with the energy δ function

$$T = \int_{-\infty}^{\infty} dt = 2\pi\delta(2m - 2m),$$

and we can identify the static potential between a pair of nucleons as

$$V(\vec{x}-\vec{y}) = \lambda^2 \int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{|g_k|^2}{\omega_k}. \quad (6.20)$$

All of these results are well known, having been previously obtained by more familiar methods.[19]

VII. RELATION OF THIS FORMULATION TO SOME OTHER PATH-INTEGRAL REPRESENTATIONS OF QUANTUM AMPLITUDES

Our construction of the path integral for the U matrix emphasizes its composition of distinct kinematical and dynamical parts. The dynamical part is, of course, evident in the interaction exponential factor of the differential element of the path integral for the infinitesimal time interval. The kinematical part appears in the differential path element as the measure with which the dynamical effects are integrated. This kinematical measure has the coherent-state overlap factor $\langle\phi_{j+1}(0)|\phi_j(0)\rangle$ as a weight functional times $D^2\phi_j$.

It is also of interest to point out the relation of our path-integral formulation to the usual one. The variable of functional integration over the jth complete set of intermediate states, which we take as coherent states of the free field, is the coherent-state wave function ϕ_j which is associated with the time $t_j = t + j\epsilon$. By reabsorbing the time dependence back into the free-field coherent states the jth differential element of the path integral may be written

$$\langle\phi_{j+1}(0)|\phi_j(0)\rangle \exp[-i\epsilon V(z^{-1/2}\phi^*_{j+1}(t_{j+1}), \phi_j(t_j)z^{*-1/2})]D^2\phi_j$$
$$= \langle\phi_{j+1}(t_j+\epsilon)|\phi_j(t_j)\rangle \exp[-i\epsilon H(z^{-1/2}\phi^*_{j+1}(t_j+\epsilon), \phi_j(t_j)z^{*-1/2})]D^2\phi_j. \quad (7.1)$$

The kinematical weight functional now has the form

$$\exp\{|z|^{-1}[-\tfrac{1}{2}(\phi_{j+1}(t_j+\epsilon), \phi_{j+1}(t_j+\epsilon)) - \tfrac{1}{2}(\phi_j(t_j), \phi_j(t_j)) + (\phi_{j+1}(t_j+\epsilon), \phi_j(t_j))]\}. \quad (7.2)$$

The two-time inner product is the obvious generalization of the inner product defined earlier. For canonical systems of fields there inner products are always of the form

$$(\phi_j(t_j), \phi_i(t_i)) = \int d^3x\, J_0(\phi_j(\vec{x}, t_j), \phi_i(\vec{x}, t_i))$$
$$= -i\int d^3x\, \pi_j(t_j)\phi_i(t_i), \quad (7.3)$$

where $\pi(\vec{x}, t)$ is the field canonically conjugate to $\phi(\vec{x}, t)$ in some Lagrangian $\pi = \partial\mathcal{L}/\partial\dot{\phi}$. The kinematical weight factor is now, to order ϵ, the exponential of

$$i\epsilon \int d^3x[\tfrac{1}{2}\pi_j(\vec{x}, t_j)\dot{\phi}_j(\vec{x}, t_j) - \tfrac{1}{2}\dot{\pi}_j(\vec{x}, t_j)\phi_j(\vec{x}, t_j)]$$
$$= i\epsilon \int d^3x\left\{\pi_j(\vec{x}, t_j)\dot{\phi}_j(\vec{x}, t_j) - \frac{1}{2}\frac{d}{dt_j}[\pi_j(\vec{x}, t_j)\phi_j(\vec{x}, t_j)]\right\}. \quad (7.4)$$

The sum of this with the term $-i\epsilon H = -i\epsilon \int d^3x\, \mathcal{H}(\phi_j(\vec{x}, t_j), \pi_j(\vec{x}, t_j))$, we define to be the action

$$i\epsilon \int d^3x\, \mathcal{L}(\phi_j(\vec{x}, t)),$$

with

$$\mathcal{L}(\phi(\vec{x}, t)) = \pi(\vec{x}, t)\dot{\phi}(\vec{x}, t) - \mathcal{H}(\phi(\vec{x}, t), \pi(\vec{x}, t)) - \frac{1}{2}\frac{d}{dt}[\pi(\vec{x}, t)\phi(\vec{x}, t)]. \quad (7.5)$$

This differs from the usual expression only by the total time derivative which is inconsequential for the determination of the equations of motion.

Thus the path integral in our formulation is equivalent to the usual path-integral representation for quantum field theories that are of canonical form. However, our formulation is in fact more general and can be applied to theories which have no canonical form and no Lagrangian action with which to formulate the usual path-integral representation.[20,17] We simply do not have to assume the canonical form for the bilinear current J_μ.

The operators of our formulation are based on the bilinear conserved currents which are determined solely from the field equations, rather than on a canonical quantization procedure.[8,9,10,17] These bilinear operators are essentially self-adjoint on a dense domain spanned by a set of coherent states and have been shown to be general.[11] These operators describe the whole q-number theory, ensure its covariance, and give conditions on equal-time commutation relations, if any exist in the theory; all independently of any separate postulation of a locality or a quasilocality condition.

Examples of such noncanonical, non-Lagrangian quantum field theories are ultralocal models which can be obtained by omitting spatial gradient terms from the Hamiltonians of certain covariant theories. These models are interesting because, although they are noncovariant, they are exactly solvable.[8,21] Interacting ultralocal systems are not continuously connected to the corresponding free systems in the limit as the interaction coupling constant is turned off. The dynamics of these systems can be well defined. However, $\pi = i\dot{\phi}$ does not share a dense domain in common with the Hamiltonian, and thus is not well behaved. Since no canonical momentum operator exists for the interacting theory no canonical form exists and the Legendre transformation to the Lagrangian does not exist.[22] Therefore the path-integral representation in the usual form with the Lagrangian action does not exist. On the other hand our formulation of a path-integral representation for the U-matrix does exist.

Another, less exotic, example occurs for field theories which contain ghost fields. Such fields are introduced into a theory whenever, in addition to the usual field equations, auxiliary conditions must also be satisfied by one or more of the fields. The ghost fields "act as Lagrange multipliers which enforce these constraints. Without the use of these ghost fields, a Lagrangian cannot be constructed which generates the field equations and the auxiliary conditions. In other words, because the physical fields are not linearily independent the theory is not canonical. These extra fields "clutter up" the usual path-integral formalism because they must be treated as dummy variables which are integrated out. The extra integrations can be avoided in the formalism presented here because it is unnecessary to postulate the ghost fields in the first instance. The particular example of a system of a Dirac field minimally coupled to a massive vector field, as well as the ultralocal models, is treated in Ref. 8.

In the usual path-integral formulation of a quantum field theory the generating functional of the vacuum Green's functions is integrated with the field interacting with an auxiliary external source current. This source current is purely auxiliary serving as a variational parameter for generating the Green's functions; after which it is set to zero. However, the calculations are done before this external source is set to zero. This source usually violates some of the conservation laws, so that the calculation is deprived of some of the symmetries that could otherwise facilitate the calculation. The symmetries are recovered only after setting the external sources to zero. Also, this treatment is biased in favor of the consideration of transitions between initial and final states that are the Fock vacuum or states near it in Fock space.

However, a totally different perspective can now be taken for this analysis because we have shown within the Lehmann-Symanzik-Zimmermann framework, that for canonical theories the usual generating functional, with the external source set to zero, *is* the S matrix in the coherent-state representation. Consequently, Eq. (3.4) applies, and the S-matrix elements can be obtained directly by using the coherent-state wave functions as variational parameters.

On the other hand, one need not take this approach when considering problems with nonvacuum ground states such as condensates of solitons, superfluids, etc., since it is the coherent-state S-matrix element itself which is then of interest.

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*Permanent address: Physics Department, Washington University, St. Louis, Missouri 63130.

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